Chapter 3: VECTORS

Scalars and Vectors *Scalars,* such as temperature, have magnitude only. They are specified by a number with a unit (10°C) and obey the rules of arithmetic and ordinary algebra. *Vectors,* such as displacement, have both magnitude and direction (5 m, north) and obey the rules of vector algebra.

Adding Vectors Geometrically Two vectors $\vec{a}$ and $\vec{b}$ may be added geometrically by drawing them to a common scale and placing them head to tail. The vector connecting the tail of the first to the head of the second is the vector sum $\vec{s}$. To subtract $\vec{b}$ from $\vec{a}$, reverse the direction of $\vec{b}$ to get $-\vec{b}$; then add $-\vec{b}$ to $\vec{a}$. Vector addition is commutative

$$\vec{a} + \vec{b} = \vec{b} + \vec{a} \qquad (3\text{-}2)$$

and obeys the associative law

$$(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c}). \qquad (3\text{-}3)$$

Components of a Vector The (scalar) *components* a_x and a_y of any two-dimensional vector $\vec{a}$ along the coordinate axes are found by dropping perpendicular lines from the ends of $\vec{a}$ onto the coordinate axes. The components are given by

$$a_x = a\cos\theta \quad \text{and} \quad a_y = a\sin\theta, \qquad (3\text{-}5)$$

where θ is the angle between the positive direction of the x axis and the direction of $\vec{a}$. The algebraic sign of a component indicates its direction along the associated axis. Given its components, we can find the magnitude and orientation (direction) of the vector $\vec{a}$ by using

$$a = \sqrt{a_x^2 + a_y^2} \quad \text{and} \quad \tan\theta = \frac{a_y}{a_x} \qquad (3\text{-}6)$$

Unit-Vector Notation *Unit vectors* $\hat{\text{i}}$, $\hat{\text{j}}$, and $\hat{\text{k}}$ have magnitudes of unity and are directed in the positive directions of the *x, y,* and *z* axes, respectively, in a right-handed coordinate system (as defined by the vector products of the unit vectors). We can write a vector $\vec{a}$ in terms of unit vectors as

$$\vec{a} = a_x\hat{\text{i}} + a_y\hat{\text{j}} + a_z\hat{\text{k}}, \qquad (3\text{-}7)$$

in which $a_x\hat{\text{i}}$, $a_y\hat{\text{j}}$, and $a_z\hat{\text{k}}$ are the **vector components** of $\vec{a}$ and a_x, a_y, and a_z are its **scalar components.**

Adding Vectors in Component Form To add vectors in component form, we use the rules

$$r_x = a_x + b_x \qquad r_y = a_y + b_y \qquad r_z = a_z + b_z. \qquad \text{(3-10 to 3-12)}$$

Here $\vec{a}$ and $\vec{b}$ are the vectors to be added, and $\vec{r}$ is the vector sum. Note that we add components axis by axis. We can then express the sum in unit-vector notation or magnitude-angle notation.

Product of a Scalar and a Vector The product of a scalar s and a vector $\vec{v}$ is a new vector whose magnitude is sv and whose direction is the same as that of $\vec{v}$ if s is positive, and opposite that of $\vec{v}$ if s is negative. (The negative sign reverses the vector.) To divide $\vec{v}$ by s, multiply $\vec{v}$ by $1/s$.

The Scalar Product The **scalar** (or **dot**) **product** of two vectors $\vec{a}$ and $\vec{b}$ is written $\vec{a}\cdot\vec{b}$ and is the *scalar* quantity given by

$$\vec{a}\cdot\vec{b} = ab\cos\phi, \qquad \text{(3-20)}$$

in which ϕ is the angle between the directions of $\vec{a}$ and $\vec{b}$. A scalar product is the product of the magnitude of one vector and the scalar component of the second vector along the direction of the first vector. Note that $\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}$, which means that the scalar product obeys the commutative law.

In unit-vector notation,

$$\vec{a}\cdot\vec{b} = (a_x\hat{\mathrm{i}} + a_y\hat{\mathrm{j}} + a_z\hat{\mathrm{k}})\cdot(b_x\hat{\mathrm{i}} + b_y\hat{\mathrm{j}} + b_z\hat{\mathrm{k}}), \qquad \text{(3-22)}$$

which may be expanded according to the distributive law.

The Vector Product The **vector** (or **cross**) **product** of two vectors $\vec{a}$ and $\vec{b}$ is written $\vec{a}\times\vec{b}$ and is a *vector* $\vec{c}$ whose magnitude c is given by

$$c = ab\sin\phi, \qquad \text{(3-24)}$$

in which ϕ is the smaller of the angles between the directions of $\vec{a}$ and $\vec{b}$. The direction of $\vec{c}$ is perpendicular to the plane defined by $\vec{a}$ and $\vec{b}$ and is given by a right-hand rule, as shown in Fig. 3-19. Note that $\vec{a}\times\vec{b} = -(\vec{b}\times\vec{a})$, which means that the vector product does not obey the commutative law.

In unit-vector notation,

$$\vec{a}\times\vec{b} = (a_x\hat{\mathrm{i}} + a_y\hat{\mathrm{j}} + a_z\hat{\mathrm{k}})\times(b_x\hat{\mathrm{i}} + b_y\hat{\mathrm{j}} + b_z\hat{\mathrm{k}}), \qquad \text{(3-26)}$$

which we may expand with the distributive law.

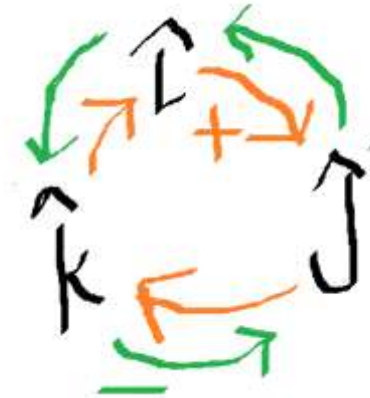
Products of Vectors

Let $\hat{\mathrm{i}}$, $\hat{\mathrm{j}}$, and $\hat{\mathrm{k}}$ be unit vectors in the x, y, and z directions. Then

$$\hat{\mathrm{i}}\cdot\hat{\mathrm{i}}=\hat{\mathrm{j}}\cdot\hat{\mathrm{j}}=\hat{\mathrm{k}}\cdot\hat{\mathrm{k}}=1, \quad \hat{\mathrm{i}}\cdot\hat{\mathrm{j}}=\hat{\mathrm{j}}\cdot\hat{\mathrm{k}}=\hat{\mathrm{k}}\cdot\hat{\mathrm{i}}=0,$$

$$\hat{\mathrm{i}}\times\hat{\mathrm{i}}=\hat{\mathrm{j}}\times\hat{\mathrm{j}}=\hat{\mathrm{k}}\times\hat{\mathrm{k}}=0,$$

$$\hat{\mathrm{i}}\times\hat{\mathrm{j}}=\hat{\mathrm{k}}, \quad \hat{\mathrm{j}}\times\hat{\mathrm{k}}=\hat{\mathrm{i}}, \quad \hat{\mathrm{k}}\times\hat{\mathrm{i}}=\hat{\mathrm{j}}$$



Any vector $\vec{a}$ with components a_x, a_y, and a_z along the x, y, and z axes can be written as

$$\vec{a}=a_x\hat{\mathrm{i}}+a_y\hat{\mathrm{j}}+a_z\hat{\mathrm{k}}.$$

Let $\vec{a}$, $\vec{b}$, and $\vec{c}$ be arbitrary vectors with magnitudes a, b, and c. Then

$$\vec{a}\times(\vec{b}+\vec{c})=(\vec{a}\times\vec{b})+(\vec{a}\times\vec{c})$$

$$(s\vec{a})\times\vec{b}=\vec{a}\times(s\vec{b})=s(\vec{a}\times\vec{b}) \qquad (s = \text{a scalar}).$$

Let θ be the smaller of the two angles between $\vec{a}$ and $\vec{b}$. Then

$$\vec{a}\cdot\vec{b}=\vec{b}\cdot\vec{a}=a_xb_x+a_yb_y+a_zb_z=ab\cos\theta$$

$$\vec{a}\times\vec{b}=-\vec{b}\times\vec{a}=\begin{vmatrix}\hat{\mathrm{i}} & \hat{\mathrm{j}} & \hat{\mathrm{k}}\\ a_x & a_y & a_z\\ b_x & b_y & b_z\end{vmatrix}$$

$$=\hat{\mathrm{i}}\begin{vmatrix}a_y & a_z\\ b_y & b_z\end{vmatrix}-\hat{\mathrm{j}}\begin{vmatrix}a_x & a_z\\ b_x & b_z\end{vmatrix}+\hat{\mathrm{k}}\begin{vmatrix}a_x & a_y\\ b_x & b_y\end{vmatrix}$$

$$=(a_yb_z-b_ya_z)\hat{\mathrm{i}}+(a_zb_x-b_za_x)\hat{\mathrm{j}}$$

$$+(a_xb_y-b_xa_y)\hat{\mathrm{k}}$$

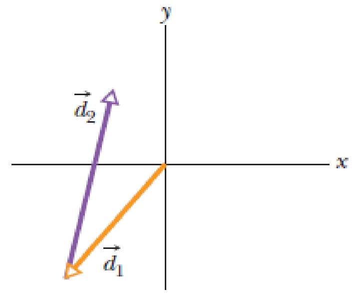
$$|\vec{a}\times\vec{b}|=ab\sin\theta$$

$$\vec{a}\cdot(\vec{b}\times\vec{c})=\vec{b}\cdot(\vec{c}\times\vec{a})=\vec{c}\cdot(\vec{a}\times\vec{b})$$

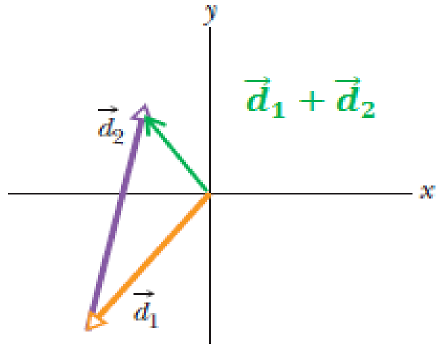
$$\vec{a}\times(\vec{b}\times\vec{c})=(\vec{a}\cdot\vec{c})\vec{b}-(\vec{a}\cdot\vec{b})\vec{c}$$

Chapter 3: VECTORS

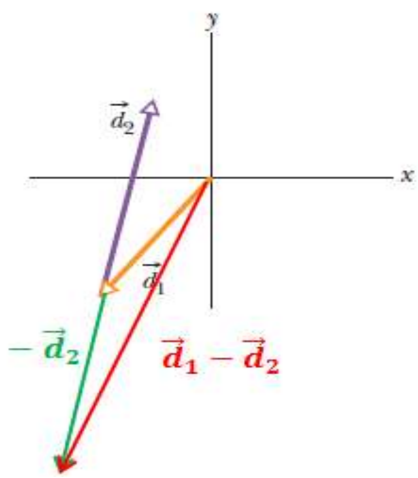
Q-2) The two vectors shown in the below figure lie in an xy plan. What are the signs of the x and y components, respectively, of (a) $\vec{d}_1 + \vec{d}_2$, (b) $\vec{d}_1 - \vec{d}_2$, and (c) $\vec{d}_2 - \vec{d}_1$?



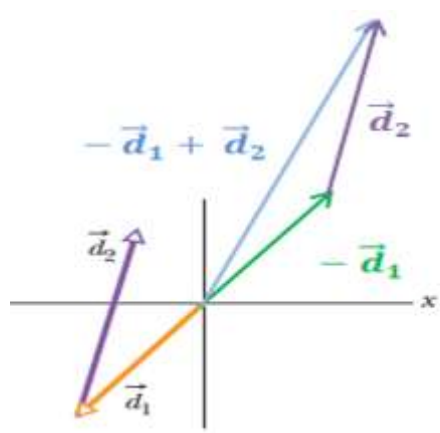
(a) The x component is negative and the y component is positive, $\vec{d}_1 + \vec{d}_2$ lies in the second quadrant



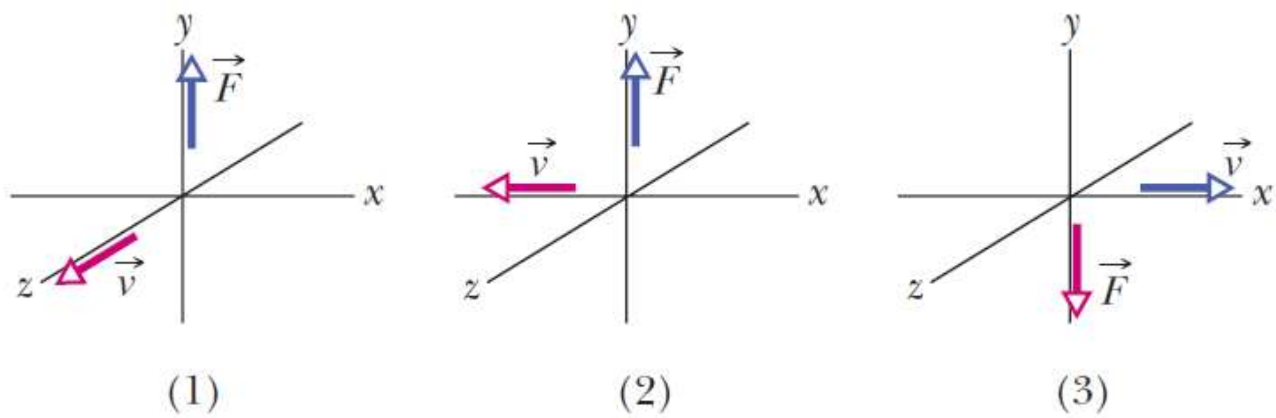
(b) $\vec{d}_1 - \vec{d}_2$ lies in the third quadrant in which its both components are negative



(c) $\vec{d}_2 - \vec{d}_1$ lies in the first quadrant so the both components are positive



Q-9) If $\vec{F} = q(\vec{v} \times \vec{B})$ and $\vec{v}$ is perpendicular to $\vec{B}$, then what is the direction of $\vec{B}$ in the three situations shown in the below figure when constant q is (a) positive and (b) negative?



$\vec{F} = q(\vec{v} \times \vec{B})$

(a) q is positive:

1) $\vec{F}: +\hat{j}, \vec{v}: +\hat{k} \rightarrow\rightarrow \vec{B}: +\hat{i}$ (Positive x axis)

2) $\vec{F}: +\hat{j}, \vec{v}: -\hat{i} \rightarrow\rightarrow \vec{B}: +\hat{k}$ (Positive z axis)

3) $\vec{F}: -\hat{j}, \vec{v}: +\hat{i} \rightarrow\rightarrow \vec{B}: +\hat{k}$ (Positive z axis)

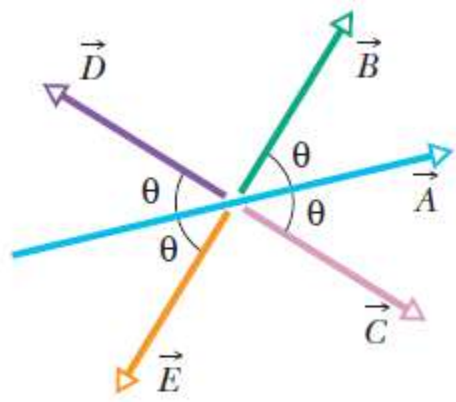
(b) q is negative:

1) $\vec{F}: +\hat{j}, \vec{v}: +\hat{k} \rightarrow\rightarrow \vec{B}: -\hat{i}$ (Negative x axis)

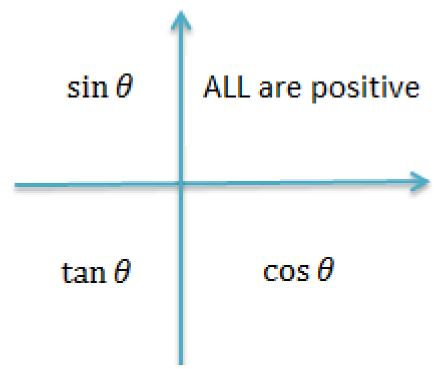
2) $\vec{F}: +\hat{j}, \vec{v}: -\hat{i} \rightarrow\rightarrow \vec{B}: -\hat{k}$ (Negative z axis)

3) $\vec{F}: -\hat{j}, \vec{v}: +\hat{i} \rightarrow\rightarrow \vec{B}: -\hat{k}$ (Negative z axis)

Q-10) The below figure shows vector $\vec{A}$ and four other vectors that have the same magnitude but differ in orientation. (a) which of those other four vectors have the same dot product with $\vec{A}$? (b) which have a negative dot product with $\vec{A}$?



$\vec{A}.\vec{B} = |\vec{A}||\vec{B}|\cos\theta$



(a) $\vec{A}.\vec{B} = \vec{A}.\vec{C}$

$\vec{A}.\vec{D} = \vec{A}.\vec{E}$

(b) Negative dot product:

$\vec{A}.\vec{D}$ and $\vec{A}.\vec{E}$ are negative

$\vec{A}.\vec{D} = |\vec{A}||\vec{D}|\cos(180^\circ - \theta) = -|\vec{A}||\vec{D}|\cos(\theta)$

P-5) A ship sets out to sail to a point 120 km due north. An unexpected storm blows the ship to a point 100 km due east of its starting point. (a) How far and (b) in what direction must it now sail to reach its original destination?

$$\vec{d}_{final} = (120\ Km)\ \hat{j}$$
$$\vec{d}_{storm} = (100\ Km)\ \hat{i}$$
$$\vec{d}_{sail} = ?$$

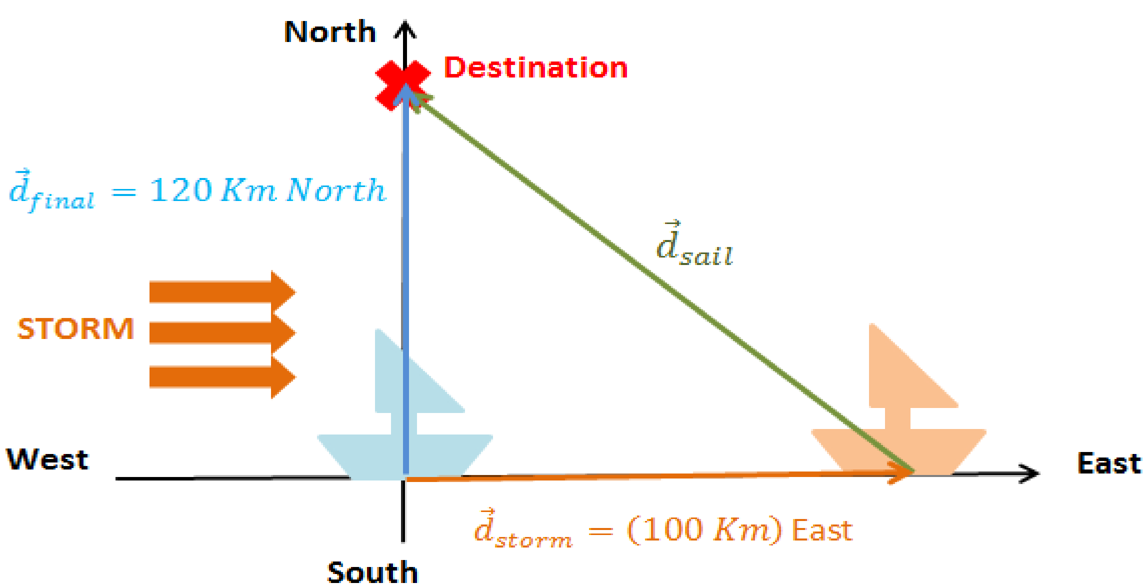
$$\vec{d}_{final} = \vec{d}_{storm} + \vec{d}_{sail}$$
$$(120\ Km)\ \hat{j} = (100\ Km)\ \hat{i} + \vec{d}_{sail}$$
$$\vec{d}_{sail} = (-100\ Km)\ \hat{i} + (120\ Km)\ \hat{j}$$

$$|\vec{d}_{sail}| = \sqrt{(-100\ Km)^2 + (120\ Km)^2} = 156.2\ Km$$

$\theta = \tan^{-1}\left(\frac{120}{100}\right) = 50.2^{\circ}$ but $\vec{d}_{sail}$ has negative x component and positive y component. That means $\vec{d}_{sail}$ lies in the second quadrant ($180^{\circ} - 50.2^{\circ} = 129.8^{\circ}$)

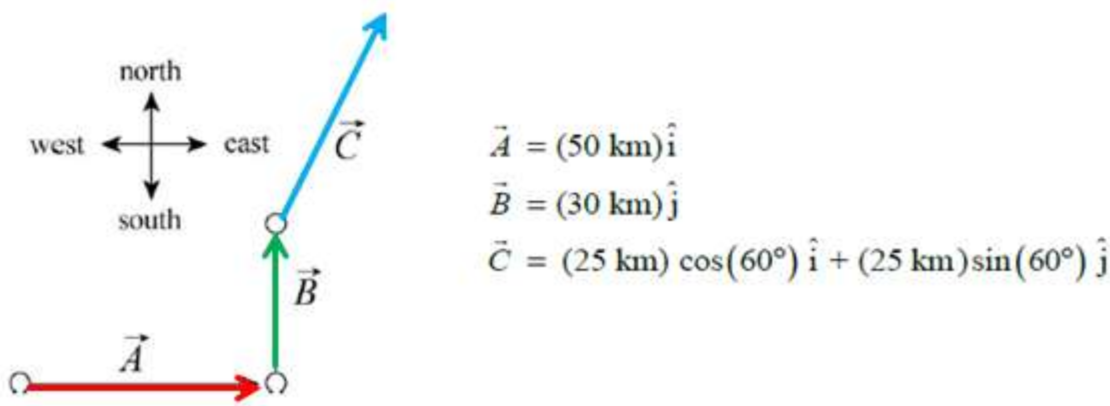
A ship must sail for 156.2 Km with 129.8° counter clockwise the positive x axis to reach its original destination.

Note: 129.8° counterclockwise from east → 39.8° west from north → 50.2° north from west.



P-12) A car is driven east for a distance of 50 km, then north for 30 km, and then in a direction 30° east of north for 25 km. Sketch the vector diagram and determine (a) the magnitude and (b) the angle of the car's total displacement from its starting point.

The angle between $\vec{C}$ and the positive x axis is 60°



(a) The car's total displacement from its starting point:

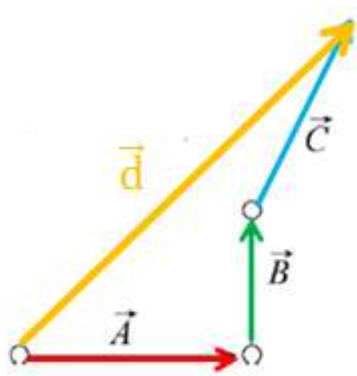
$$\vec{d} = \vec{A} + \vec{B} + \vec{C} = (62.5\ \text{Km})\,\hat{\imath} + (51.7\ \text{Km})\,\hat{\jmath}$$

The magnitude of the car's displacement from its starting point:

$$|\vec{d}| = \sqrt{(62.5\ \text{Km})^2 + (51.7\ \text{Km})^2} = 81.1\ \text{Km}$$

(b) The angle $\theta = \tan^{-1}\left(\frac{51.7}{62.5}\right) = 39.6^{\circ}$ (counter clockwise from the + x axis)

The car 81 Km far away from its starting point and it points 40° North of East



P-36) If $\vec{d}_1 = 3\hat{i} - 2\hat{j} + 4\hat{k}$ and $\vec{d}_2 = -5\hat{i} + 2\hat{j} - \hat{k}$, then what is the $(\vec{d}_1 + \vec{d}_2).(\vec{d}_1 \times 4\vec{d}_2)$?

$\vec{d}_1 + \vec{d}_2 = -2\hat{i} + 3\hat{k}$

$4\vec{d}_2 = -20\hat{i} + 8\hat{j} - 4\hat{k}$

$$\vec{d}_1 \times 4\vec{d}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -2 & 4 \\ -20 & 8 & -4 \end{vmatrix} = \begin{vmatrix} -2 & 4 \\ 8 & -4 \end{vmatrix}\hat{i} - \begin{vmatrix} 3 & 4 \\ -20 & -4 \end{vmatrix}\hat{j} + \begin{vmatrix} 3 & -2 \\ -20 & 8 \end{vmatrix}\hat{k}$$

$\vec{d}_1 \times 4\vec{d}_2 = -24\,\hat{\imath} - 68\,\hat{\jmath} - 16\,\hat{k}$

$(\vec{d}_1 + \vec{d}_2).(\vec{d}_1 \times 4\vec{d}_2) = (-2\hat{\imath} + 3\hat{k}).(-24\,\hat{\imath} - 68\,\hat{\jmath} - 16\,\hat{k}) = 48 - 48 = 0$

Without doing any calculations, **The DOT PRODUCT (Scalar Product) of PERPENDICULAR Vectors is ZERO.**

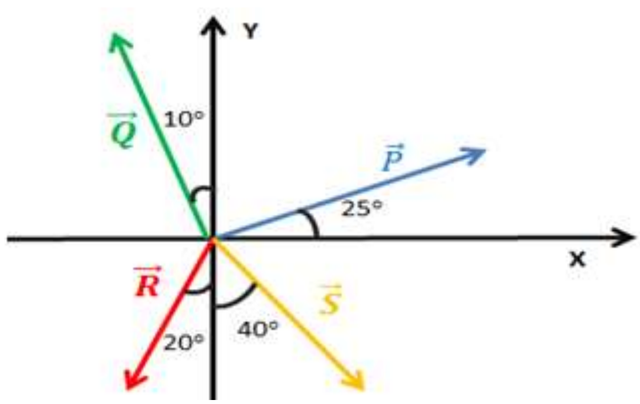
P-56) Find the sum of the following four vectors in (a) unit-vector notation, and as (b) a magnitude and (c) an angle relative to + x.

$\vec{P}$: 10.0 m, at 25.0° counterclockwise from + x

$\vec{Q}$: 12.0 m, at 10.0° counterclockwise from + y

$\vec{R}$: 8.00 m, at 20.0° clockwise from - y

$\vec{S}$: 9.00 m, at 40.0° counterclockwise from –y



$$\vec{P} = 10.0\ m\ (\cos 25^{\circ}\ \hat{\imath} + \sin 25^{\circ}\ \hat{\jmath}) = 9.06\ \hat{\imath} + 4.23\ \hat{\jmath}$$

$$\vec{Q} = 12.0\ m\ (\cos 100^{\circ}\ \hat{\imath} + \sin 100^{\circ}\ \hat{\jmath}) = 12.0\ m\ (-\sin 10^{\circ}\ \hat{\imath} + \cos\ 10^{\circ}\ \hat{\jmath})$$

$$\vec{Q} = -2.08\ \hat{\imath} + 11.8\ \hat{\jmath}$$

$$\vec{R} = 8.0\ m\ (\cos 250^{\circ}\ \hat{\imath} + \sin 250^{\circ}\ \hat{\jmath}) = 8.0\ m\ (-\sin 20^{\circ}\ \hat{\imath} - \cos\ 20^{\circ}\ \hat{\jmath})$$

$$\vec{R} = -2.74\ \hat{\imath} - 7.52\ \hat{\jmath}$$

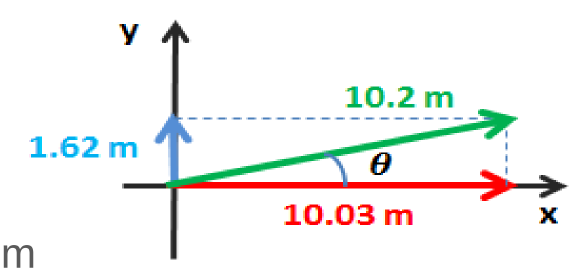
$$\vec{S} = 9.0\ \text{m}\ (\cos 310^{\circ}\ \hat{\imath} + \sin 310^{\circ}\ \hat{\jmath}) = 9.0\ \text{m}\ (\sin 40^{\circ}\ \hat{\imath} - \cos\ 40^{\circ}\ \hat{\jmath})$$

$$\vec{S} = 5.79\ \text{m}\ \hat{\imath} - 6.89\ \text{m}\ \hat{\jmath}$$

(a) $\vec{P} + \vec{Q} + \vec{R} + \vec{S} = (10.03\ \text{m})\ \hat{\imath} + (1.62\ \text{m})\ \hat{\jmath}$

(b) $|\vec{P} + \vec{Q} + \vec{R} + \vec{S}| = \sqrt{(10.03)^2 + (1.62)^2} = 10.2\ \text{m}$

(c) $\theta = \tan^{-1}\left(\frac{1.62}{10.03}\right) = 9.2^{\circ}$ (counter clockwise from the + x axis)



P-73) Two vectors are given by $\vec{a} = 3.0\hat{i} + 5.0\hat{j}$ and $\vec{b} = 2.0\hat{i} + 4.0\hat{j}$. Find (a) $\vec{a} \times \vec{b}$, (b) $\vec{a} \cdot \vec{b}$, (c) $(\vec{a} + \vec{b}) \cdot \vec{b}$, and (d) the component of $\vec{a}$ along the direction of $\vec{b}$?

(a) $\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 5 & 0 \\ 2 & 4 & 0 \end{vmatrix} = 2.0\,\hat{k}$

(b) $\vec{a} \cdot \vec{b} = (3.0\hat{i} + 5.0\hat{j}) \cdot (2.0\hat{i} + 4.0\hat{j}) = 26.0$

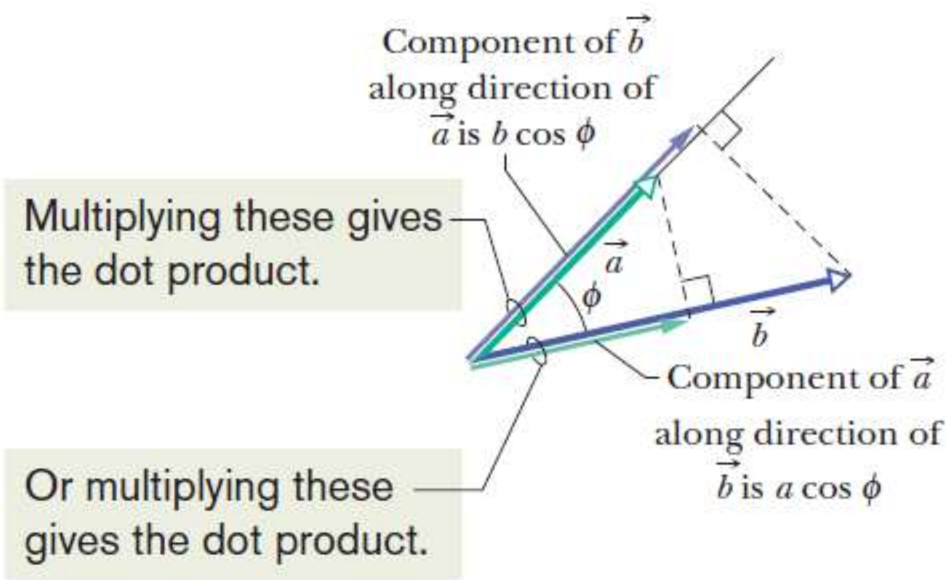
(c) $(\vec{a} + \vec{b}) \cdot \vec{b} = \left((3.0\hat{i} + 5.0\hat{j}) + (2.0\hat{i} + 4.0\hat{j})\right) \cdot (2.0\hat{i} + 4.0\hat{j})$

$$= (5.0\hat{i} + 9.0\hat{j}) \cdot (2.0\hat{i} + 4.0\hat{j}) = 46$$

(d) The component of $\vec{a}$ along the direction of $\vec{b} = |\vec{a}| \cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{b}|}$

$|\vec{b}| = 4.47$, $|\vec{a}| = 5.83$

$$\frac{\vec{a} \cdot \vec{b}}{|\vec{b}|} = \frac{26.0}{4.47} = 5.82$$



$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta \rightarrow\rightarrow \cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|} = \frac{26.0}{4.47 \times 5.83} \rightarrow\rightarrow \theta = 3.89^{\circ}$$