## Exercises:

### 1.2.0:

a. If $a<b$ and $c<d<0$, then $ac>bd$. False.

counter exp: $a=\frac{2}{3}$, $b=1$, $C=-2$, $d=-1$

$ac<bd$ not $ac>bd$.

b. If $a\leq b$ and $c\geq 1$, then $|a+c|\leq|b+c|$ False

$a=-4$, $b=-1$, and $C=2$.

c. If $a\leq b$ and $b\leq a+c$, then $|a-b|\leq C$. True.

$|a-b|=|b-a|$ Thm 2 (ii)

$|a-b|=b-a$ $b\geq a$ (hypothisis).

$\underset{-a}{b}\leq a+C-a$ (hypothisis).

$b-a\leq \cancel{a}+C-\cancel{a}$

$b-a\leq C$ #

d. If $a<b-\varepsilon$ for all $\varepsilon>0$, then $a<0$. true.

No $a\in\mathbb{R}$ satisfies $a<b-\varepsilon$ for all $\varepsilon>0$ so the inequality is vacuously satisfied.

proof: If $b\leq 0$ then $a<b-\varepsilon<0$

If $b>0$ then for $\varepsilon=b$ $\to$ $a<b-\varepsilon=0$.

$a<0$.

1.2.1 : $a, b, c \in \mathbb{R}$ and $\underset{\smile}{a \leq b}$.

a. prove that $a+c \leq b+c$.

case 1 : If $\underset{+c}{a} < \underset{+c}{b}$ (Additive property).

$$\boxed{a+c < b+c}.$$

case 2 : If $\underset{+c}{a} = \underset{+c}{b}$ since $+$ is function.

$$\boxed{a+c = b+c}$$

Thus, $a+c \leq b+c$ holds for all $a \leq b$.

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b. If $\underset{\smile}{c \geq 0}^{\text{2 cases}}$, prove that $a.c \leq b.c$

If $c=0$ then $a.c = 0 = b.c$

So we may spse $c > 0$.

If $a < b$ then $ac < bc$ (multiplicative property

If $a=b$ then $ac = bc$ since $\cdot$ is function.

Thus, $a.c \leq b.c$ $\forall$ $c \geq 0$ and $a \leq b$.

1.2.3:

a. prove that $a = a^+ - a^-$ and $|a| = a^+ + a^-$

By Def: $a^+ - a^- = \dfrac{|a| + a - (|a| - a)}{2}$

$= \dfrac{\cancel{|a|} + a - \cancel{|a|} + a}{2} = \dfrac{\cancel{2}a}{\cancel{2}} = \boxed{a}$ #

and $a^+ + a^- = \dfrac{|a| + \cancel{a} + |a| - \cancel{a}}{2} = \dfrac{\cancel{2}|a|}{\cancel{2}} = |a|$ #

b. prove that $a^+ = \begin{cases} a, & a \geq 0 \\ 0, & a \leq 0 \end{cases}$ and $a^- = \begin{cases} 0, & a \geq 0 \\ -a, & a \leq 0 \end{cases}$

If $a \geq 0$ then $a^+ = \dfrac{(a+a)}{2} = \dfrac{2a}{2} = a$.

If $a < 0$ then $a^+ = \dfrac{(-a+a)}{2} = 0$.

similarly:

If $a \geq 0$ then $a^- = 0$

If $a < 0$ then $a^- = -a$

1.2.4: Solve each of the following inequalities for $x \in \mathbb{R}$.

a. $|2x+1| < 7 \rightarrow \left(\underset{-1}{-7} < \underset{-1}{2x+1} < \underset{-1}{7}\right) \div 2$

$-4 < x < 3$.

b. $|2-x| < 2 \rightarrow \underset{-2}{-2} < \underset{-2}{2-x} < \underset{-2}{2}$

$(-4 < -x < 0) \times -1$

$4 > x > 0$.

c. $|x^3-3x+1| < x^3 \rightarrow \underset{-x^3}{-x^3} < \underset{-x^3}{\cancel{x^3}}-3x+1 < \underset{-x^3}{x^3}$

$(-2x^3 < -3x+1 < 0) \times -1$

$2x^3 > 3x-1 > 0$

$3x-1 > 0 \rightarrow x > \frac{1}{3}$

$2x^3-3x+1 > 0 \rightarrow (x-1)(2x^2+2x-1) > 0 \rightarrow x=1, \frac{-1 \pm \sqrt{3}}{2}$

$\rightarrow \frac{-1-\sqrt{3}}{2} < x < \frac{-1+\sqrt{3}}{2}$ or $x > 1$

so The sol. $\left(\frac{1}{3}, \frac{(\sqrt{3}-1)}{2}\right) \cup (1, \infty)$.

2 casses

d. $\frac{x}{x-1} \cancel{<} \frac{1}{1}$

We cannot multiply by $x-1$ unless we consider its sign $\therefore$ 2 casses.

Case 1 : $x-1>0$ Then $x<x-1$ so $0<-1$ i.e this case is empty.

case 2 : $x-1<0$ Then $x>x-1$ so $0>-1$ i.e every number from this case works. Thus the solution is $(-\infty, 1)$.

e. $\frac{x^2}{4x^2-1} \cancel{<} \frac{1}{4}$ زي فوق لازم أشتغل حسب إشارة المقام

case 1 : $4x^2-1>0 \rightarrow 4x^2<4x^2-1 \rightarrow \underline{0<-1}$ i.e this case is emp[ty]

case 2 : $4x^2-1<0 \rightarrow 4x^2>4x^2-1 \rightarrow 0>-1$

Thus The solution $\left(-\frac{1}{2}, \frac{1}{2}\right)$

$\therefore 4x^2 \underset{+1}{-1} < 0+1$

$\sqrt{x^2} < \sqrt{\frac{1}{4}}$

$|x| < \frac{1}{2}$

Done

1.2.5: Let $a, b \in \mathbb{R}$.

a. prove that if $a > 2$ and $b = 1 + \sqrt{a-1}$, then $2 < b < a$.

suppose $a > 2$

Then $a - 1 > 1$

So $1 < \sqrt{a-1} < a-1$ By $0 < a < 1 \to 0 < a^2 < a$

and $a > 1$ implise $a^2 > a$.

There for $2 < \sqrt{a-1} + 1 < a - 1 + 1$

$2 < 1 + \sqrt{a-1} < a$

$2 < b < a$ ∎

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b. prove that if $2 < a < 3$ and $b = 2 + \sqrt{a-2}$, then $0 < a < b$.

suppose $2 < a < 3$

Then $0 < a - 2 < 1$

So $0 < a - 2 < \sqrt{a-2} < 1$ By $0 < a < 1 \to 0 < a^2 < a$

and $a > 1 \to a^2 > a$.

Therfor
add 2 to each side
$2 < a < 2 + \sqrt{a-2}$

$2 < a < b$ ∎

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C. prove that if $0 < a < 1$ and $b = 1 - \sqrt{1-a}$ then $0 < b < a$.

suppose $0 < a < 1$ Then $0 > -a > -1$ So $0 < 1 - a < 1$ ($1 > 1 - a > 0$)

Hence, $\sqrt{1-a}$ is real and By $0 < a < 1 \to 0 < a^2 < a$ and $a > 1 \to a^2 > a$

$\to 1 - a < \sqrt{1-a}$

Therefor: $b = 1 - \sqrt{1-a}$

$b < 1 - (1 - a)$

$b < a$

d. prove that if $3<a<5$ and $b=2+\sqrt{a-2}$, then $3<b<a$.

suppose $3<a<5$ Then $1<a-2<3$

So $1<\sqrt{a-2}<a-2$ By $0<a<1 \to 0<a^2<a$ and $a>1 \to a^2>a$.

Therefor $3<\sqrt{a-2}+2<a-2+2$

$3<b<a$

1.2.6 The arithmetic mean of $a,b \in \mathbb{R}$ is $\boxed{A(a,b)=\frac{(a+b)}{2}}$, and the geometric mean of $a,b \in [0,\infty)$ is $\boxed{G(a,b)=\sqrt{ab}}$. If $0 \le a \le b$, *prove that

$a \le G(a,b) \le A(a,b) \le b$.

*prove that $G(a,b)=A(a,b)$ iff $a=b$.

$a+b-2\sqrt{ab}=(\sqrt{a}-\sqrt{b})^2 \ge 0 \quad \forall\, a,b \in [0,\infty)$

$a+b-2\sqrt{ab} \ge 0$
$+2\sqrt{ab} \quad +2\sqrt{ab}$
$2\sqrt{ab} \le a+b$

Thus, $2\sqrt{ab} \le a+b$ and ① $G(a,b) \le A(a,b)$

on the other hand since $0 \le a \le b$ we have $A(a,b)=\frac{a+b}{2} \le \frac{2b}{2}=b$

$\Longrightarrow$ ② $A(a,b) \le b$

And $G(a,b)=\sqrt{ab} \ge \sqrt{a^2}=a$

$\Longrightarrow$ ③ $G(a,b) \ge a$

1 and 2 and 3 $\longrightarrow$ $a \le G(a,b) \le A(a,b) \le b$. ✓

* $G(a,b)=A(a,b)$

$\sqrt{ab}=\frac{(a+b)}{2}$ So $G(a,b)=A(a,b) \Leftrightarrow a=b$.

$2\sqrt{ab}=a+b$

Iff $(\sqrt{a}-\sqrt{b})^2=0$

Iff $\sqrt{a}=\sqrt{b}$

Iff $a=b$.

1.2.7: let $x \in \mathbb{R}$

a. prove that $|x| \leq 2$ implies $|x^2-4| \leq 4|x-2|$.

$|x^2-4| = |(x+2)(x-2)|$

$= |x+2||x-2|$ By multiplicative.

But $|x+2| \leq |x|+|2| \rightarrow |x+2| \leq |x|+2$ (By Triangle inquality).

$|x^2-4| \leq (|x|+2)|x-2|$

$\leq (2+2)|x-2|$ $|x| \leq 2$. By Assumption.

$\leq 4|x-2|$ ☒

b. prove that $|x| \leq 1$ implies $|x^2+2x-3| \leq 4|x-1|$.

$|x^2+2x-3| = |(x+3)(x-1)|$

$= |x+3||x-1|$ By multiplicative.

But $|x+3| \leq |x|+3$ By Triangle inquality.

Now $|x^2+2x-3| \leq (|x|+3)(|x-1|)$.

$\leq (1+3)|x-1|$ $|x| \leq 1$ By Assumption.

$\leq 4|x-1|$ ☒

c. prove that $-3 \leq x \leq 2$ implies $|x^2+x-6| \leq 6\,|x-2|$.

S: $|x^2+x-6| = |(x-2)(x+3)|$

$= |x+3||x-2|$. By multiplicative

But $|x+3| \leq |x| + 3$ By Triangle inequality.

$|x^2+x-6| \leq (|x|+3)\,|x-2|$

$\leq 6\,|x-2|$ $\qquad -3 \leq x \leq 2 \rightarrow 3 \leq |x| \leq 2$.

By Assumption.

d. prove that $-1 < x < 0$ implies $|x^3 - 2x + 1| < 1.26\,|x-1|$.

Natural

1.2.8 : For each of the following, find all values of $n \in \mathbb{N}$ satisfy the given inquality.

a. $\frac{1-n}{1-n^2} < 0.01$

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$\frac{\cancel{(1-n)}}{(1+n)\cancel{(1-n)}} < 0.01$

$\frac{1}{1+n} < 0.01 \rightarrow \frac{1}{1+n} < \frac{1}{100}$

since $n > 0 \rightarrow 1+n > 0$ for all $n \in \mathbb{N}$

So $n+1 > 100$
$\quad -1 \qquad -1$

$n > 99$

b. $\frac{n^2+2n+3}{2n^3+5n^2+8n+3} < 0.025$

$\frac{\cancel{(n^2+2n+3)}}{(2n+1)\cancel{(n^2+2n+3)}} < \frac{1}{40}$

$\frac{1}{2n+1} < \frac{1}{40}$ $\qquad 2n+1 > 0, \forall n \in \mathbb{N}$

$(2n+1) > 40$

$n > \frac{39}{2}$

$n > 19.5$ But $n \in \mathbb{N}$ so

$n > 20$

c. $\frac{(n-1)}{n^3-n^2+n-1} < \underset{\sim}{0.002} = \frac{1}{500}$

$\frac{\cancel{(n-1)}}{(n^2+1)(n-1)} < \frac{1}{500}$

$\frac{1}{n^2+1} \cancel{<} \frac{1}{500}$

$n^2+\cancel{1} > 500$
$\quad -1 \qquad -1$

$n^2 > 499$

$n > \sqrt{499} \approx 22.33$ But $n \in \mathbb{N}$

$\boxed{n \geq 23}$

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1.2.9 :

$n, q, L \neq 0$

a. Interpreting a rational $\frac{m}{n}$ as $m.n^{-1} \in \mathbb{R}$ use postulate 1 to prove that $m, n, p, q, L \in \mathbb{Z}$

$\frac{m}{n} + \frac{p}{q} = \frac{mq+np}{nq}$, $\frac{m}{n} \cdot \frac{p}{q} = \frac{mp}{nq}$, $\frac{-m}{n} = -\frac{m}{n}$ and $\left(\frac{L}{n}\right)^{-1} = \frac{n}{L}$.

① $mn^{-1} + pq^{-1} = mqn^{-1}q^{-1} + npn^{-1}q^{-1}$. Notes: $n^{-1}q^{-1}nq = 1$

$n^{-1}q^{-1}(m+p) = n^{-1}q^{-1}(mq+np)$ * → uniqueness of multiplicative inverse

$(nq)^{-1}(m+p) = (nq)^{-1}(mq+np)$ implies $(nq)^{-1} = n^{-1}q^{-1}$.

Therefore $mn^{-1} + pq^{-1} = (mq+np)(nq)^{-1}$.

$\boxed{1.9.2}$ a. ② simmilarly to ①

$$mn^{-1}(pq^{-1}) = mpn^{-1}q^{-1}.$$

$$mp(nq)^{-1} = mp(nq)^{-1}$$

$\blacksquare$

③ $\frac{m}{n} + \frac{-m}{n} = \frac{m-m}{n} = \frac{0}{n} = 0$

Therefore By uniqueness of additive inverse : $-\left(\frac{m}{n}\right) = \frac{-m}{n}$

④ simmilarly $\left(\frac{m}{n}\right)\left(\frac{n}{m}\right) = \frac{mn}{nm} = mn(mn)^{-1} = 1$ So $\left(\frac{m}{n}\right)^{-1} = \frac{n}{m}$

↳ By the uniqueness of multiplicative inverses.

b. using Remark 1.1, prove that postulate 1 holds with $\mathbb{Q}$ in place of $\mathbb{R}$.

Any subset of $\mathbb{R}$ which contains 0 and 1 will satisfy the [1] Associative and [2] commutative properties, the [3] distributed law and [4] have Identity 0 and a [5] multiplicative inverse.

By 1.2.9.a : $\mathbb{Q}$ satisfies the closuer property, has additive inverse and every nonzero $q \in \mathbb{Q}$ has multiplicative inverse.

There for $\mathbb{Q}$ satisfies Postulate 1 $\blacksquare$

1.2.9

c. prove that the sum of rational and an irrational is always irrational. What can you say about the product of a rational and an irrational?

If $r \in Q$ and $x \in Q^c$ By contradiction.

let $q = r + x$, $q \in Q$

Then $x = q - r$ $\qquad q - r = Q - Q = Q$

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$x = q - r \in Q$

But $x \in Q^c$ $\Rightarrow$ X.

Simmilarly:

If $r \in Q$ and $x \in Q^c$ and let $q = rx$, $q \in Q$, $r \neq 0$

Then $q = rx \rightarrow x = \frac{q}{r} \rightarrow x \in Q$ X with $x \in Q^c$.

However: the product of any irrational with 0 is rational.

d. let $\frac{m}{n}, \frac{p}{q} \in \mathbb{R}$ with $nq > 0$. prove that $\frac{m}{n} < \frac{p}{q}$ iff $mq < np$.

By the first multiplicative property

$nq * (mn^{-1} < pq^{-1})$ iff $mq = mn^{-1}qn < pq^{-1}nq = np$

$$mn^{-1} < pq^{-1} \iff mq < np$$

1.2.10 : prove that $(ab+cd)^2 \leq (a^2+c^2)(b^2+d^2)$ for all $a,b,c,d \in \mathbb{R}$

let $(cb-ad)^2 \geq 0$

$$c^2b^2 - 2cbad + a^2d^2 \geq 0$$

$$c^2b^2 + a^2d^2 \geq 2cbad$$ Adding $a^2b^2 + c^2d^2$ to both side

$$a^2b^2 + c^2d^2 + c^2b^2 + a^2d^2 \geq a^2b^2 + c^2d^2 + 2cbad$$

$$a^2(b^2+d^2) + c^2(b^2+d^2) \geq (ab+cd)^2$$

$$(a^2+c^2)(b^2+d^2) \geq (ab+cd)^2$$

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$P := \mathbb{R}^+$

1.2.11 (a) let $\mathbb{R}^+$ represent the collection of positive real numbers. prove that $\mathbb{R}^+$ satisfies the following two properties

(i) For each $x \in \mathbb{R}$, one and only one of the following holds : $x \in \mathbb{R}^+$, $-x \in \mathbb{R}^+$ or $x = 0$.

(ii) given $x, y \in \mathbb{R}^+$ both $x+y$ and $x.y$ belong to $\mathbb{R}^+$.

let $x \in \mathbb{R}$, By Trichotomy property either $x>0$ or $-x>0$ or $x=0$

Thus $\mathbb{R}^+$ satisfies i.

If $x>0$ and $y>0$ then by Additive property $x+y>0$

and by First multiplicative property : $xy>0$

Thus satisfies (ii)

b. suppose that $\mathbb{R}$ contains a subset $\mathbb{R}^+$ (not necessarily the set of positive numbers) which satisfies (i) and (ii). Define $x \lessdot y$ by $y-x \in \mathbb{R}^+$. prove that Postulate 2 holds with $\lessdot$, in place of $<$.

→ To prove the Trichotomy property suppose $a, b \in \mathbb{R}$.
By (i) either $a-b \in \mathbb{R}^+$, $b-a \in \mathbb{R}^+$ or $a-b=0$.
Thus either $a > b$, $b > a$ or $a=b$.

→ To prove Transitive property, suppose $a<b$ and $b<c$
Then $b-a, c-b \in \mathbb{R}^+$ and its follows from (ii) That
$c-a = b-a+c-b \in \mathbb{R}^+ \rightarrow$ i.e $c>a$.

→ since $b-a = (b+c)-(a+c)$ its clear that the Additive property holds.

→ suppose $a<b$ i.e $b-a \in \mathbb{R}^+$.
If $c>0$ then $c \in \mathbb{R}^+$ and its follows from (ii) that $bc-ac=(b-a)c \in \mathbb{R}^+$
i.e $bc > ac$.
If $c<0$ then $-c \in \mathbb{R}^+$ so $ac-bc = (b-a)(-c) \in \mathbb{R}^+$, i.e $ac > bc$.