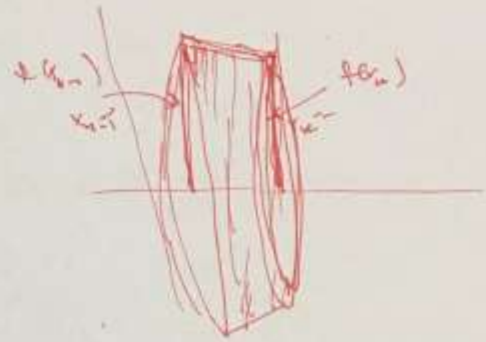
6.4 Area of surface of Revolution

Frustum surface area



$$A = 2\pi\left(\frac{r_1+r_2}{2}\right)L$$

$$A = 2\pi\,\frac{f(x_{k-1})+f(x_k)}{2}\,\Delta s$$

$$\sum 2\pi\, \frac{f(x_{k-1})+f(x_k)}{2}\sqrt{(\Delta x_k)^2+(\Delta y_k)^2}$$

$$\boxed{S=\int_a^b 2\pi\, f(x)\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx}$$

$$=\int_a^b 2\pi y\sqrt{1+(y')^2}\,dx$$

Ex Find the area of the surface generated by revolving the curve $y=2\sqrt{x}$, $1\le x\le 2$ about the x-axis

$y'=2\cdot\frac{1}{2}x^{-1/2}$

$(y')^2=\frac{1}{x}$

$(1+x)^{1/2}$

Area of the surface $=\int_1^2 2\pi\, 2\sqrt{x}\sqrt{1+\frac{1}{x}}\,dx$

$$=4\pi\int_1^2\sqrt{1+x}\,dx=4\pi\,\frac{(1+x)^{3/2}}{\frac{3}{2}}\Big|_1^2$$

Exz (# 18 from the exercises)

Find the area of the surface generated by revolving the curve $y = \frac{x^3}{9}$, $0 \le x \le 2$ about the x-axis.

$$S = 2\pi \int_a^b f(x)\sqrt{1+(f'(x))^2}\, dx$$

$$y = \frac{x^3}{9}$$

$$y' = \frac{3x^2}{9} = \frac{x^2}{3}$$

$$(y')^2 = \frac{x^4}{9}$$

$$1+(y')^2 = 1+\frac{x^4}{9} = \frac{x^4+9}{9}$$

$$S = 2\pi \int_0^2 \frac{x^3}{9}\sqrt{\frac{x^4+9}{9}}\, dx$$

$$= \frac{2\pi}{27}\int_0^2 x^3\sqrt{x^4+9}\, dx$$

$$= \frac{\pi}{54}\left.\frac{(x^4+9)^{3/2}}{\frac{3}{2}}\right|_0^2$$

$$= \frac{2}{3}\frac{\pi}{54}\left((25)^{3/2} - 9^{3/2}\right)$$

$$= \frac{\pi}{81}(125-27) = \frac{98}{81}\pi$$

We can get a similar formula by revolving about y-axis

$$\boxed{S = 2\pi \int_c^d x(y)\sqrt{1+(x'(y))^2}\, dy}.$$

Ex 3) Find the are of the surface generated by revolving the curve $x = 1-y$, $0 \le y \le 1$ about y-axis.

$x = 1-y$

$x(y) = 1-y$

$x' = -1$

$(x')^2 = 1$

$1+(x')^2 = 2$

$$S = 2\pi \int_0^1 (1-y)\sqrt{2}\, dy$$

$$= 2\sqrt{2}\pi \left(y - \frac{y^2}{2}\Big|_0^1\right)$$

$$= 2\sqrt{2}\pi \left(1 - \frac{1}{2}\right)$$

$$= \sqrt{2}\,\pi$$

Ex 4) #18 (From Exersices)

Find the surface area generated by rev

$x(y) = \frac{1}{3} y^{3/2} - y^{1/2}$ about y-axis

$1 \le y \le 3$

Sol'n $x' = \frac{1}{3} \cdot \frac{3}{2} y^{1/2} - \frac{1}{2} y^{-1/2} = \frac{1}{2} y^{1/2} - \frac{1}{2} y^{-1/2}$

$(x')^2 = \frac{1}{4} y - \frac{1}{2} + \frac{1}{4} y^{-1}$

$1 + (x')^2 = \frac{1}{4} y + \frac{1}{2} + \frac{1}{4} y^{-1}$

$= \left(\frac{1}{2} y^{1/2} + \frac{1}{2} y^{-1/2}\right)^2$

$\sqrt{1 + (x')^2} = \frac{1}{2} y^{1/2} + \frac{1}{2} y^{-1/2}$

$S = 2\pi \int_1^3 x(y) \sqrt{1 + (x'(y))^2}\, dy$

$= 2\pi \int_1^3 \left(\frac{1}{3} y^{3/2} - y^{1/2}\right)\left(\frac{1}{2} y^{1/2} + \frac{1}{2} y^{-1/2}\right) d$

Ex5 # 26 (From Exercises), Surface area

$x = \sqrt{2y-1}$, y-axis

$5/8 \le y \le 1$

Sol

$x = (2y-1)^{1/2}$

$x' = \frac{1}{2}(2y-1)^{-1/2} \cdot 2$

$= (2y-1)^{-1/2}$

$(x')^2 = (2y-1)^{-1}$

$1 + (x')^2 = 1 + (2y-1)^{-1}$

$= 1 + \frac{1}{(2y-1)} = \frac{2y-1+1}{2y-1} = \frac{2y}{2y-1}$

$$S = 2\pi \int_{5/8}^{1} x(y) \sqrt{1 + (x'(y))^2} \, dy$$

$$= 2\pi \int_{5/8}^{1} \cancel{\sqrt{2y-1}} \cdot \frac{\sqrt{2y}}{\cancel{\sqrt{2y-1}}} \, dy$$

$$= 2\sqrt{2}\pi \int_{5/8}^{1} y^{1/2} \, dy$$

$$= 2\sqrt{2}\pi \left(\frac{2}{3} y^{3/2} \Big|_{5/8}^{1} \right)$$

$$= 2\sqrt{2}\pi \left(\frac{2}{3} - \frac{2}{3}\left(\frac{5}{8}\right)^{3/2} \right)$$