Key

Birzeit University
Mathematics Department
Math331-Section (1)
Quiz#3

Instructor: Dr. Ala Talahmeh
Time: 50 minutes
Name:...

First Semester 2024/2025
Date: 21/12/2024
Number:..............

Question#1 [**2 marks**]. Find the **minimum radius of convergence** of the power series solutions for the DE: $(x^2+x+1)y''-3y=0$ about the ordinary point $x_0=1$.

0.5 $x^2+x+1=0 \Rightarrow x=-\frac{1}{2}\pm\frac{\sqrt{3}}{2}i$

0.5 $\rho_1 = \text{dist}\{-\frac{1}{2}+\frac{\sqrt{3}}{2}i,\ 1\} = \sqrt{(-\frac{1}{2}+1)^2+(\frac{\sqrt{3}}{2})^2}=\sqrt{3}$

0.5 $\rho_2 = \text{dist}\{-\frac{1}{2}-\frac{\sqrt{3}}{2}i,\ 1\} = \sqrt{3}$.

0.5 $\therefore \rho=\sqrt{3}$.

Question#2 [**4 marks**]. Determine the **singular points** of the differential equation

$$(x+2)^2(x-1)y''+2xy'+6y=0.$$

Determine whether they are **regular** or **irregular**.

1 $(x+2)^2(x-1)=0 \Rightarrow x=-2,\ x=1$ are singular points

$\boxed{x=-2}$ 1 , $\lim_{x\to-2}(x+2)\frac{2x}{(x+2)^2(x-1)}$ infinite

0.5 $\therefore x=-2$ is irregular singular point.

$\boxed{x=1}$ 1 , $\lim_{x\to1}(x-1)\frac{2x}{(x+2)^2(x-1)}=\frac{2}{9}$ finite

$\lim_{x\to1}(x-1)^2\frac{6}{(x+2)^2(x-1)}=0$ finite

$\therefore x=1$ 0.5 is regular singular point

Question#3 [4 marks]. Solve the initial-value problem:

$$x^2y'' + xy' + y = 0, \;\; y(e^{\pi/2}) = 1, \;\; y'(e^{\pi/2}) = \frac{2}{\pi}.$$

$y = x^m$, $x>0$. The aux. eq. is $m^2+1=0$ (1)
$m = \pm i$

(1) $y = c_1 \cos(\ln x) + c_2 \sin(\ln x)$.

(1)
$1 = y(e^{\pi/2}) = c_1 \cos\frac{\pi}{2} + c_2 \sin(\frac{\pi}{2}) \Rightarrow \boxed{c_2 = 1}$

$y' = -c_1 \frac{\sin(\ln x)}{x} + \frac{\cos(\ln x)}{x}$

$\frac{2}{\pi} = y'(e^{\pi/2}) = -e^{-\pi/2} c_1 \Rightarrow \boxed{c_1 = -\frac{2}{\pi} e^{\pi/2}}$ (1)

$\therefore y = -\frac{2}{\pi} e^{\pi/2} \cos(\ln x) + \sin(\ln x)$.

Question#4 [5 marks]. Suppose that $y = \sum_{n=0}^{\infty} a_n x^n$ is a solution of the initial-value problem: $y'' - e^x y' - y\cos x = 0$, $y(0) = 2$, $y'(0) = 1$. Find a_0, a_1, a_2, a_3, then write the solution.

(1) $\boxed{a_0 = y(0) = 2}$, (1) $\boxed{a_1 = y'(0) = 1}$

$a_2 = \frac{y''(0)}{2!}$, $y''(0) = e^0 y'(0) + y(0)\cos 0$
$= 2 + 1 = 3$.

(1) $\boxed{a_2 = \frac{3}{2}}$

$a_3 = \frac{y'''(0)}{3!}$, $y''' = e^x y' + e^x y'' + y'\cos x - y\sin x$

(1) $\boxed{a_3 = \frac{5}{6}}$ $\quad y'''(0) = 2y'(0) + y''(0)$
$= 2 + 3$
$= 5$

(1) $\therefore y = 2 + x + \frac{3}{2}x^2 + \frac{5}{6}x^3 + \cdots$

Question#5 [5 marks]. Consider the differential equation:

$$y'' - 2xy' + 2y = 0, \quad -\infty < x < \infty.$$

Find **two linearly independent** series solutions y_1 and y_2 near an ordinary point $x_0 = 0$.

0.5 $y = \sum_{n=0}^{\infty} a_n x^n$, $y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$, $y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$

$\Rightarrow \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - 2x \sum_{n=1}^{\infty} n a_n x^{n-1} + 2 \sum_{n=0}^{\infty} a_n x^n = 0.$

0.5 $\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=1}^{\infty} 2n a_n x^n + \sum_{n=0}^{\infty} 2 a_n x^n = 0$

0.5 $(2a_2 + 2a_0) + \sum_{n=1}^{\infty} \left[(n+2)(n+1) a_{n+2} - 2(n-1) a_n\right] x^n = 0$

$2a_2 + 2a_0 = 0$, $(n+2)(n+1) a_{n+2} = 2(n-1) a_n$, $n = 1, 2, \ldots$

0.5

0.5 $\boxed{a_2 = -a_0}$, $\boxed{a_{n+2} = \frac{2(n-1)}{(n+2)(n+1)} a_n, \ n = 1, 2, \ldots}$

0.5 $\boxed{n=1}$, $a_3 = 0$, $\boxed{n=2}$, 0.5 $a_4 = \frac{2a_2}{(4)(3)} = \frac{-a_2}{6}$

$\boxed{n=3}$, 0.5 $a_5 = \frac{4a_3}{20} = 0$

$\therefore y = a_0 + a_1 x + a_2 x^2 + \cdots$

0.5 $= a_0 + a_1 x - a_0 x^2 + 0x^3 + \frac{-a_2}{6} x^4 + \cdots$

Good Luck

$= a_0 \underbrace{\left(1 - x^2 - \frac{1}{6}x^4 + \cdots\right)}_{y_1} + a_1 \underbrace{(x)}_{y_2}$

0.5 $W(y_1, y_2)(0) = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \neq 0$