# 6.3 : Arc Length.

Length $$L = \int_{\square}^{\square} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$L = \int_{\square}^{\square} \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

$\boxed{2}$ Find the Length of the curve $y = x^{3/2}$ from $x=0$ to $x=4$

$$\frac{dy}{dx} = \frac{3}{2} x^{1/2}$$

$$\left(\frac{dy}{dx}\right)^2 = \frac{9}{4} x$$

$$\left(\frac{dy}{dx}\right)^2 + 1 = \frac{9x+4}{4}$$

By substitution:-

let $U = 9x+4$
$dU = 9\, dx$

$x=0 \longrightarrow U=4$
$x=4 \longrightarrow U=40$

$$L = \int_0^4 \sqrt{\frac{9x+4}{4}}\, dx$$

$$= \int_0^4 \frac{1}{2}\sqrt{9x+4}\, dx$$

$$= \int_4^{40} \frac{1}{2}\sqrt{U}\, \frac{dU}{9} = \frac{1}{18}\int_4^{40} \sqrt{U}\, dU$$

$$= \frac{1}{18}\cdot\frac{2}{3}\left[U^{3/2}\right]_4^{40}$$

$$= \frac{1}{27}\left[40^{3/2} - 4^{3/2}\right]$$

$$= \frac{1}{27}\left[8\sqrt{1000} - 8\right] = \frac{8}{27}\left[\sqrt{1000} - 1\right]$$

$\boxed{6}$ Find the length of the curve

$x = \frac{y^3}{6} + \frac{1}{2y}$ from $y=2$ to $y=3$

$$\frac{dx}{dy} = \frac{3y^2}{6} + \frac{-1}{2y^2} = \frac{y^2}{2} - \frac{1}{2y^2}$$

$$\left(\frac{dx}{dy}\right)^2 = \frac{y^4}{4} - \frac{1}{2} + \frac{1}{4y^4}$$

$$\left(\frac{dx}{dy}\right)^2 + 1 = \frac{y^4}{4} + \frac{1}{2} + \frac{1}{4y^4}$$

$$L = \int_2^3 \sqrt{\frac{y^4}{4} + \frac{1}{2} + \frac{1}{4y^4}}\, dy$$

$$\int_2^3 \sqrt{\left(\frac{y^2}{2} + \frac{1}{2y^2}\right)^2}\, dy = \int_2^3 \frac{y^2}{2} + \frac{1}{2y^2}\, dy$$

$$= \left[\frac{y^3}{6} - \frac{1}{2y}\right]_2^3$$

$$\text{Length} = \boxed{\frac{39}{12}}$$

$\boxed{7}$ $y = \frac{3}{4} x^{4/3} - \frac{3}{8} x^{2/3} + 5 \qquad 1 \le x \le 8$

$$L = \int_1^8 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$\frac{dy}{dx} = x^{1/3} - \frac{1}{4} x^{-1/3}$$

$$\left(\frac{dy}{dx}\right)^2 = x^{2/3} - \frac{2}{4} x^{1/3} x^{-1/3} + \frac{1}{16 x^{2/3}} = x^{2/3} - \frac{1}{2} + \frac{1}{16 x^{2/3}}$$

$$\left(\frac{dy}{dx}\right)^2 + 1 = x^{2/3} + \frac{1}{2} + \frac{1}{16 x^{2/3}}$$

$$L = \int_1^8 \sqrt{x^{2/3} + \frac{1}{2} + \frac{1}{16 x^{2/3}}}\, dx$$

$$L = \int_1^8 \sqrt{\left(x^{1/3} + \frac{1}{4x^{1/3}}\right)^2}\, dx = \int_1^8 x^{1/3} + \frac{1}{4x^{1/3}}\, dx$$

$$= \frac{3}{4} x^{4/3} + \frac{3}{8} x^{2/3} \Big]_1^8$$

$$= \frac{3}{4} \cdot 16 + \frac{3}{8}(4) - \frac{3}{4} - \frac{3}{8}$$

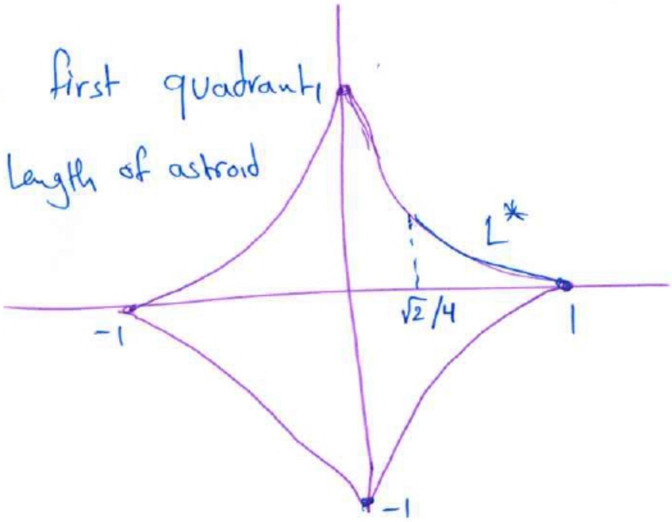
$$= 12 + \frac{3}{2} - \frac{3}{4} - \frac{3}{8}$$

$$= \frac{99}{8}$$

$\boxed{22}$ $x^{2/3} + y^{2/3} = 1$ is called astroid

• $L^*$: the Length of half the first quadrant portion $\longrightarrow$ then to find the Length of astroid

$$\boxed{L = 8L^*}$$




$$L^* = \int_{\frac{\sqrt{2}}{4}}^{1} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$y^{2/3} = 1 - x^{2/3}$

$y = \left(1 - x^{2/3}\right)^{3/2}$ first quadrant

$$\frac{dy}{dx} = \frac{3}{2}\left(1 - x^{2/3}\right)^{\frac{1}{2}} \cdot \frac{-2}{3} x^{-1/3}$$

$$= \frac{-\sqrt{1 - x^{2/3}}}{x^{1/3}}$$

solution

$$\boxed{L = 8L^* = 8 \cdot \frac{3}{4} = 6}$$

$$\left(\frac{dy}{dx}\right)^2 = \frac{1 - x^{2/3}}{x^{2/3}}$$

$$\left(\frac{dy}{dx}\right)^2 + 1 = \frac{1 - x^{2/3} + x^{2/3}}{x^{2/3}} = \frac{1}{x^{2/3}} = x^{-2/3}$$

$$L^* = \int_{\frac{\sqrt{2}}{4}}^{1} \sqrt{x^{-2/3}}\, dx = \frac{3}{2} \cdot x^{2/3}\Big]_{\frac{\sqrt{2}}{4}}^{1} = \frac{3}{2}\left[1 - \left(\frac{\sqrt{2}}{4}\right)^{2/3}\right] = \frac{3}{2}\left(1 - \frac{1}{2}\right)$$

$$= \frac{3}{4}$$

# 6.4 : Surface Area

$\boxed{15}$ find the surface area that generated by revolving the curve $y = \sqrt{2x - x^2}$ $\quad \frac{1}{2} \leq x \leq \frac{3}{2}$ about x-axis

$$A = 2\pi \int_{\square}^{\square} y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx$$

$$\frac{dy}{dx} = \frac{2 - 2x}{2\sqrt{2x - x^2}} = \frac{1 - x}{\sqrt{2x - x^2}}$$

$$\left(\frac{dy}{dx}\right)^2 = \frac{(1-x)^2}{2x - x^2} = \frac{1 - 2x + x^2}{2x - x^2}$$

$$\left(\frac{dy}{dx}\right)^2 + 1 = \frac{1 - 2x + x^2 + 2x - x^2}{2x - x^2} = \frac{1}{2x - x^2}$$

$$A = 2\pi \int_{1/2}^{3/2} \sqrt{2x - x^2} \sqrt{\frac{1}{2x - x^2}} \, dx$$

$$A = 2\pi \int_{1/2}^{3/2} 1 \, dx$$

$$= 2\pi \, x \Big]_{1/2}^{3/2} = \boxed{2\pi}$$

$\boxed{17}$ find the area of the surface generated by revolving the curve

$$x = \frac{y^3}{3} \qquad 0 \leq y \leq 1 \qquad \text{about } y\text{-axis}$$

$$A = 2\pi \int_{\square}^{\square} x \sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy$$

$$\frac{dx}{dy} = y^2$$

$$\left(\frac{dx}{dy}\right)^2 = y^4$$

$$\left(\frac{dx}{dy}\right)^2 + 1 = y^4 + 1$$

$$A = 2\pi \int_0^1 \frac{y^3}{3} \sqrt{y^4+1}\, dy$$

Let $u = y^4 + 1$

$du = 4y^3\, dy$

$y = 0 \longrightarrow u = 1$

$y = 1 \longrightarrow u = 2$

$$A = 2\pi \int \frac{\cancel{y^3}}{3} \sqrt{u} \frac{du}{4\cancel{y^3}}$$

$$= \frac{2\pi}{12} \int_1^2 \sqrt{u}\, du$$

$$\frac{\pi}{6} \frac{2}{3} u^{3/2} \Big]_1^2 = \boxed{\frac{\pi}{9} \left[\sqrt{8} - 1\right]}$$

$\boxed{22}$ Find the area of the surface generated by revolving the curve $y = \frac{1}{3}\left(x^2+2\right)^{3/2}$ $0 \le x \le \sqrt{2}$, y-axis

Hint Express $ds = \sqrt{dx^2 + dy^2}$ in terms of $dx$, & evaluate the integral $S = \int 2\pi x \, ds$

$$y = \frac{1}{3}\left(x^2+2\right)^{3/2}$$

$$\frac{dy}{dx} = \frac{1}{2}\left(x^2+2\right)^{1/2} \cdot 2x = x\left(x^2+2\right)^{1/2}$$

$$dy = x\left(x^2+2\right)^{1/2} dx$$

$$ds = \sqrt{dx^2 + dy^2} = \sqrt{dx^2 + x^2(x^2+2)dx^2} = \sqrt{1+x^4+2x^2}\, dx$$

$$ds = \sqrt{1+x^4+2x^2}\, dx$$

$$S = \int 2\pi x \, ds$$

$$= \int_0^{\sqrt{2}} 2\pi x \cdot \sqrt{x^4+2x^2+1}\, dx$$

$$\int_0^{\sqrt{2}} 2\pi x \sqrt{(x^2+1)^2}\, dx = 2\pi \int_0^{\sqrt{2}} x(x^2+1)\, dx$$

$$= 2\pi \left[\frac{x^4}{4} + \frac{x^2}{2}\right]_0^{\sqrt{2}}$$

$$= 2\pi \left[1 + 1 - 0\right] = \boxed{4\pi}$$