3.4 Basis and Dimension

Definition

The vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ form a **basis** for a vector space V if and only if

(i) $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are linearly independent.
(ii) $\mathbf{v}_1, \ldots, \mathbf{v}_n$ span V.

EXAMPLE 1 The *standard basis* for $\mathbb{R}^3$ is $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$; however, there are many bases that we could choose for $\mathbb{R}^3$. For example,

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \right\} \quad \text{and} \quad \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

are both bases for $\mathbb{R}^3$. We will see shortly that any basis for $\mathbb{R}^3$ must have exactly three elements. ■

EXAMPLE 2 In $\mathbb{R}^{2\times 2}$, consider the set $\{E_{11}, E_{12}, E_{21}, E_{22}\}$, where

$$E_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad E_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

$$E_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad E_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

If

$$c_1E_{11} + c_2E_{12} + c_3E_{21} + c_4E_{22} = O$$

then

$$\begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

so $c_1 = c_2 = c_3 = c_4 = 0$. Therefore, E_{11}, E_{12}, E_{21}, and E_{22} are linearly independent. If A is in $\mathbb{R}^{2\times 2}$, then

$$A = a_{11}E_{11} + a_{12}E_{12} + a_{21}E_{21} + a_{22}E_{22}$$

Thus, $E_{11}, E_{12}, E_{21}, E_{22}$ span $\mathbb{R}^{2\times 2}$ and hence form a basis for $\mathbb{R}^{2\times 2}$. ■

Standard Bases

In Example 1, we referred to the set $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ as the *standard basis* for $\mathbb{R}^3$. We refer to this basis as the standard basis because it is the most natural one to use for representing vectors in $\mathbb{R}^3$. More generally, the standard basis for $\mathbb{R}^n$ is the set $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$.

The most natural way to represent matrices in $\mathbb{R}^{2\times 2}$ is in terms of the basis $\{E_{11}, E_{12}, E_{21}, E_{22}\}$ given in Example 2. This, then, is the standard basis for $\mathbb{R}^{2\times 2}$.

The standard way to represent a polynomial in P_n is in terms of the functions $1, x, x^2, \ldots, x^{n-1}$, and consequently, the standard basis for P_n is $\{1, x, x^2, \ldots, x^{n-1}\}$.

EXAMPLE 9 Determine $N(A)$ if

$$A = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{bmatrix}$$

Solution

Using Gauss–Jordan reduction to solve $A\mathbf{x} = \mathbf{0}$, we obtain

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 0 \\ 2 & 1 & 0 & 1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & -2 & 1 & 0 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{cccc|c} 1 & 0 & -1 & 1 & 0 \\ 0 & -1 & -2 & 1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 2 & -1 & 0 \end{array}\right]$$

The reduced row echelon form involves two free variables, x_3 and x_4.

$$x_1 = x_3 - x_4$$
$$x_2 = -2x_3 + x_4$$

Thus, if we set $x_3 = \alpha$ and $x_4 = \beta$, then

$$\mathbf{x} = \begin{pmatrix} \alpha - \beta \\ -2\alpha + \beta \\ \alpha \\ \beta \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} -1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

Remark:

In Example 9 of Section 3.2, we saw that $N(A)$ is the subspace of $\mathbb{R}^4$ spanned by the vectors

$$\begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

Since these two vectors are linearly independent, they form a basis for $N(A)$.

Theorem 3.4.1 *If $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is a spanning set for a vector space V, then any collection of m vectors in V, where $m > n$, is linearly dependent.*

Corollary 3.4.2 *If both $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ and $\{\mathbf{u}_1, \dots, \mathbf{u}_m\}$ are bases for a vector space V, then $n = m$.*

Proof Let $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ and $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_m$ both be bases for V. Since $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ span V and $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_m$ are linearly independent, it follows from Theorem 3.4.1 that $m \leq n$. By the same reasoning $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_m$ span V, and $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ are linearly independent, so $n \leq m$. ■

In view of Corollary 3.4.2, we can now refer to the number of elements in any basis for a given vector space. This leads to the following definition.

Definition

Let V be a vector space. If V has a basis consisting of n vectors, we say that V has **dimension** n. The subspace $\{\mathbf{0}\}$ of V is said to have dimension 0. V is said to be **finite dimensional** if there is a finite set of vectors that spans V; otherwise, we say that V is **infinite dimensional**.

Example (Dimensions of Some Familiar Vector Spaces).

$$\dim(\mathbb{R}^n) = n$$
$$\dim(\mathbb{P}_n) = n$$
$$\dim(M_{mn}) = mn$$

Example (Dimension of span(S)).

If $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_r\}$ is linear independent and $W = \text{span}(S)$, then

$$\dim(W) = r$$

EXAMPLE 3 Let P be the vector space of all polynomials. We claim that P is infinite dimensional. If P were finite dimensional, say of dimension n, any set of $n + 1$ vectors would be linearly dependent. However, $1, x, x^2, \dots, x^n$ are linearly independent, since $W[1, x, x^2, \dots, x^n] > 0$. Therefore, P cannot be of dimension n. Since n was arbitrary, P must be infinite dimensional. The same argument shows that $C[a, b]$ is infinite dimensional. ■

Theorem 3.4.3 *If V is a vector space of dimension $n > 0$, then*

(I) *any set of n linearly independent vectors spans V.*

(II) *any n vectors that span V are linearly independent.*

EXAMPLE 4 Show that $\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}$ is a basis for $\mathbb{R}^3$.

Solution

Since $\dim \mathbb{R}^3 = 3$, we need only show that these three vectors are linearly independent. This follows, since

$$\begin{vmatrix} 1 & -2 & 1 \\ 2 & 1 & 0 \\ 3 & 0 & 1 \end{vmatrix} = 2$$

■

Theorem 3.4.4 *If V is a vector space of dimension $n > 0$, then*

(*) no set of larger than n vectors can be linearly independent.

(i) *no set of fewer than n vectors can span V.*

(ii) *any subset of fewer than n linearly independent vectors can be extended to form a basis for V.*

(iii) *any spanning set containing more than n vectors can be pared down to form a basis for V.*

EXAMPLE The following three sets in $\mathbb{R}^3$ show how a linearly independent set can be enlarged to a basis and how further enlargement destroys the linear independence of the set. Also, a spanning set can be shrunk to a basis, but further shrinking destroys the spanning property.

$$\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}2\\3\\0\end{bmatrix}\right\} \qquad \left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}2\\3\\0\end{bmatrix}, \begin{bmatrix}4\\5\\6\end{bmatrix}\right\} \qquad \left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}2\\3\\0\end{bmatrix}, \begin{bmatrix}4\\5\\6\end{bmatrix}, \begin{bmatrix}7\\8\\9\end{bmatrix}\right\}$$

Linearly independent but does not span $\mathbb{R}^3$ — A basis for $\mathbb{R}^3$ — Spans $\mathbb{R}^3$ but is linearly dependent

SECTION 3.4 EXERCISES

3. Consider the vectors

$$\mathbf{x}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} 4 \\ 3 \end{bmatrix}, \quad \mathbf{x}_3 = \begin{bmatrix} 7 \\ -3 \end{bmatrix}$$

(a) Show that $\mathbf{x}_1$ and $\mathbf{x}_2$ form a basis for $\mathbb{R}^2$.

(b) Why must $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ be linearly dependent?

(c) What is the dimension of Span$(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$?

5. Let

$$\mathbf{x}_1 = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix}, \quad \mathbf{x}_3 = \begin{bmatrix} 2 \\ 6 \\ 4 \end{bmatrix}$$

(a) Show that $\mathbf{x}_1, \mathbf{x}_2$, and $\mathbf{x}_3$ are linearly dependent.

(b) Show that $\mathbf{x}_1$ and $\mathbf{x}_2$ are linearly independent. $x_1 \neq \lambda x_2 \quad \forall \lambda \in \mathbb{R}$

(c) What is the dimension of Span$(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$?

$S' = \text{span}(x_1, x_2, x_3) = \text{span}(x_1, x_2)$

Basis of S' is $\{x_1, x_2\}$ $\therefore \dim S' = 2$

3. (a) Since

$$\begin{vmatrix} 2 & 4 \\ 1 & 3 \end{vmatrix} = 2 \neq 0$$

it follows that $\mathbf{x}_1$ and $\mathbf{x}_2$ are linearly independent and hence form a basis for R^2.

(b) It follows from Theorem 3.4.1 that any set of more than two vectors in R^2 must be linearly dependent.

5. (a) Since

$$\begin{vmatrix} 2 & 3 & 2 \\ 1 & -1 & 6 \\ 3 & 4 & 4 \end{vmatrix} = 0$$

it follows that $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_3$ are linearly dependent.

(b) If $c_1\mathbf{x}_1 + c_2\mathbf{x}_2 = \mathbf{0}$, then

$$\begin{aligned} 2c_1 + 3c_2 &= 0 \\ c_1 - c_2 &= 0 \\ 3c_1 + 4c_2 &= 0 \end{aligned}$$

and the only solution to this system is $c_1 = c_2 = 0$. Therefore $\mathbf{x}_1$ and $\mathbf{x}_2$ are linearly independent.

7. Find a basis for the subspace S of $\mathbb{R}^4$ consisting of all vectors of the form $(a+b, a-b+2c, b, c)^T$, where a, b, and c are all real numbers. What is the dimension of S?

8. Given $\mathbf{x}_1 = (1,\ 1,\ 1)^T$ and $\mathbf{x}_2 = (3,\ -1,\ 4)^T$:

(a) Do $\mathbf{x}_1$ and $\mathbf{x}_2$ span $\mathbb{R}^3$? Explain.

dim $\mathbb{R}^3 = 3$. $\{x_1, x_2\}$ can't span $\mathbb{R}^3$ since their number is less than 3.

(b) Let $\mathbf{x}_3$ be a third vector in $\mathbb{R}^3$ and set $X = (\mathbf{x}_1\ \ \mathbf{x}_2\ \ \mathbf{x}_3)$. What condition(s) would X have to satisfy in order for $\mathbf{x}_1$, $\mathbf{x}_2$, and $\mathbf{x}_3$ to form a basis for $\mathbb{R}^3$?

(c) Find a third vector $\mathbf{x}_3$ that will extend the set $\{\mathbf{x}_1, \mathbf{x}_2\}$ to a basis for $\mathbb{R}^3$.

$$x_3 = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$|X| = \begin{vmatrix} 1 & 3 & a \\ 1 & -1 & b \\ 1 & 4 & c \end{vmatrix} =$$

⑦ $\mathbb{R}^4 = \{ (a, b, c, d)^T : a, b, c, d \in \mathbb{R} \}$

$S' = \{ (a+b, a-b+2c, b, c)^T : a, b, c \in \mathbb{R} \} \subset \mathbb{R}^4$

$$\begin{pmatrix} a+b \\ a-b+2c \\ b \\ c \end{pmatrix} = a \underbrace{\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}}_{V_1} + b \underbrace{\begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \end{pmatrix}}_{V_2} + c \underbrace{\begin{pmatrix} 0 \\ 2 \\ 0 \\ 1 \end{pmatrix}}_{V_3}$$

$S' = \text{span}(\underline{V_1, V_2, V_3})$

$\underline{AX = 0}$

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \xrightarrow{R_2 - R_1} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & -2 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \xrightarrow{-\frac{1}{2}R_2} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \xrightarrow{R_3 - R_2}$$

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \xrightarrow{R_4 - R_3} \begin{array}{cccc} c_1 & c_2 & c_3 & \\ \end{array}$$
$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$\therefore \quad c_1 = c_2 = c_3 = 0$

$\therefore \{ v_1, v_2, v_3 \}$ is L.I. set

$B = \{ v_1, v_2, v_3 \}$ is a basis for S'.

$\dim S' = 3$.

8 (a) Since the dimension of R^3 is 3, it takes at least three vectors to span R^3. Therefore $\mathbf{x}_1$ and $\mathbf{x}_2$ cannot possibly span R^3.

(b) The matrix X must be nonsingular or satisfy an equivalent condition such as $\det(X) \neq 0$.

(c) If $\mathbf{x}_3 = (a, b, c)^T$ and $X = (\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$ then

$$\det(X) = \begin{vmatrix} 1 & 3 & a \\ 1 & -1 & b \\ 1 & 4 & c \end{vmatrix} = \underline{5a - b - 4c}$$

$$x_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \checkmark$$

If one chooses a, b, and c so that

$$5a - b - 4c \neq 0$$

$$x_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

then $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ will be a basis for R^3.

$$x_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

10. The vectors

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} 2 \\ 5 \\ 4 \end{bmatrix},$$

$$\mathbf{x}_3 = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}, \quad \mathbf{x}_4 = \begin{bmatrix} 2 \\ 7 \\ 4 \end{bmatrix}, \quad \mathbf{x}_5 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

span $\mathbb{R}^3$. Pare down the set $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4, \mathbf{x}_5\}$ to form a basis for $\mathbb{R}^3$.

$X = (x_1 \quad x_2 \quad x_3)$

$|X| \neq 0$

$|X| = 0$

11. Let S be the subspace of P_3 consisting of all polynomials of the form $ax^2 + bx + 2a + 3b$. Find a basis for S.

$P_3 = \{ax^2 + bx + c : a, b, c \in \mathbb{R}\}$

$S = \{ax^2 + bx + 2a + 3b, \quad a, b \in \mathbb{R}\} \subset P_3$

$ax^2 + bx + 2a + 3b = a\underbrace{(x^2 + 2)}_{v_1} + b\underbrace{(x + 3)}_{v_2}$

① $S = \text{span}(v_1, v_2)$ ② $\{v_1, v_2\}$ is L.I. set since $v_1 \neq \lambda v_2 \; \forall \lambda \in \mathbb{R}$

$B = \{v_1, v_2\}$. dim $S = 2$

10. We must find a subset of three vectors that are linearly independent. Clearly $\mathbf{x}_1$ and $\mathbf{x}_2$ are linearly independent, but

$$\mathbf{x}_3 = \mathbf{x}_2 - \mathbf{x}_1$$

so $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_3$ are linearly dependent. Consider next $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_4$. If $X = (\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_4)$ then

$$\det(X) = \begin{vmatrix} 1 & 2 & 2 \\ 2 & 5 & 7 \\ 2 & 4 & 4 \end{vmatrix} = 0$$

so these three vectors are also linearly dependent. Finally if we set $X = (\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_5)$ then

$$\det(X) = \begin{vmatrix} 1 & 2 & 1 \\ 2 & 5 & 1 \\ 2 & 4 & 0 \end{vmatrix} = -2$$

so the vectors $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_5$ are linearly independent and hence form a basis for R^3.

14. In each of the following, find the dimension of the subspace of P_3 spanned by the given vectors:

(a) $x, x-1, x^2+1$ If $S' = \text{span}(x, x-1, x^2+1) \Rightarrow \dim S' = 3$.

(b) $x, x-1, x^2+1, x^2-1$ $S' = \text{span}(x, x-1, x^2+1, x^2-1)$
$= \text{span}(x, x-1, x^2+1) \Rightarrow \dim S' = 3$

(c) $x^2, x^2-x-1, x+1$ (d) $2x, x-2$

15. Let S be the subspace of P_3 consisting of all polynomials $p(x)$ such that $p(0) = 0$, and let T be the subspace of all polynomials $q(x)$ such that $q(1) = 0$. Find bases for

(a) S (b) T (c) $S \cap T$

$$W = \begin{vmatrix} x & x-1 & x^2+1 \\ 1 & 1 & 2x \\ 0 & 0 & 2 \end{vmatrix}$$

$$= 2 \begin{vmatrix} x & x-1 \\ 1 & 1 \end{vmatrix}$$

$$= 2[x-(x-1)]$$

$$= 2 \neq 0$$

$\{x, x-1, x^2+1\}$ is L.I. set.

(S) $p(0) = 0 \Rightarrow a_0 = 0$

(T) $q(1) = 0 \Rightarrow a_2 + a_1 + a_0 = 0 \Rightarrow a_0 = -a_2 - a_1$

$(S \cap T)$ $\left.\begin{array}{l} p(0) = 0 \\ p(1) = 0 \end{array}\right\} \Rightarrow \left.\begin{array}{l} a_0 = 0 \\ a_2 + a_1 + a_0 = 0 \end{array}\right\} \Rightarrow \left.\begin{array}{l} a_0 = 0 \\ a_2 + a_1 = 0 \end{array}\right\} \Rightarrow \begin{array}{l} a_0 = 0 \\ a_2 = -a_1 \end{array}$

$$P_3 = \{ a_2 x^2 + a_1 x + a_0 \ : \ a_0, a_1, a_2 \in \mathbb{R} \}$$

$$S = \{ a_2 x^2 + a_1 x \ : \ a_2, a_1 \in \mathbb{R} \}$$

$$T = \{ a_2 x^2 + a_1 x - (a_1 + a_2) : a_2, a_1 \in \mathbb{R} \}$$

$$S \cap T = \{ a_2 x^2 - a_2 x \ : a_2 \in \mathbb{R} \}$$

ⓐ $$a_2 x^2 + a_1 x = a_2 (\underbrace{x^2}_{v_1}) + a_1 (\underbrace{x}_{v_2})$$

$\therefore$ span $(v_1, v_2) = S'$

$\because$ $\{x, x^2\}$ is L.I set.

$\therefore$ $\{x, x^2\}$ is a basis for S'.

(b) $a_2 x^2 + a_1 x - (a_1 + a_2) = a_1 (\underbrace{x - 1}_{V_1}) + a_2 (\underbrace{x^2 - 1}_{V_2})$

$\therefore$ $\text{span}(V_1, V_2) = T$

$\because$ $\{x-1, x^2-1\}$ is L.I. set

$\therefore$ $\{x-1, x^2-1\}$ is a basis for T.

(c) $a_2 x^2 - a_2 x = a_2 (\underbrace{x^2 - x}_{V_1})$

$\therefore$ $\text{span}(V_1) = S' \cap T$

$\because$ $\{x^2 - x\}$ is L.I.

$\therefore$ $\{x^2 - x\}$ is a basis for $S' \cap T$.