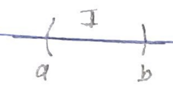
3.2 : one sided limits and limits at infinity :

Df 1: let $a \in \mathbb{R}$ and f be a real function

i. $f(x)$ is said to be converge to L as x approaches a from the right iff f is definded on some open interval I with left endpoint a and for every ($\varepsilon > 0$ $\exists$ a $\delta > 0$ (which ingeneral depends on ε, I, f and a) such that $a + \delta \in I$ and $a < x < a + \delta \Rightarrow |f(x) - L| < \varepsilon$.)



$\lim\limits_{x \to a^+} f(x)$

لازم a تكون طرف يسار

Henc, L is called the right hand limit of f at a and denote it by

$$f(a^+) := L := \lim_{x \to a^+} f(x)$$

$|x-a| < \delta \rightarrow |f(x) - L| < \varepsilon$

$-\delta < x - a < \delta$

$a - \delta < x < \delta + a$

لا بدي أحكي عن $\lim\limits_{x \to a^+} f(x)$ فالجزء الثاني ما بيلزم

$\Longrightarrow a < x < a + \delta$

ii. $f(x)$ is said to be converge to L as x approachs a from the left iff f is defined on some open interval I with right endpoint a and for every ($\varepsilon > 0$, $\exists$ a $\delta > 0$ s.t $a - \delta \in I$ and $a - \delta < x < a \Rightarrow |f(x) - L| < \varepsilon$).

Here, L is called the left hand limit of f at a and denote by

$$f(a^-) := L := \lim_{x \to a^-} f(x)$$

Expl:

1. Prove that $f(x) = \begin{cases} x+1 & , x \geq 0 \\ x-1 & , x < 0 \end{cases}$ has one sided limits at $x=0$ but $\lim_{x\to 0} f(x)$ DNE.

$x-1 \quad 0 \quad x+1$

$\lim_{x\to 0^+} f(x) = 1$, $\lim_{x\to 0^-} f(x) = -1$, $\lim_{x\to 0} f(x)$ DNE. $\rightarrow$ دا بحكي انه عند اليمين ≠ عند اليسار فهي مش موجودة عموماً !!

$\rightarrow \lim_{x\to 0^+} f(x) = 1$:

let $\varepsilon > 0$ and set $\delta = \varepsilon$

لأنه x محصورة بين 0 و δ و موجبة

If $0 < x < \delta$ then $|f(x) - L| = |(x+1) - 1| = |x| = x < \varepsilon$

$< \delta$

$< \varepsilon$

$\therefore \lim_{x\to 0^+} f(x) = 1$ exist

$\rightarrow \lim_{x\to 0^-} f(x) = -1$

let $\varepsilon > 0$ and set $\delta = \varepsilon$

محصورة بين δ- وصفر سالبة

If $-\delta < x < 0$ then $|f(x) - L| = |(x-1) - (-1)| = |x| = -x < \delta = \varepsilon$

$\delta > -x$

$-x < \delta$

By Thm 2 $\rightarrow \lim_{x\to 0} f(x)$ DNE

By sandwich theorem

let $x_n = \frac{(-1)^n}{n} \to 0$ as $n \to \infty$, $f(x_n) = \begin{cases} \frac{1}{n} + 1 & , n \text{ even} \\ \frac{-1}{n} - 1 & , n \text{ odd} \end{cases}$

$f(x_n) = (-1)^n \left(1 + \frac{1}{n}\right)$ does not conv. as $n \to \infty$

Hence, By the sequential charactrization of limits:

$\lim_{x\to 0} f(x)$ DNE

exp 1:

ii. $\lim_{x \to 0^+} \sqrt{x} = 0$ $\qquad a < x < \delta + a \quad , a = 0$

let $\varepsilon > 0$ set $\delta = \varepsilon^2$. الجذر موجب فبروح القيمة المطلقة

If $0 < x < \delta$ then $|f(x) - L| = |\sqrt{x} - 0| = \sqrt{x}$ بدي اياها اقل من ε

$< \sqrt{\delta}$ فلازم اختار δ

$< \sqrt{\varepsilon^2}$

$= \varepsilon$ بدون || bc ε موجبة

Thm 1: let f be a real function Then the limit $\lim_{x \to a} f(x)$ exists and equals L iff $L = \lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x)$.

proof:

$\Rightarrow$ suppose that $\lim_{x \to a} f(x) = L$ exists. Then given $\varepsilon > 0$, $\exists\, \delta > 0$ s.t.

$0 < |x - a| < \delta \longrightarrow |f(x) - L| < \varepsilon \quad *$

If $a < x < a + \delta$, then $a - \delta < x < a + \delta$ (i.e. $|x - a| < \delta$)

and $*$ implies $|f(x) - L| < \varepsilon$. Hence, $\lim_{x \to a^+} f(x) = L$ exists.

similarly, if $a - \delta < x < a$, then $|x - a| < \delta$ and $*$ implies $|f(x) - L| < \varepsilon$

which means $\lim_{x \to a^-} f(x) = L$ exists cont.

Conversly, supose $\lim_{x\to a^+} f(x) = \lim_{a\to -} f(x) = L$, Then given

$\varepsilon > 0$, $\exists\ \delta_1, \delta_2 > 0$ s.t

$a < x < a + \delta_1 \Rightarrow |f(x) - L| < \varepsilon$ ... 1

$a - \delta_2 < x < a \Rightarrow |f(x) - L| < \varepsilon$ ... 2

$a - \delta_2 < x < a + \delta_1$

set $\delta = \min\{\delta_1, \delta_2\}$, Then

$|x - a| < \delta \Rightarrow a - \delta < x < a + \delta$ whic implies

$a < x < a + \delta_1$ or $a - \delta_2 < x < a$

Hence 1 and 2 give $|f(x) - L| < \varepsilon$

That is $\lim_{x\to a} f(x) = L$

$\boxed{}$

Df 2: limits at infinity $x \to \pm\infty$

let $a, L \in \mathbb{R}$ and let f be a real function

i. $f(x)$ is said to be converge to L as $x \to \infty$ iff $\exists\ c > 0$ s.t $(c, \infty) \subset Dom(f)$ and given ($\varepsilon > 0$ there is an $M \in \mathbb{R}$ s.t $x > M \Rightarrow |f(x) - L| < \varepsilon$) in which case we shall write $\lim_{x\to\infty} f(x) = L$ or $f(x) \to L$ as $x \to \infty$

similarly, $f(x) \to L$ as $x \to -\infty$ iff $\exists$ a $c > 0$ s.t $(-\infty, -c) \subset Dom(f)$ and given $\varepsilon > 0$ there is an $M \in \mathbb{R}$ s.t $x < M \Rightarrow |f(x) - L| < \varepsilon$ in which case we shall write $\lim_{x\to -\infty} f(x) = L$ or $f(x) \to L$ as $x \to -\infty$

ii. $f(x)$ is said to be converge to ∞ as $x \to a$ (i.e $\lim_{x \to a} f(x) = \infty$) iff there is an open interval I containing a s.t $I \setminus \{a\} \subset Dom(f)$ and given $M \in \mathbb{R}$ there is a $\delta > 0$ s.t $\boxed{0 < |x-a| < \delta \Rightarrow f(x) > M}$ in which case we write $\lim_{x \to a} f(x) = \infty$ or $f(x) \to \infty$ as $x \to a$.

similarly $f(x)$ is said to converge to $-\infty$ as $x \to a$ (i.e $\lim_{x \to a} f(x) = -\infty$) iff $\exists$ an open interval I containing a s.t $I \setminus \{a\} \subset Dom(f)$ and given $M \in \mathbb{R}$ there is a $\delta > 0$ s.t $0 < |x-a| < \delta \Rightarrow f(x) < M$.

note

RMK: obvious modifications of this Df, we define $f(x) \to \pm\infty$ as $x \to a^+$ and $x \to a^-$, and $f(x) \to \pm\infty$ as $x \to \pm\infty$.

exp:

i. prove that $\lim_{x \to \infty} \frac{1}{x} = 0$.

let $\varepsilon > 0$ be given, set $M = \frac{1}{\varepsilon}$

If $\underset{\frac{1}{x} < \frac{1}{M}}{x > M}$ then $\underbrace{|f(x) - L| = \left|\frac{1}{x} - 0\right| = \frac{1}{x}}$

$C > 0$ يعني Dom في المنطقة الموجبة حسب التعريف فبشيل القيمة المطلقة

$< \frac{1}{M}$ we want $< \varepsilon$ so choose M

$< \frac{1}{(1 \setminus \varepsilon)} = \varepsilon$ ☺

RMK:

$\lim_{x \to a^+} f(x) = \infty$; given $M \in \mathbb{R}$ $\exists\, \delta > 0$ s.t

$a < x < a + \delta \Rightarrow f(x) > M$

$\lim_{x \to a^-} f(x) = \infty$: given $M \in \mathbb{R}$, $\exists\, \delta > 0$ s.t.

$a - \delta < x < a \Rightarrow f(x) > M$

3.2.1 (c)

exp: prove that $\lim_{x\to -1^+} \frac{1}{x^2-1} = -\infty$ $\frac{1}{(x-1)(x+1)}$ $-1<x<-1+\delta$

let $M\in\mathbb{R}$ we need to find a $\delta>0$ s.t $-1<x<-1+\delta$

$\Rightarrow f(x) = \frac{1}{x^2-1} < M$

suppose without lose of generality (WLOG) that $M<0$.

set $\delta = \min\{1, \frac{-1}{2M}\}$ → M negative

$x+1<\delta$

If $-1<x<-1+\delta$ then $-1<x<0$ since $\delta\le 1$

Hence, $1<1-x<2$ and $x^2-1<0$

$(x+1)(x-1)<0$

Thus, $0<1-x^2=(1-x)(1+x)$

$<(1-x)\delta$

$<2\delta$ Now choose δ s.t $2\delta < -\frac{1}{M}$ → $1-x^2 < -\frac{1}{M}$

$<2\left(-\frac{1}{2M}\right)$ $x^2-1 > \frac{1}{M}$

$= -\frac{1}{M}$ $\frac{1}{x^2-1} < M$

i.e, $1-x^2 < -\frac{1}{M}$ we conclude that $f(x)=\frac{1}{x^2-1}<M$

Notation : $\lim\limits_{\substack{x \to a \\ x \in I}} f(x)$ $(*)$ [a is an extended real number] .

- $*$ will denote $\lim\limits_{x \to a} f(x)$ (when it exists) .

- If a is a finite left endpoint of I , then $*$ will denote $\lim\limits_{x \to a^+} f(x)$ (when it exists).

- If a is a finite right endpoint of I, then $*$ will denote $\lim\limits_{x \to a^-} f(x)$ (when it exists).

- If $a = \pm\infty$ is an endpoint of I , then $*$ will denote $\lim\limits_{x \to \pm\infty} f(x)$ (when each exists).

كل الحالات

$L = \pm\infty$

$a \to a^+$, a^-

$x \to \pm\infty$

$\rightarrow$ using this notation , we can state a sequantial characterization of limits valid for two-sided , one-sided and infinite limits .

Thm 2 : let a be an extended real number , and let I be a nondegenerate open interval which either contains a or has a as one of its endpoints . suppose farther that f is a real function defined on I except possibly at a . Then $\lim\limits_{\substack{x \to a \\ x \in I}} f(x)$ exists and equals L iff $f(x_n) \to L$ for all seq. $x_n \in I$ which satisfy $x_n \neq a$ and $x_n \to a$ as $n \to \infty$

proof $\rightarrow$

proof :

case, $\lim_{x \to a} f(x) = \infty$ iff $f(x_n) \to \infty$ as $n \to \infty$ for any seq. $x_n \in I$ which converges to a and $x_n \neq a$ $\forall n \in \mathbb{N}$.

$\Rightarrow$ spse $\lim_{x \to a} f(x) = \infty$ if $x_n \in I$, $x_n \to a$ as $n \to \infty$ and $x_n \neq a$

Then given $M \in \mathbb{R}$, $\exists$ a $\delta > 0$ s.t

$0 < |x - a| < \delta \Rightarrow f(x) > M$ i

and $\exists$ an $N \in \mathbb{N}$ s.t $n \geq N \Rightarrow |x_n - a| < \delta$ ii

Consequatly (i and ii), $n \geq N \Rightarrow f(x_n) > M$

(i.e) $\lim_{n \to \infty} f(x_n) = \infty$ ☒

$\Leftarrow$ conversly, spse that $f(x_n) \to \infty$ as $n \to \infty$ for any seq. $x_n \in I$ which $x_n \to a$ as $n \to \infty$ and $x_n \neq a$ but $\lim_{x \to a} f(x) \neq \infty$

By the def'n of "convergence" to ∞, there are numbers $M_0 \in \mathbb{R}$ and $x_n \in I$ s.t $|x_n - a| < \frac{1}{n}$ and $f(x_n) \leq M_0$. $\forall n > N$

We have a seq. $x_n \to a$ but $f(x_n) \not\to \infty$ as $n \to \infty$

which is a contradiction. Thus, $\lim_{x \to a} f(x) = \infty$

☒

RMK: using Thm 2, we can prove limit Theorems represented in sec 3.1. These limits thms can be used to evelvate infinite limits and limits at $\pm\infty$.

درجة البسط = درجة المقام بين دول معاملين

exp 3: prove that $\lim\limits_{x\to\infty} \dfrac{2x^2-1}{1-x^2} = -2$ بدون def

$$\lim_{x\to\infty} \frac{2x^2-1}{1-x^2} = \lim_{x\to\infty} \frac{2-\frac{1}{x^2}}{\frac{1}{x^2}-1} = \frac{2-\lim\limits_{x\to\infty}\frac{1}{x^2}}{\lim\limits_{x\to\infty}\frac{1}{x^2}-1} = \frac{2-0}{0-1} = -2$$

Try By def'n.

let $\varepsilon > 0$ and set M

If $x > M$ then $\left|\dfrac{2x^2-1}{1-x^2} + 2\right|$

$$= \left|\frac{\cancel{2x^2}-1+2-\cancel{2x^2}}{1-x^2}\right|$$

$$= \left|\frac{1}{(1-x)(1+x)}\right|$$