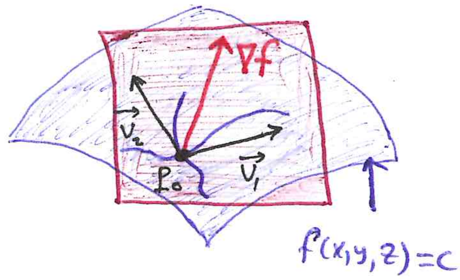
14.6 Tangent Planes and Differentials

* If $\vec{r} = g(t)\,\vec{i} + h(t)\,\vec{j} + k(t)\,\vec{k}$ is a smooth curve on the level surface $f(x,y,z)=c$ of a diff. function f, then

$$f\big(g(t), h(t), k(t)\big) = c$$



$$\frac{d}{dt} f\big(g(t), h(t), k(t)\big) = \frac{d}{dt}\, c$$

$$\frac{\partial f}{\partial x}\frac{dg}{dt} + \frac{\partial f}{\partial y}\frac{dh}{dt} + \frac{\partial f}{\partial z}\frac{dk}{dt} = 0$$

$$\left(f_x\,\vec{i} + f_y\,\vec{j} + f_z\,\vec{k}\right)\cdot\left(\frac{dg}{dt}\,\vec{i} + \frac{dh}{dt}\,\vec{j} + \frac{dk}{dt}\,\vec{k}\right) = 0$$

$$\nabla f \cdot \frac{dr}{dt} = 0$$

" ∇f is $\perp$ to the curve's velocity vector"

Def • The tangent plane at the point $P_0(x_0, y_0, z_0)$ on the level surface $f(x,y,z)=c$ of a diff function f is the plane through P_0 and $\perp$ to $\nabla f(P_0)$. That is

$$f_x(P_0)\,x + f_y(P_0)\,y + f_z(P_0)\,z = f_x(P_0)\,x_0 + f_y(P_0)\,y_0 + f_z(P_0)\,z_0$$

• The normal line of the surface at P_0 is the line through P_0 and $\parallel$ to $\nabla f(P_0)$. That is,

$$x = x_0 + f_x(P_0)\,t, \quad y = y_0 + f_y(P_0)\,t, \quad z = z_0 + f_z(P_0)\,t$$

Exp Find the tangent plane and normal line of the surface $f(x,y,z) = x^2 + y^2 + z - 9 = 0$ at the point $P_0(1,2,4)$

• $\nabla f = f_x\,\vec{i} + f_y\,\vec{j} + f_z\,\vec{k}$

$= (2x)\,\vec{i} + (2y)\,\vec{j} + \vec{k}$

$\nabla f(1,2,4) = 2\,\vec{i} + 4\,\vec{j} + \vec{k}$

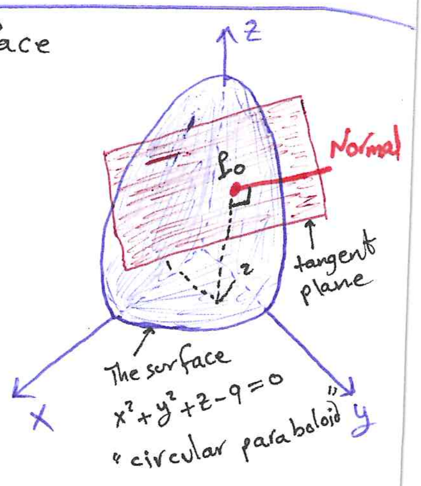
Tangent plane: $2x + 4y + z = 2 + 8 + 4$

$2x + 4y + z = 14$

• Normal line:

$x = 1 + 2t, \quad y = 2 + 4t$

$z = 4 + t$



Exp (Plane Tangent to a surface $z = f(x,y)$ at $(x_0, y_0, f(x_0,y_0))$)

Find an equation for the plane that is tangent to the surface $z = 4x^2 + y^2$ at the point $(1,1,5)$

$f(x,y,z) = 4x^2 + y^2 - z = 0$

- $\nabla f = f_x \vec{i} + f_y \vec{j} + f_z \vec{k} = (8x)\vec{i} + (2y)\vec{j} - \vec{k}$

$\nabla f(1,1,5) = 8\vec{i} + 2\vec{j} - \vec{k}$

- Tangent plane: $8x + 2y - z = (8)(1) + (2)(1) + (-1)(5) = 5$

Exp Find the parametric equation for the line tangent to the curve of intersection of the surfaces

$f(x,y,z) = x^2 + y^2 - 2 = 0$ and $g(x,y,z) = x + z - 4 = 0$ at $P_0(1,1,3)$

- $\nabla f = f_x \vec{i} + f_y \vec{j} + f_z \vec{k}$

$= (2x)\vec{i} + (2y)\vec{j}$

$\nabla f(1,1,3) = 2\vec{i} + 2\vec{j}$

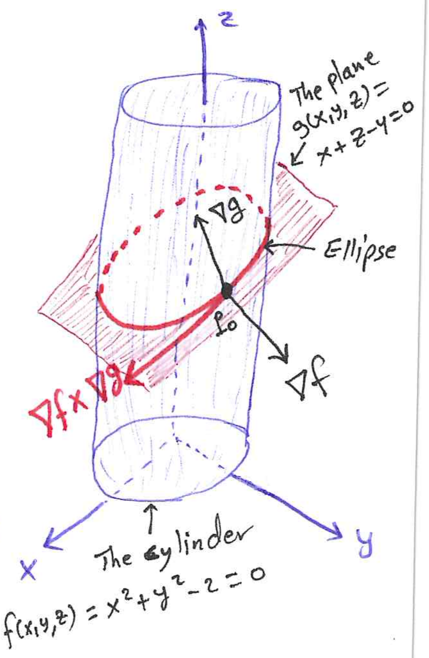
- $\nabla g = g_x \vec{i} + g_y \vec{j} + g_z \vec{k}$

$= \vec{i} + \vec{k}$

$\nabla g(1,1,3) = \vec{i} + \vec{k}$

- $\vec{v} = \nabla f(1,1,3) \times \nabla g(1,1,3)$ is // to the tangent line:

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 2 & 0 \\ 1 & 0 & 1 \end{vmatrix} = 2\vec{i} - 2\vec{j} - 2\vec{k}$$



- Hence, the tangent line:

$x = 1 + 2t$, $y = 1 - 2t$, $z = 3 - 2t$

Estimating the change in f in a Direction $\vec{u}$

The estimating change in the value of a diff function f when we move a small distance ds from a point P_0 in the direction $\vec{u}$ is

$$df = (D_{\vec{u}} f)(P_0)\, ds = \underbrace{\left(\nabla f(P_0) \cdot \vec{u}\right)}_{\text{Directional derivative}} \underbrace{ds}_{\text{distance increment}}$$

<u>Exp</u> By about how much will $f(x,y,z) = e^x \cos yz$ change as the point $P(x,y,z)$ moves from the origin a distance $ds = 0.1$ unit in the direction of $2\vec{i} + 2\vec{j} - 2\vec{k}$? $P_0(0,0,0)$

- $\nabla f = f_x \vec{i} + f_y \vec{j} + f_z \vec{k} = (e^x \cos yz)\vec{i} - (e^x z \sin yz)\vec{j} - (e^x y \sin yz)\vec{k}$

  $\nabla f(0,0,0) = \vec{i}$

- Direction: $\vec{u} = \dfrac{2\vec{i} + 2\vec{j} - 2\vec{k}}{\sqrt{4+4+4}} = \dfrac{1}{\sqrt{3}}\vec{i} + \dfrac{1}{\sqrt{3}}\vec{j} - \dfrac{1}{\sqrt{3}}\vec{k}$

- $(D_{\vec{u}} f)(0,0,0) = \nabla f(0,0,0) \cdot \vec{u}$

$$= \vec{i} \cdot \left[\frac{1}{\sqrt{3}}\vec{i} + \frac{1}{\sqrt{3}}\vec{j} - \frac{1}{\sqrt{3}}\vec{k}\right]$$

$$= \frac{1}{\sqrt{3}}$$

Hence, $df = (D_{\vec{u}} f)(0,0,0)\, ds$

$$= \left(\frac{1}{\sqrt{3}}\right)(0.1)$$

$$\approx 0.058$$

Linearization of $f(x,y)$

Def The linearization of a function $f(x,y)$ at a point (x_0,y_0) where f is diff is the function

$$L(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0).$$

The approximation $f(x,y) \approx L(x,y)$ is the standard linear approximation of f at (x_0, y_0).

Note. The linearization of function of two variables is a tangent plane
" " " " " " single " " " line.

The Error in the Standard Linear Approximation:

$$|E(x,y)| \leq \frac{M}{2}\left[|x-x_0| + |y-y_0|\right]^2$$, where M is any upper bound for $|f_{xx}|$, $|f_{yy}|$, $|f_{xy}|$

Exp Let $f(x,y) = x^2 - 3xy + 5$ and $P_0(2,1)$

1. Find the linearization $L(x,y)$ of the function $f(x,y)$ at P_0

$$L(x,y) = f(2,1) + f_x(2,1)(x-2) + f_y(2,1)(y-1)$$
$$= 3 + x - 2 + (-6)(y-1)$$
$$= x - 6y + 7$$

$f_x = 2x - 3y$

$f_y = -3x$

2. Find an upper bound for the magnitude of the error E in approximation $f(x,y) \approx L(x,y)$ over the rectangle $R: |x-2| \leq 0.1$, $|y-1| \leq 0.1$

$f_{xx} = 2$, $f_{yy} = 0$, $f_{xy} = -3$ $\Rightarrow M = 3$

$$|E(x,y)| \leq \frac{M}{2}\left[|x-2| + |y-1|\right]^2 \leq \frac{3}{2}\left[0.1 + 0.1\right]^2 = 0.06$$

3. Express the upper bound as a percentage of $f(2,1)$, the value of f at the center of the rectangle.

As a percentage of $f(2,1) = 3$, the error is no greater than $\frac{0.06}{3} \times 100 = 2\%$

Exp (Functions of more than two variables)

Let $f(x,y,z) = x^2 - xy + 3\sin z$ and $P_0(2,1,0)$.

[1] Find the linearization $L(x,y,z)$ of $f(x,y,z)$ at P_0.

$$L(x,y,z) = f(2,1,0) + f_x(2,1,0)(x-2) + f_y(2,1,0)(y-1) + f_z(2,1,0)(z-0)$$

$$= 2 + 3(x-2) - 2(y-1) + 3z$$

$$= 3x - 2y + 3z - 2$$

$f_x = 2x - y$
$f_y = -x$
$f_z = 3\cos z$

[2] Find an upper bound for the error incurred in replacing f by L on the region R: $|x-2| \le 0.01$, $|y-1| \le 0.02$, $|z| \le 0.01$

- $f_{xx} = 2$, $f_{yy} = 0$, $f_{zz} = -3\sin z$, $f_{xy} = -1$, $f_{xz} = 0$, $f_{yz} = 0$
- Note that $|z| \le 0.01 \Leftrightarrow -0.01 \le z \le 0.01$
  $\Rightarrow |f_{zz}| = |-3\sin z| \le 3\sin 0.01 \approx 0.03$

$\Longrightarrow M = 2$

- $$|E(x,y,z)| \le \frac{M}{2}\left[|x-2| + |y-1| + |z|\right]^2 \le \frac{2}{2}\left[0.01 + 0.02 + 0.01\right]^2$$
  $$= 0.0016.$$

Differentials

Def If we move from (x_0, y_0) to $(x_0 + dx, y_0 + dy)$ nearby, then the resulting estimate change (total differential of f) is

$$df = f_x(x_0, y_0)\,dx + f_y(x_0, y_0)\,dy$$ where $dx = \Delta x = x_2 - x_1$, $dy = \Delta y = y_2 - y_1$

Exp A cylindrical can is designed to have $r = 3$ cm with off $dr = 0.08$ and $h = 12$ cm with off $dh = -0.3$. Estimate the resulting absolute change in the volume of the can.

$$V = \pi r^2 h$$

$V_r = 2\pi r h$
$V_h = \pi r^2$

$$dV = V_r(r_0, h_0)\,dr + V_h(r_0, h_0)\,dh$$
$$= 2\pi r_0 h_0\,dr + \pi r_0^2\,dh$$
$$= 2\pi(3)(12)(0.08) + \pi(3)^2(-0.3)$$
$$= 3.06\pi \approx 9.61 \text{ cm}^3$$

Exp A circular cylindrical tank with $h = 8\,m$ and $r = 2\,m$. How sensitive are the tank's volumes to small variations in height and radius? $V = \pi r^2 h$

Total differenential gives the approximation for the change in volume: $dV = V_r(2,8)\,dr + V_h(2,8)\,dh$

$$= 2\pi r h\Big|_{(2,8)} dr + \pi r^2\Big|_{(2,8)} dh$$

$$= 32\pi\, dr + 4\pi\, dh$$

- One unit change in r will change V by 32π units.
- " " " " h " " " " 4π " .
- The tank's volume is 8 times more sensitive to small change in r than to small change in h.

Exp Let $V = \pi r^2 h$. Suppose that r is measured with error of no more than 2 % and h with error of no more than 0.5 %. Estimate the resulting percentage error in calculating V.

, We are given that $\left|\frac{dr}{r}\right| \le 0.02$ and $\left|\frac{dh}{h}\right| \le 0.005$

, $$\frac{dV}{V} = \frac{2\pi r h\,dr + \pi r^2\,dh}{\pi r^2 h} = \frac{2dr}{r} + \frac{dh}{h}$$

$$\left|\frac{dV}{V}\right| \le 2\left|\frac{dr}{r}\right| + \left|\frac{dh}{h}\right|$$

$$\le 2(0.02) + 0.005$$

$$= 0.045.$$

Hence, the estimate error in calculating V is at most 4.5 %
