

# Chapter 6. The Laplace Transform.

## 6.1 Definition of the Laplace Transform.

Review: (Section 8.7 Calculus I)

Improper Integral is defined as:

$$\int_a^{\infty} f(t) dt = \lim_{A \rightarrow \infty} \int_a^A f(t) dt, \text{ where } A > 0 \text{ (real)}$$

If  $\int_a^A f(t) dt$  exists for  $A > a$ , and the limit:

as  $A \rightarrow \infty$  exists, then the improper integral is said to converge to the limit value.

Otherwise the integral is said to be diverge.

Example: 
$$\int_0^{\infty} e^{\alpha t} dt = \lim_{A \rightarrow \infty} \int_0^A e^{\alpha t} dt$$

$$= \lim_{A \rightarrow \infty} \left. \frac{e^{\alpha t}}{\alpha} \right|_0^A = \lim_{A \rightarrow \infty} \frac{e^{\alpha A}}{\alpha} - \frac{1}{\alpha},$$

$\alpha \neq 0$

$$= \lim_{A \rightarrow \infty} \frac{e^{\alpha A} - 1}{\alpha} = \begin{cases} -\frac{1}{\alpha} & , \alpha < 0 \\ \text{div.} & , \alpha > 0 \end{cases}$$

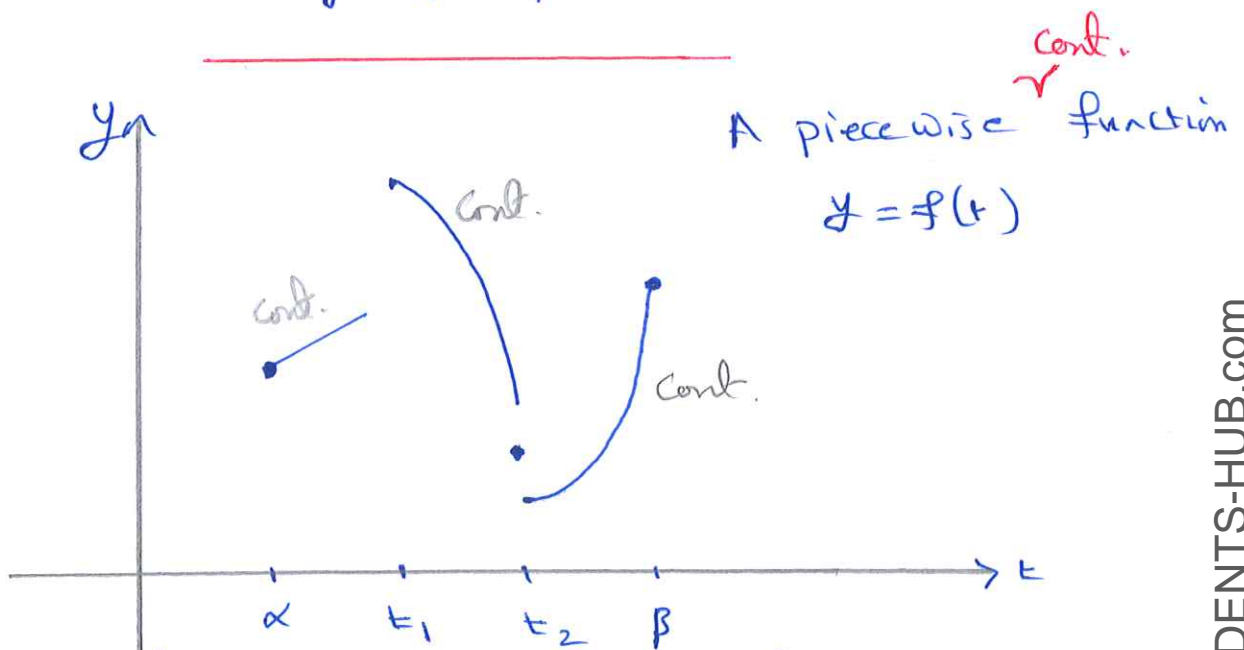
$$\text{If } \alpha = 0 \Rightarrow \int_0^{\infty} e^{\alpha t} dt = \int_0^{\infty} 1 dt = \infty \text{ (div.)}$$

Example:  $\int_1^{\infty} \frac{1}{t} dt = \lim_{A \rightarrow \infty} \int_1^A \frac{1}{t} dt$

$$= \lim_{A \rightarrow \infty} \ln |t| \Big|_1^A = \lim_{A \rightarrow \infty} (\ln A - \ln 1) = \infty \text{ (div.)}$$

Def: A function  $f$  is said to be piecewise continuous on  $\alpha \leq t \leq \beta$  if it is continuous except for a finite number of jump discontinuities.

Example:



$$\int_{\alpha}^{\beta} f(t) dt = \int_{\alpha}^{t_1} f(t) dt + \int_{t_1}^{t_2} f(t) dt + \int_{t_2}^{\beta} f(t) dt \quad (2)$$

Remark: If  $f$  is a piecewise continuous on

$a \leq t \leq b$ , then  $\int_a^b f(t) dt$  exists. However,

piecewise continuity is not enough to ensure convergence

of  $\int_a^b f(t) dt$ . In this case we use Comparison tests. If  $|f(t)| \leq g(t)$  &  $\int_a^\infty g(t) dt$  converges then  $\int_a^\infty f(t) dt$  converges. If  $f(t) \geq g(t)$  &  $\int_a^\infty g(t) dt$  diverges, then  $\int_a^\infty f(t) dt$  diverges.

The Laplace Transform.

One of the tools to solve Linear Differential equations is Laplace Transform

Def: An integral transform is a relation of the form

$$F(s) = \int_{\alpha}^{\beta} f(t) K(s, t) dt, \quad \dots (1)$$

where  $K(s, t)$  is called the kernel of the transformation and  $\alpha, \beta$  are also given.

It is possible that  $\alpha = -\infty$ , or  $\beta = \infty$ , or both.

Eq. (1) transforms  $f$  into another function  $F$ , which is called the transform of  $f$ .

Def: (Laplace Transform) [L.T].

The Laplace transform of  $f$  is defined as

$$\mathcal{L}\{f(t)\} = \int_0^{\infty} f(t) e^{-st} dt = F(s) \quad \dots (2)$$

provided that the improper integral converges.

Thm: Existence of Laplace Transform:

suppose that

1)  $f$  is piecewise continuous on  $0 \leq t \leq A$ ,  $\forall A > 0$ .

2)  $|f(t)| \leq k e^{at}$ ,  $t \geq M$ , where  $k, a, M \in \mathbb{R}$

and  $k$  and  $M$  are positive. Then the Laplace

transform  $\mathcal{L}(f(t)) = F(s)$  defined by eq. (2)

exists for  $s > a$ .

## Examples:

$$1) \mathcal{L}(1) = \int_0^{\infty} 1 e^{-st} dt = \lim_{A \rightarrow \infty} \int_0^A e^{-st} dt$$

$$= \lim_{A \rightarrow \infty} \left. \frac{e^{-st}}{-s} \right|_0^A = \lim_{A \rightarrow \infty} \frac{-e^{-sA} + 1}{s}, \quad s > 0.$$

$$\Rightarrow \mathcal{L}(1) = \frac{1}{s}, \quad s > 0.$$

In General  $\mathcal{L}(k) = \frac{k}{s}$ ,  $s > 0$  and  $k$  is Constant

$$\text{Ex. } \mathcal{L}(2023) = \frac{2023}{s}, \quad s > 0.$$

$$2) \mathcal{L}(e^{kt}) = \int_0^{\infty} e^{kt} e^{-st} dt = \lim_{A \rightarrow \infty} \int_0^A e^{(k-s)t} dt$$

$$= \lim_{A \rightarrow \infty} \left. \frac{e^{(k-s)t}}{(k-s)} \right|_0^A = \lim_{A \rightarrow \infty} \left. \frac{-e^{-(s-k)t}}{-(s-k)} \right|_0^A$$

$$= \lim_{A \rightarrow \infty} \frac{1 - e^{-(s-k)A}}{(s-k)} = \frac{1}{s-k}, \quad s > k.$$

$$\Rightarrow \mathcal{L}(e^{kt}) = \frac{1}{s-k}, \quad s > k$$

ex.  $\mathcal{L}(e^{2t}) = \frac{1}{s-2}, \quad s > 2.$

$$3) \mathcal{L}(t^2) = \int_0^{\infty} t^2 e^{-st} dt = \lim_{A \rightarrow \infty} \int_0^A t^2 e^{-st} dt$$

$$= \lim_{A \rightarrow \infty} \left( -\frac{t^2}{s} e^{-st} - \frac{2t}{s^2} e^{-st} - \frac{2e^{-st}}{s^3} \right) \Big|_0^A$$

$$= \lim_{A \rightarrow \infty} \left[ \frac{-\frac{A^2}{s} - \frac{2A}{s^2} - \frac{2}{s^3}}{e^{sA}} + \frac{2}{s^3} \right]$$

L.H.  $= 0 + \frac{2}{s^3} = \frac{2!}{s^3}, \quad s > 0$

In General:

$$\mathcal{L}(t^n) = \frac{n!}{s^{n+1}}, \quad s > 0$$

ex.  $\mathcal{L}(t^4) = \frac{4!}{s^5}, \quad s > 0$

$$\mathcal{L}(t) = \frac{1}{s^2}, \quad s > 0$$

$$4) \mathcal{L}(\sin at) = \int_0^{\infty} \sin at e^{-st} dt$$

$$= \lim_{A \rightarrow \infty} \int_0^A \sin at e^{-st} dt \quad \dots (*)$$

Integrating by parts:  $u = e^{-st}$   $dv = \sin at dt$

$$du = -s e^{-st} dt, \quad v = -\frac{\cos at}{a}$$

$$(*) \Rightarrow \lim_{A \rightarrow \infty} \left[ -\frac{\cos at}{a} e^{-st} \Big|_0^A - \int_0^A \frac{s}{a} e^{-st} \cos at dt \right], s > 0$$

Again by parts.  
⋮  
Continue.

$$\Rightarrow F(s) = \frac{1}{a} - \frac{s^2}{a^2} \int_0^{\infty} e^{-st} \sin at dt$$

$$F(s) = \frac{1}{a} - \frac{s^2}{a^2} F(s)$$

$$\Rightarrow F(s) \left( 1 + \frac{s^2}{a^2} \right) = \frac{1}{a}$$

$$\Rightarrow F(s) = \frac{1}{a} \cdot \frac{a^2}{s^2 + a^2} = \frac{a}{s^2 + a^2}$$

$$\therefore \mathcal{L}(\sin at) = \frac{a}{s^2 + a^2}, \quad s > 0$$

$$5) \mathcal{L}(\cos at) = \frac{s}{s^2 + a^2}, \quad s > 0.$$

$$6) \mathcal{L}(\cosh(at)) = \frac{s}{s^2 - a^2}, \quad s > |a|$$

$$7) \mathcal{L}(\sinh(at)) = \frac{a}{s^2 - a^2}, \quad s > |a|$$

$$8) \mathcal{L}(e^{at} \sin bt) = \frac{b}{(s-a)^2 + b^2}, \quad s > a$$

$$9) \mathcal{L}(e^{at} \cos bt) = \frac{s-a}{(s-a)^2 + b^2}, \quad s > a$$

$$10) \mathcal{L}(t^n e^{at}) = \frac{n!}{(s-a)^{n+1}}, \quad n = 0, 1, 2, \dots$$

$$11) \mathcal{L}(t^n f(t)) = (-1)^n \frac{d^n}{ds^n} (\mathcal{L}(f(t))) = F^{(n)}(s) (-1)^n$$

$$12) \mathcal{L}(\alpha f(t) \pm \beta g(t)) = \alpha \mathcal{L}(f(t)) \pm \beta \mathcal{L}(g(t)).$$

Proof (12):  $\int_0^{\infty} (\alpha f(t) \pm \beta g(t)) e^{-st} dt$

$$= \alpha \int_0^{\infty} f(t) e^{-st} dt \pm \beta \int_0^{\infty} g(t) e^{-st} dt = \alpha \mathcal{L}(f(t)) \pm \beta \mathcal{L}(g(t))$$



Proof (7):  $\mathcal{L}(\sinh at) = \mathcal{L}\left(\frac{e^{at} - e^{-at}}{2}\right)$

$$= \frac{1}{2} \mathcal{L}(e^{at}) - \frac{1}{2} \mathcal{L}(e^{-at})$$

$$= \frac{1}{2} \left(\frac{1}{s-a}\right) - \frac{1}{2} \left(\frac{1}{s+a}\right) = \frac{1}{2} \left(\frac{s+a - (s-a)}{s^2 - a^2}\right)$$

$$= \frac{1}{2} \left(\frac{2a}{s^2 - a^2}\right) = \frac{a}{s^2 - a^2} \quad , \quad s > |a|$$

Proof (8):  $\mathcal{L}(\cosh at) = \mathcal{L}\left(\frac{e^{at} + e^{-at}}{2}\right)$

$$= \frac{1}{2} [\mathcal{L}(e^{at}) + \mathcal{L}(e^{-at})] = \frac{1}{2} \left(\frac{1}{s-a} + \frac{1}{s+a}\right)$$

$$= \frac{1}{2} \left(\frac{s+a + s-a}{s^2 - a^2}\right) = \frac{1}{2} \left(\frac{2s}{s^2 - a^2}\right) = \frac{s}{s^2 - a^2} \quad , \quad |a| < s$$

Example: Find  $\mathcal{L}(4e^{-2t} - 3\sin 4t)$

$$= 4 \mathcal{L}(e^{-2t}) - 3 \mathcal{L}(\sin 4t)$$

$$= \frac{4}{s+2} - \frac{3 \cdot 4}{s^2 + (4)^2} = \frac{4}{s+2} - \frac{12}{s^2 + 16}$$

Example:  $\mathcal{L}(\sin^2 t) = \mathcal{L}\left(\frac{1 - \cos 2t}{2}\right)$

$$= \mathcal{L}\left(\frac{1}{2}\right) - \frac{1}{2} \mathcal{L}(\cos 2t)$$

$$= \frac{\frac{1}{2}}{s} - \frac{1}{2} \cdot \frac{s}{s^2 + 4} = \frac{1}{2s} - \frac{s}{2(s^2 + 4)}$$

Example:  $\mathcal{L}(t^2 e^t) = (-1)^2 \frac{d^2}{ds^2} (\mathcal{L}(e^t)) = F''(s)$

$$F(s) = \mathcal{L}(e^t) = \frac{1}{s-1}$$

$$\Rightarrow F'(s) = \frac{-1}{(s-1)^2} = -(s-1)^{-2}$$

$$\Rightarrow F''(s) = 2(s-1)^{-3}$$

$$\Rightarrow \mathcal{L}(t^2 e^t) = 2(s-1)^{-3} = \frac{2}{(s-1)^3}$$

Example:  $\mathcal{L}(\cosh(6t)) = \frac{s}{s^2 - 36}, s > 6$

$$\text{Example: } \mathcal{L}(e^{2t} \sinh 6t) = \frac{6}{(s-2)^2 + 36}$$

$a = 2, b = 6$  in Rule #8

Example:  $\mathcal{L}\left(\sin\left(2t + \frac{\pi}{3}\right)\right)$

$$= \mathcal{L}\left(\sin 2t \cos \frac{\pi}{3} + \cos 2t \sin \frac{\pi}{3}\right)$$

$$= \mathcal{L}\left(\frac{1}{2} \sin 2t + \frac{\sqrt{3}}{2} \cos 2t\right)$$

$$= \frac{1}{2} \mathcal{L}(\sin 2t) + \frac{\sqrt{3}}{2} \mathcal{L}(\cos 2t)$$

$$= \frac{1}{2} \left(\frac{2}{s^2+4}\right) + \frac{\sqrt{3}}{2} \left(\frac{s}{s^2+4}\right)$$

$$= \frac{1 + \frac{\sqrt{3}}{2}s}{s^2+4}$$

proof (5)  $\mathcal{L}(\cos at) = \mathcal{L}\left(\frac{e^{at} + e^{-at}}{2}\right)$

$$= \frac{1}{2} \mathcal{L}(e^{at} + e^{-at}) = \frac{1}{2} \left(\frac{1}{s-a} + \frac{1}{s+a}\right)$$

$$= \frac{1}{2} \left(\frac{s+a + s-a}{s^2 - (a)^2}\right) = \frac{1}{2} \left(\frac{2s}{s^2+a^2}\right) = \frac{s}{s^2+a^2}$$

Note that:  $\left. \begin{aligned} \cos \theta &= e^{i\theta} - i \sin \theta \\ \cos \theta &= e^{-i\theta} + i \sin \theta \end{aligned} \right\} \Rightarrow \text{Euler formulas}$

$$\Rightarrow 2 \cos \theta = e^{i\theta} + e^{-i\theta} \Rightarrow \cos \theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

(11)

## 6.2 Solutions of Initial Value Problem.

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In this section we will show how the Laplace transform can be used to solve IVPs for linear DE with constant coefficients.

The inverse of Laplace Transform.

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$$\mathcal{L}(f(t)) = F(s) \Rightarrow f(t) = \mathcal{L}^{-1}(F(s))$$

Example: Find the Inverse of Laplace Transform.

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$$1) \mathcal{L}^{-1}\left(\frac{2023}{s}\right) = 2023$$

$$2) \mathcal{L}^{-1}\left(\frac{1}{s^3}\right) = \frac{1}{2!} \mathcal{L}^{-1}\left(\frac{2!}{s^3}\right) = \frac{1}{2} t^2.$$

$$\begin{aligned} 3) \mathcal{L}^{-1}\left(\frac{2s-3}{s^2+4}\right) &= 2 \mathcal{L}^{-1}\left(\frac{s}{s^2+4}\right) - 3 \mathcal{L}^{-1}\left(\frac{1}{s^2+4}\right) \\ &= 2 \mathcal{L}^{-1}\left(\frac{s}{s^2+4}\right) - \frac{3}{2} \mathcal{L}^{-1}\left(\frac{2}{s^2+4}\right) \\ &= 2 \cos 2t - \frac{3}{2} \sin 2t. \end{aligned}$$

(12)

$$\begin{aligned}
 4) \quad \mathcal{L}^{-1} \left( \frac{1}{(s-4)^4} \right) &= \frac{1}{3!} \mathcal{L}^{-1} \left( \frac{3!}{(s-4)^4} \right) \\
 &= \frac{1}{3!} e^{4t} \mathcal{L}^{-1} \left( \frac{3!}{s^4} \right) \\
 &= \frac{1}{6} e^{4t} \cdot t^3.
 \end{aligned}$$


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$$\begin{aligned}
 5) \quad \mathcal{L}^{-1} \left( \frac{s}{(s-2)^2 + 9} \right) &= \mathcal{L}^{-1} \left( \frac{(s-2) + 2}{(s-2)^2 + 9} \right) \\
 &= \mathcal{L}^{-1} \left( \frac{(s-2)}{(s-2)^2 + 9} \right) + \mathcal{L}^{-1} \left( \frac{2}{(s-2)^2 + 9} \right) \\
 &= e^{2t} \mathcal{L}^{-1} \left( \frac{s}{s^2 + 9} \right) + \frac{2}{3} e^{2t} \mathcal{L}^{-1} \left( \frac{3}{s^2 + 9} \right) \\
 &= e^{2t} \cos 3t + \frac{2}{3} e^{2t} \sin 3t.
 \end{aligned}$$


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$$\begin{aligned}
 6) \quad \mathcal{L}^{-1} \left( \frac{2s+2}{s^2+2s+6} \right) &= \mathcal{L}^{-1} \left( \frac{2(s+1)}{(s+1)^2+5} \right) \quad \text{z-p dkt!} \\
 &= 2 \mathcal{L}^{-1} \left( \frac{(s+1)}{(s+1)^2+5} \right) = 2 e^{-t} \mathcal{L}^{-1} \left( \frac{s}{s^2+5} \right) \\
 &= 2 e^{-t} \cos(\sqrt{5} t).
 \end{aligned}$$

$$7) \mathcal{L}^{-1}\left(\frac{1}{s^2 - 2s - 3}\right) = \mathcal{L}^{-1}\left(\frac{1}{(s-3)(s+1)}\right)$$

Using Partial Fraction:

$$\frac{1}{(s-3)(s+1)} = \frac{A}{s-3} + \frac{B}{s+1} \quad \text{or let}$$

$$\Rightarrow A(s+1) + B(s-3) = 1 \quad \begin{matrix} \nearrow s = -1 \Rightarrow B = -\frac{1}{4} \\ \searrow s = 3 \Rightarrow A = \frac{1}{4} \end{matrix}$$

$$\Rightarrow (A+B)s + (A-3B) = 1 \quad \text{(cover up method for linear factors)}$$

$$\Rightarrow \left. \begin{matrix} A+B=0 \\ A-3B=1 \end{matrix} \right\} \boxed{B = -\frac{1}{4}} \ \& \ \boxed{A = \frac{1}{4}}$$

$$\begin{aligned} \therefore \mathcal{L}^{-1}\left(\frac{1}{s^2 - 2s - 3}\right) &= \mathcal{L}^{-1}\left(\frac{\frac{1}{4}}{s-3}\right) + \mathcal{L}^{-1}\left(\frac{-\frac{1}{4}}{s+1}\right) \\ &= \frac{1}{4} \mathcal{L}^{-1}\left(\frac{1}{s-3}\right) - \frac{1}{4} \mathcal{L}^{-1}\left(\frac{1}{s+1}\right) \\ &= \frac{1}{4} e^{3t} \mathcal{L}^{-1}\left(\frac{1}{s}\right) - \frac{1}{4} e^{-t} \mathcal{L}^{-1}\left(\frac{1}{s}\right) \\ &= \frac{1}{4} e^{3t} - \frac{1}{4} e^{-t} \end{aligned}$$



$$8) \mathcal{L}^{-1} \left( \frac{s}{(s+2)(s^2+4)} \right)$$

$$\frac{s}{(s+2)(s^2+4)} = \frac{A}{s+2} + \frac{Bs+C}{s^2+4}$$

$$\Rightarrow s = A(s^2+4) + (Bs+C)(s+2)$$

$$\Rightarrow s = As^2 + 4A + Bs^2 + 2Bs + Cs + 2C$$

$$\Rightarrow s = (A+B)s^2 + (2B+C)s + (4A+2C)$$

$$\Rightarrow \left. \begin{array}{l} A+B=0 \\ 2B+C=1 \\ 4A+2C=0 \end{array} \right\} \Rightarrow \begin{array}{l} A = -\frac{1}{4} \\ B = \frac{1}{4} \\ C = \frac{1}{2} \end{array}$$

$$\therefore \mathcal{L}^{-1} \left( \frac{s}{(s+2)(s^2+4)} \right) = \mathcal{L}^{-1} \left( \frac{-\frac{1}{4}}{s+2} \right) + \mathcal{L}^{-1} \left( \frac{\frac{1}{4}s + \frac{1}{2}}{s^2+4} \right)$$

$$= -\frac{1}{4} \mathcal{L}^{-1} \left( \frac{1}{s+2} \right) + \frac{1}{4} \mathcal{L}^{-1} \left( \frac{s}{s^2+4} \right) + \frac{1}{2} \mathcal{L}^{-1} \left( \frac{1}{s^2+4} \right)$$

$$= -\frac{1}{4} e^{-2t} + \frac{1}{4} \cos 2t + \frac{1}{4} \sin 2t.$$

# Laplace of Derivatives.

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$$1) \mathcal{L}(y(t)) = Y(s)$$

$$2) \mathcal{L}(y'(t)) = sY(s) - y(0)$$

$$3) \mathcal{L}(y''(t)) = s^2 Y(s) - sy(0) - y'(0)$$

$$4) \mathcal{L}(y'''(t)) = s^3 Y(s) - s^2 y(0) - sy'(0) - y''(0)$$

⋮

In General:

$$\mathcal{L}(y^{(n)}(t)) = s^n Y(s) - s^{n-1} y(0) - s^{n-2} y'(0) - \dots - y^{(n-1)}(0).$$

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Proof (2):  $\mathcal{L}(y'(t)) = \lim_{A \rightarrow \infty} \int_0^A e^{-st} y'(t) dt$

$$= \lim_{A \rightarrow \infty} \left( e^{-st} y(t) \Big|_0^A + \int_0^A s e^{-st} y(t) dt \right)$$

$$= \lim_{A \rightarrow \infty} \left( e^{-sA} y(A) - y(0) \right) + s \lim_{A \rightarrow \infty} \int_0^A e^{-st} y(t) dt$$

$$= -y(0) + s \mathcal{L}(y(t)) = sY(s) - y(0)$$

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$\left( \begin{array}{l} u = e^{-st} \\ \vdots \\ \text{by parts} \end{array} \right)$   
 $\left( \begin{array}{l} dv = y' dt \\ \vdots \end{array} \right)$



Proof (3):  $\mathcal{L}(y''(t))$

Let  $g(t) = y'(t)$ , then  $g'(t) = y''(t)$

$$\begin{aligned} \Rightarrow \mathcal{L}(g'(t)) &= s \mathcal{L}(g(t)) - g(0) \\ &= s \mathcal{L}(y'(t)) - y'(0) \\ &= s \left( s Y(s) - y(0) \right) - y'(0) \\ &= s^2 Y(s) - s y(0) - y'(0) \end{aligned}$$

Now, we are ready to use Laplace Transform to solve IVP.

Example: Use Laplace Transform to solve:

$$y'' - y' - 2y = 0, \quad y(0) = 1, \quad y'(0) = 0$$

Sol: Take Laplace Transform for both sides.

$$\mathcal{L}(y'' - y' - 2y) = \mathcal{L}(0)$$

$$\Rightarrow \mathcal{L}(y'') - \mathcal{L}(y') - 2 \mathcal{L}(y) = 0$$

$$\Rightarrow [s^2 Y(s) - s y(0) - y'(0)] - [s Y(s) - y(0)] - 2Y(s) = 0$$

$$\Rightarrow [s^2 - s - 2] Y(s) - \underbrace{s y(0)}_{=1} + \underbrace{y(0)}_{=1} - \underbrace{y'(0)}_{=0} = 0$$

$$\Rightarrow [s^2 - s - 2] Y(s) = s - 1$$

$$\Rightarrow Y(s) = \frac{s-1}{s^2 - s - 2} = \frac{s-1}{(s+1)(s-2)}$$

$$\therefore y(t) = \mathcal{L}^{-1}(Y(s)) = \mathcal{L}^{-1}\left(\frac{s-1}{(s+1)(s-2)}\right)$$

$$\frac{s-1}{(s+1)(s-2)} = \frac{A}{s+1} + \frac{B}{s-2}$$

$$\Rightarrow s-1 = (s-2)A + (s+1)B$$

$$\text{Let } s = -1 \Rightarrow A = \frac{2}{3}$$

$$\text{Let } s = 2 \Rightarrow B = \frac{1}{3}$$

$$\begin{aligned} \therefore y(t) &= \mathcal{L}^{-1}\left(\frac{\frac{2}{3}}{s+1}\right) + \mathcal{L}^{-1}\left(\frac{\frac{1}{3}}{s-2}\right) \\ &= \frac{2}{3} e^{-t} + \frac{1}{3} e^{2t} \end{aligned}$$

Example: solve  $y'' + y = \sin 2t$ ,  $y(0) = 2$ ,  $y'(0) = 1$

Sol:  $\mathcal{L}(y'' + y) = \mathcal{L}(\sin 2t)$

$$\Rightarrow [s^2 Y(s) - s y(0) - y'(0)] + Y(s) = \frac{2}{s^2 + 4}$$

$$\Rightarrow (s^2 + 1)Y(s) - 2s - 1 = \frac{2}{s^2 + 4}$$

$$\Rightarrow (s^2 + 1)Y(s) = \frac{2}{s^2 + 4} + (2s + 1) \cdot \left( \frac{s^2 + 4}{s^2 + 4} \right)$$

$$\Rightarrow (s^2 + 1)Y(s) = \frac{2 + 2s^3 + 8s + s^2 + 4}{(s^2 + 4)}$$

$$\Rightarrow Y(s) = \frac{(2s^3 + s^2 + 8s + 6)}{(s^2 + 1)(s^2 + 4)}$$

Using Partial Fraction:

$$\frac{2s^3 + s^2 + 8s + 6}{(s^2 + 1)(s^2 + 4)} = \frac{(As + B)}{(s^2 + 1)} + \frac{(Cs + D)}{(s^2 + 4)}$$

$$\begin{aligned} \Rightarrow 2s^3 + s^2 + 8s + 6 &= (As + B)(s^2 + 4) + (Cs + D)(s^2 + 1) \\ &= (A + C)s^3 + (B + D)s^2 + (4A + C)s + (4B + D) \end{aligned}$$

$$\left. \begin{array}{l} A + C = 2 \\ B + D = 1 \\ 4A + C = 8 \\ 4B + D = 6 \end{array} \right\} \Rightarrow \begin{array}{l} A = 2 \\ B = 5/3 \\ C = 0 \\ D = -2/3 \end{array}$$

$$\begin{aligned} \Rightarrow y(t) &= \mathcal{L}^{-1}(Y(s)) = \mathcal{L}^{-1}\left(\frac{2s + 5/3}{s^2 + 1}\right) + \mathcal{L}^{-1}\left(\frac{-2/3}{s^2 + 4}\right) \\ &= 2 \mathcal{L}^{-1}\left(\frac{s}{s^2 + 1}\right) + \frac{5}{3} \mathcal{L}^{-1}\left(\frac{1}{s^2 + 1}\right) - \frac{1}{3} \mathcal{L}^{-1}\left(\frac{2}{s^2 + 4}\right) \\ &= 2 \cos t + \frac{5}{3} \sin t - \frac{1}{3} \sin 2t. \end{aligned}$$

Example: Solve  $y' = t$ ,  $y(0) = 1$

Sol:  $\mathcal{L}(y') = \mathcal{L}(t)$

$$\Rightarrow sY(s) - y(0) = \frac{1}{s^2}$$

$$\Rightarrow sY(s) = \frac{1}{s^2} + 1 = \frac{1 + s^2}{s^2}$$

$$\Rightarrow Y(s) = \frac{1}{s^3} + \frac{1}{s}$$

$$\therefore y(t) = \mathcal{L}^{-1}\left(\frac{1}{s^3}\right) + \mathcal{L}^{-1}\left(\frac{1}{s}\right) = \frac{1}{2} t^2 + 1$$

Example: Solve  $y'' + 2y' + y = 4e^{-t}$ ,  $y(0) = 2$ ,  $y'(0) = -$

Using Laplace Transform.

Sol:  $\mathcal{L}(y'' + 2y' + y) = \mathcal{L}(4e^{-t})$

$$\Rightarrow [s^2 Y(s) - sy(0) - y'(0)] + 2[sY(s) - y(0)] + Y(s) = \frac{4}{s+1}$$

$$\Rightarrow (s^2 + 2s + 1)Y(s) = \frac{4}{s+1} + 2s - 1 + 4$$

$$\Rightarrow \underbrace{(s^2 + 2s + 1)}_{(s+1)(s+1)} Y(s) = \frac{2s^2 + 5s + 7}{(s+1)}$$

$$\Rightarrow Y(s) = \frac{2s^2 + 5s + 7}{(s+1)^3}$$

$$y(t) = \mathcal{L}^{-1}(Y(s)) = \mathcal{L}^{-1}\left(\frac{2s^2 + 5s + 7}{(s+1)^3}\right)$$

$$\frac{2s^2 + 5s + 7}{(s+1)^3} = \frac{A}{(s+1)} + \frac{B}{(s+1)^2} + \frac{C}{(s+1)^3}$$

$$\begin{aligned} \Rightarrow 2s^2 + 5s + 7 &= A(s+1)^2 + B(s+1) + C \\ &= A(s^2 + 2s + 1) + (Bs + B) + C \end{aligned}$$

$$\Rightarrow 2s^2 + 5s + 7 = As^2 + (2A + B)s + (A + B + C)$$

$$\Rightarrow \boxed{A = 2}$$

$$2A + B = 5 \Rightarrow \boxed{B = 1}$$

$$A + B + C = 7 \Rightarrow \boxed{C = 4}$$

$$2 \mathcal{L}^{-1} \left( \frac{2}{(s+1)^3} \right)$$

$$\begin{aligned} \Rightarrow y(t) &= \mathcal{L}^{-1} \left( \frac{2}{s+1} \right) + \mathcal{L}^{-1} \left( \frac{1}{(s+1)^2} \right) + \mathcal{L}^{-1} \left( \frac{4}{(s+1)^3} \right) \\ &= 2e^{-t} + e^{-t} \cdot t + 2e^{-t} \cdot t^2 \\ &= (2 + t + 2t^2) e^{-t} \end{aligned}$$

Example: Solve  $y^{(4)} - y = 0$ ,  $y(0) = y''(0) = y'''(0) = 0$ ,  $y'(0) = 1$

Using Laplace transform.

Sol:  $\mathcal{L}(y^{(4)}) - \mathcal{L}(y) = \mathcal{L}(0)$

$$\Rightarrow s^4 Y(s) - s^3 y(0) - s^2 y'(0) - s y''(0) - y'''(0) - Y(s) = 0$$

$$\Rightarrow (s^4 - 1) Y(s) = s^2$$

$$\therefore Y(s) = \frac{s^2}{(s^4 - 1)} = \frac{s^2}{(s-1)(s+1)(s^2+1)}$$

$$\therefore y(t) = \mathcal{L}^{-1}(Y(s)) = \mathcal{L}^{-1}\left(\frac{s^2}{(s-1)(s+1)(s^2+1)}\right)$$

$$\frac{s^2}{(s-1)(s+1)(s^2+1)} = \frac{A}{s-1} + \frac{B}{s+1} + \frac{Cs+D}{s^2+1}$$

$$\therefore s^2 = A(s+1)(s^2+1) + B(s-1)(s^2+1) + (Cs+D)(s^2-1)$$

$$\text{Let } s=1 \Rightarrow 1 = 4A \Rightarrow \boxed{A = \frac{1}{4}}$$

$$s=-1 \Rightarrow 1 = -4B \Rightarrow \boxed{B = -\frac{1}{4}}$$

$$s=0 \Rightarrow 0 = A + (-B) - D \Rightarrow \boxed{D = \frac{1}{2}}$$

$$s=2 \Rightarrow 4 = 15A + 5B + (2C + \frac{1}{2})(3) \\ \Rightarrow \boxed{C = 0}$$

$$\therefore y(t) = \mathcal{L}^{-1}\left(\frac{\frac{1}{4}}{s-1}\right) + \mathcal{L}^{-1}\left(\frac{-\frac{1}{4}}{s+1}\right) + \mathcal{L}^{-1}\left(\frac{\frac{1}{2}}{s^2+1}\right)$$

$$y(t) = \frac{1}{4}e^t - \frac{1}{4}e^{-t} + \frac{1}{2}\sin t$$

$$= \frac{\sinh t + \sin t}{2}$$

## 6.3. Step Functions.

Some of the most interesting elementary applications of Laplace Transform occur in the solution of linear differential equations with discontinuous functions.

<sup>(6.3.2)</sup> <sup>1st shifting</sup>  
Theorem: (First Translation Theorem)

If  $\mathcal{L}(f(t)) = F(s)$  and  $a \in \mathbb{R}$ , then:

$$\mathcal{L}(e^{at} f(t)) = \mathcal{L}(f(t))_{s \rightarrow s-a} = F(s-a)$$

$$\text{or } \mathcal{L}^{-1}(F(s-a)) = e^{at} \mathcal{L}^{-1}(F(s))$$

Example: ①  $\mathcal{L}(e^{5t} \cdot t^3) = \mathcal{L}(t^3)_{s \rightarrow s-5} = \frac{3!}{(s-5)^4}$

②  $\mathcal{L}(e^{-2t} \cos 4t) = \mathcal{L}(\cos 4t)_{s \rightarrow s+2} = \frac{s}{s+16} \Big|_{s \rightarrow s+2}$   
 $= \frac{(s+2)}{(s+2)^2 + 16}$



$$\begin{aligned}
 \textcircled{3} \quad \mathcal{L}^{-1} \left( \frac{2s+5}{(s-3)^2} \right) &= \mathcal{L}^{-1} \left( \frac{2(s-3) + 6+5}{(s-3)^2} \right) \\
 &= \mathcal{L}^{-1} \left( \frac{2(s-3)}{(s-3)^2} + \frac{11}{(s-3)^2} \right) = \mathcal{L}^{-1} \left( \frac{2}{(s-3)} + \frac{11}{(s-3)^2} \right) \\
 &= 2 \mathcal{L}^{-1} \left( \frac{1}{s-3} \right) + 11 \mathcal{L}^{-1} \left( \frac{1}{(s-3)^2} \right) = 2e^{3t} \cdot 1 + 11e^{3t} \cdot t \\
 &= (2 + 11t) e^{3t}
 \end{aligned}$$

OR:  $\frac{2s+5}{(s-3)^2} = \frac{A}{s-3} + \frac{B}{(s-3)^2}$

$\Rightarrow 2s+5 = A(s-3) + B$

Let  $s=3 \Rightarrow B=11$

$s=0 \Rightarrow A=2$

$$\begin{aligned}
 \therefore \mathcal{L}^{-1} \left( \frac{2s+5}{(s-3)^2} \right) &= \mathcal{L}^{-1} \left( \frac{2}{(s-3)} + \frac{11}{(s-3)^2} \right) \\
 &= 2e^{3t} \cdot 1 + 11e^{3t} \cdot t \\
 &= (2 + 11t) e^{3t}
 \end{aligned}$$

$$\begin{aligned}
 \textcircled{4} \quad \mathcal{L}^{-1} \left( \frac{\frac{1}{2}s + \frac{5}{3}}{s^2 + 4s + 6} \right) &= \mathcal{L}^{-1} \left( \frac{\frac{1}{2}(s+2) + \frac{2}{3}}{(s+2)^2 + 2} \right) \\
 &= \frac{1}{2} \mathcal{L}^{-1} \left( \frac{s+2}{(s+2)^2 + (\sqrt{2})^2} \right) + \frac{2}{3\sqrt{2}} \mathcal{L}^{-1} \left( \frac{\sqrt{2}}{(s+2)^2 + (\sqrt{2})^2} \right) \\
 &= \frac{1}{2} e^{-2t} \mathcal{L}^{-1} \left( \frac{s}{s^2 + (\sqrt{2})^2} \right) + \frac{2}{3\sqrt{2}} e^{-2t} \mathcal{L}^{-1} \left( \frac{\sqrt{2}}{s^2 + (\sqrt{2})^2} \right) \\
 &= \frac{1}{2} e^{-2t} \cos(\sqrt{2}t) + \frac{2}{3\sqrt{2}} e^{-2t} \sin(\sqrt{2}t).
 \end{aligned}$$

Example: Solve the following IVP.

$$y'' - 6y' + 9y = t^2 e^{3t}, \quad y(0) = 2, \quad y'(0) = 17$$

Using Laplace Transform.

$$\text{sol: } \mathcal{L}(y'' - 6y' + 9y) = \mathcal{L}(t^2 e^{3t}).$$

$$[s^2 Y(s) - sy(0) - y'(0)] - 6[sY(s) - y(0)] + 9Y(s)$$

$$= \frac{2!}{(s-3)^3}$$

$$\Rightarrow [s^2 - 6s + 9]Y(s) - 2s - 17 + 12 = \frac{2!}{(s-3)^3}$$

(20)

$$\therefore (s-3)^2 Y(s) = \frac{2!}{(s-3)^3} + 2s + 5$$

$$\Rightarrow Y(s) = \frac{2!}{(s-3)^5} + \frac{2s+5}{(s-3)^2}$$

$$\therefore y(t) = \mathcal{L}^{-1} \left( \frac{2!}{(s-3)^5} \right) + \mathcal{L}^{-1} \left( \frac{2s+5}{(s-3)^2} \right)$$

$$= \frac{2!}{4!} \mathcal{L}^{-1} \left( \frac{4!}{(s-3)^5} \right) + (2+11t)e^{3t}$$

example (3)

$$= \frac{1}{12} e^{3t} t^4 + (2+11t)e^{3t}$$

$$= \left( \frac{1}{12} t^4 + 11t + 2 \right) e^{3t}$$


---

Home work :

Solve the following IVP Using Laplace Transform.

$$y'' + 4y' + 6y = 1 + e^{-t}$$

$$y(0) = y'(0) = 0.$$

Def: The unit step function or Heaviside function

The unit step function or Heaviside function is defined by

$$u_c(t) = u(t-c) = \begin{cases} 0 & , t < c \\ 1 & , t \geq c \end{cases} , c \geq 0$$

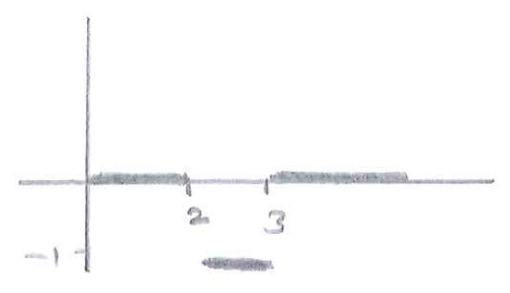
Example:  $u_5(t) = \begin{cases} 0 & , t < 5 \\ 1 & , t \geq 5 \end{cases}$



Example: sketch the graph of  $y = u_3(t) - u_2(t)$

$$u_3(t) - u_2(t) = \begin{cases} 0 & , t < 3 \\ 1 & , t \geq 3 \end{cases} - \begin{cases} 0 & , t < 2 \\ 1 & , t \geq 2 \end{cases}$$

$$= \begin{cases} 0 - 0 & , 0 \leq t < 2 \\ 0 - 1 & , 2 \leq t < 3 \\ 1 - 1 & , t \geq 3 \end{cases}$$



$$= \begin{cases} 0 & , 0 \leq t < 2 \text{ \& } t \geq 3 \\ -1 & , 2 \leq t < 3 \end{cases}$$

Remark: The Unit step function can be used to write a piecewise function in a compact form as follows:

$$f(t) = \begin{cases} g(t) & , 0 \leq t < a \\ h(t) & , a \leq t < b \\ k(t) & , t \geq b \end{cases}$$

then,  $f(t) = g(t) + (h(t) - g(t))u_a(t) + (k(t) - h(t))u_b(t)$

Example: Write the following in a compact form

$$\textcircled{1} f(t) = \begin{cases} \sin t & , 0 \leq t < \frac{\pi}{4} \\ \sin t + \cos(t - \frac{\pi}{4}) & , t \geq \frac{\pi}{4} \end{cases}$$

sol:  $f(t) = \sin t + \cos(t - \frac{\pi}{4})u_{\frac{\pi}{4}}(t)$

$$\textcircled{2} f(t) = \begin{cases} 20t & , 0 \leq t < 5 \\ 2 & , 5 \leq t < 9 \\ -1 & , t \geq 9 \end{cases}$$

sol:  $f(t) = 20t + (2 - 20t)u_5(t) - 3u_9(t)$ .

Example: Find  $\mathcal{L}(u_c(t))$

Sol:  $\mathcal{L}(u_c(t)) = \int_0^{\infty} u_c(t) e^{-st} dt$

$$= \int_c^{\infty} 1 \cdot e^{-st} dt$$

$$= \lim_{A \rightarrow \infty} \int_c^A e^{-st} dt = \lim_{A \rightarrow \infty} \left. \frac{e^{-st}}{-s} \right|_c^A$$

$$= \lim_{A \rightarrow \infty} \left( \frac{e^{-sA}}{-s} + \frac{e^{-sc}}{s} \right)$$

$$\mathcal{L}(u_c(t)) = \frac{e^{-sc}}{s}$$

Example: ①  $\mathcal{L}(u_2(t)) = \frac{e^{-2s}}{s}$

②  $\mathcal{L}^{-1}\left(\frac{e^{-6s}}{s}\right) = u_6(t)$

## Theorem (6.3.1). Second Translation Theorem.

If  $\mathcal{L}(f(t)) = F(s)$  and  $a > 0$ , then

$$\mathcal{L}(f(t-a)u_a(t)) = e^{-as}F(s), \quad s > a$$

OR:  $\mathcal{L}(f(t)u_a(t)) = e^{-as} \mathcal{L}(f(t+a))$

and  $\mathcal{L}^{-1}(e^{-as}F(s)) = f(t-a)u_a(t)$

$$= \mathcal{L}^{-1}(F(s)) \cdot u_a(t)$$

$t \rightarrow t-a$

Proof:  $\mathcal{L}(f(t-a)u_a(t)) = \int_0^{\infty} e^{-st} u_a(t) f(t-a) dt$

$$= \int_a^{\infty} e^{-st} f(t-a) dt = \lim_{A \rightarrow \infty} \int_a^A e^{-st} f(t-a) dt$$

$$\Rightarrow \lim_{A \rightarrow \infty} \int_0^A e^{-s(w+a)} f(w) dw$$

$$= \lim_{A \rightarrow \infty} \int_0^A e^{-sa} \cdot e^{-sw} f(w) dw = e^{-as} F(s)$$

Let  $w = t-a$   
 $dw = dt$   
 $t=a, w=0$   
 $t=\infty, w=\infty$

Example: Find the following:

$$\begin{aligned} \textcircled{1} \quad \mathcal{L}(t^2 u_3(t)) &= e^{-3s} \mathcal{L}(t^2)_{t \rightarrow t+3} \\ &= e^{-3s} \mathcal{L}((t+3)^2) \\ &= e^{-3s} \mathcal{L}(t^2 + 6t + 9) \\ &= e^{-3s} \left( \frac{2!}{s^3} + \frac{6}{s^2} + \frac{9}{s} \right). \end{aligned}$$

$$\textcircled{2} \quad \mathcal{L}(f(t)), \text{ where } f(t) = \begin{cases} 0 & , t < \pi \\ t - \pi & , \pi \leq t < 2\pi \\ 0 & , t \geq 2\pi. \end{cases}$$

First we write  $f(t)$  in the compact form.

$$\begin{aligned} f(t) &= 0 + (t - \pi - 0) u_{\pi}(t) + (0 - (t - \pi)) u_{2\pi}(t) \\ &= (t - \pi) u_{\pi}(t) + (\pi - t) u_{2\pi}(t) \end{aligned}$$

$$\begin{aligned} \mathcal{L}(f(t)) &= \mathcal{L}((t - \pi) u_{\pi}(t)) + \mathcal{L}((\pi - t) u_{2\pi}(t)) \\ &= e^{-\pi s} \mathcal{L}(t) + \mathcal{L}(\pi u_{2\pi}(t)) - \mathcal{L}(t u_{2\pi}(t)) \end{aligned}$$



$$= e^{-\pi s'} \cdot \frac{1}{s'^2} + \pi \cdot \frac{e^{-2\pi s'}}{s'} - e^{-2\pi s'} \mathcal{L}(t + 2\pi)$$

$$= e^{-\pi s'} \cdot \frac{1}{s'^2} + \pi \frac{e^{-2\pi s'}}{s'} - e^{-2\pi s'} \left( \frac{1}{s'^2} + \frac{2\pi}{s'} \right)$$

$$= e^{-\pi s'} \cdot \frac{1}{s'^2} + e^{-2\pi s'} \left( -\frac{1}{s'^2} - \frac{\pi}{s'} \right).$$

OR: Eq. (\*) becomes:

$$\mathcal{L}(f(t)) = e^{-\pi s'} \mathcal{L}((t+\pi) - \pi) + e^{-2\pi s'} \mathcal{L}(\pi - (t+2\pi))$$

$$= e^{-\pi s'} \mathcal{L}(t) + e^{-2\pi s'} \mathcal{L}(-t - \pi)$$

$$= e^{-\pi s'} \cdot \frac{1}{s'^2} + e^{-2\pi s'} \left( \frac{-1}{s'^2} - \frac{\pi}{s'} \right).$$

$$\textcircled{3} \quad \mathcal{L}(u_3(t) \sin 4t) = e^{-3s'} \mathcal{L}(\sin 4(t+3))$$

$$= e^{-3s'} \mathcal{L}(\sin(4t+12)) = e^{-3s'} \mathcal{L}(\sin 4t \cos 12 + \sin 12 \cos 4t)$$

$$= e^{-3s'} \cos 12 \left( \frac{4}{s'^2 + 16} \right) + e^{-3s'} \sin 12 \left( \frac{s'}{s'^2 + 16} \right)$$

Example: Find the following:

$$\begin{aligned} \text{a) } \mathcal{L}^{-1}\left(\frac{e^{-2s}}{s^2+4}\right) &= u_2(t) \mathcal{L}^{-1}\left(\frac{1}{s^2+4}\right)_{t \rightarrow t-2} \\ &= \frac{1}{2} u_2(t) \sin 2(t-2). \end{aligned}$$

$$\begin{aligned} \text{b) } \mathcal{L}^{-1}\left(\frac{e^{-4s}}{s}\right) &= \mathcal{L}^{-1}\left(e^{-4s} \cdot \frac{1}{s}\right) = u_4(t) \cdot \mathcal{L}^{-1}\left(\frac{1}{s}\right)_{t \rightarrow t-4} \\ &= u_4(t) \cdot 1 = u_4(t) \end{aligned}$$

In general:

$$\mathcal{L}^{-1}\left(\frac{e^{-cs}}{s}\right) = u_c(t).$$

$$\begin{aligned} \text{c) } \mathcal{L}^{-1}\left(\frac{1-e^{-2s}}{s^2}\right) &= \mathcal{L}^{-1}\left(\frac{1}{s^2}\right) - \mathcal{L}^{-1}\left(\frac{e^{-2s}}{s^2}\right) \\ &= t - u_2(t) \mathcal{L}^{-1}\left(\frac{1}{s^2}\right)_{t \rightarrow t-2} \\ &= t - u_2(t)(t-2) \end{aligned}$$

$$d) \mathcal{L}^{-1} \left( \frac{2(s-1)e^{-2s}}{s^2-2s+2} \right) = 2 \mathcal{L}^{-1} \left( \frac{(s-1)e^{-2s}}{(s-1)^2+1} \right)$$

$$= 2 u_2(t) \mathcal{L}^{-1} \left( \frac{s-1}{(s-1)^2+1} \right)_{t \rightarrow t-2}$$

$$= 2 u_2(t) (e^t \cos t)_{t \rightarrow t-2}$$

$$= 2 u_2(t) e^{(t-2)} \cos(t-2).$$

$$e) \mathcal{L}^{-1} \left( \frac{e^{-s}}{s^2+9} \right) = u_1(t) \mathcal{L}^{-1} \left( \frac{1}{s^2+9} \right)_{t \rightarrow t-1}$$

$$= u_1(t) \cdot \frac{1}{3} (\sin(3t))_{t \rightarrow t-1} = \frac{1}{3} u_1(t) \sin 3(t-1).$$

$$f) \mathcal{L}^{-1} \left( \frac{(1-3s)e^{-2s}}{s^2+6s+25} \right) = \mathcal{L}^{-1} \left( \frac{(1-3s)e^{-2s}}{(s+3)^2+16} \right)$$

$$= \mathcal{L}^{-1} \left( \left( \frac{-3(s+3)+10}{(s+3)^2+16} \right) e^{-2s} \right) = \mathcal{L}^{-1} \left( \frac{-3(s+3)}{(s+3)^2+16} + \frac{10}{(s+3)^2+16} \right) u_2(t)_{t \rightarrow t-2}$$

$$= u_2(t) \left[ -3 e^{-3t} \cos 4t + \frac{10}{4} e^{-3t} \sin 4t \right]_{t \rightarrow t-2}$$

$$= u_2(t) \left[ -3 e^{-3(t-2)} \cos 4(t-2) + \frac{10}{4} e^{-3(t-2)} \sin 4(t-2) \right]$$

# 6.4 Differential Equations with discontinuous Forcing

## Functions

Example: Solve the following IVP Using Laplace Transform.

$$y'' + 4y = \sin t \, u_{2\pi}(t), \quad y(0) = 1, \quad y'(0) = 0.$$

Sol:  $\mathcal{L}(y'' + 4y) = \mathcal{L}(\sin t \, u_{2\pi}(t))$

$$s^2 Y(s) - s y(0) - \overset{0}{y'(0)} + 4Y(s) = e^{-2\pi s} \mathcal{L}(\sin t)_{t \rightarrow t+2\pi}$$

$$(s^2 + 4)Y(s) = s + e^{-2\pi s} \cdot \mathcal{L}(\sin(t+2\pi))$$

$$= s + e^{-2\pi s} \mathcal{L}(\sin t) \quad \text{periodic of period } 2\pi.$$

$$\Rightarrow Y(s) = \frac{s}{(s^2 + 4)} + e^{-2\pi s} \cdot \frac{1}{(s^2 + 1)(s^2 + 4)}$$

$$\therefore y(t) = \mathcal{L}^{-1}\left(\frac{s}{s^2 + 4}\right) + \mathcal{L}^{-1}\left(\frac{1}{(s^2 + 1)(s^2 + 4)}\right) \cdot u_{2\pi}(t)$$

$t \rightarrow t - 2\pi$

$$y(t) = \cos 2t + u_{2\pi}(t) \mathcal{L}^{-1}\left(\frac{As + B}{s^2 + 1} + \frac{Cs + D}{s^2 + 4}\right)$$

$t \rightarrow t - 2\pi$

Now,  $\frac{1}{(s^2+1)(s^2+4)} = \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+4}$

$\Rightarrow 1 = (As+B)(s^2+4) + (Cs+D)(s^2+1)$

$1 = As^3 + 4As + Bs^2 + 4B + Cs^3 + Cs + Ds^2 + D$

$\Rightarrow \left. \begin{aligned} 0 &= A + C && (\text{coef. } s^3) \\ 0 &= B + D && (\text{coef. } s^2) \\ 0 &= 4A + C && (\text{coef. } s^1) \\ 1 &= 4B + D && (\text{coef. of } s^0) \end{aligned} \right\} \Rightarrow \begin{aligned} A &= 0 \\ B &= \frac{1}{3} \\ C &= 0 \\ D &= -\frac{1}{3} \end{aligned}$

$\therefore y(t) = \cos 2t + u_{2\pi}(t) \int^{-1} \left( \frac{\frac{1}{3}}{s^2+1} + \frac{-\frac{1}{3}}{s^2+4} \right)$   
 $t \rightarrow t - 2\pi$

$= \cos 2t + \frac{1}{3} u_{2\pi}(t) \sin(t-2\pi) - \frac{1}{6} u_{2\pi}(t) \sin(2(t-2\pi))$

$= \cos 2t + u_{2\pi}(t) \left[ \frac{1}{3} \sin t - \frac{1}{6} \sin 2t \right]$

$= \begin{cases} \cos 2t & , 0 \leq t < 2\pi \\ \cos 2t + \frac{1}{3} \sin t - \frac{1}{6} \sin 2t & , t \geq 2\pi. \end{cases}$

Example: solve  $y'' + y = f(t)$ ,  $y(0) = 0$ ,  $y'(0) = 2$ ,

$$\text{where } f(t) = \begin{cases} \frac{1}{2}t & , 0 \leq t < 6 \\ 3 & , t \geq 6 \end{cases}$$

Sol:  $f(t) = \frac{1}{2}t + (3 - \frac{1}{2}t)u_6(t)$

$$\mathcal{L}(f(t)) = \frac{1}{2s^2} + e^{-6s} \mathcal{L}\left(3 - \frac{1}{2}t\right)_{t \rightarrow t+6}$$

$$= \frac{1}{2s^2} + e^{-6s} \mathcal{L}\left(3 - \frac{1}{2}(t+6)\right)$$

$$= \frac{1}{2s^2} + e^{-6s} \mathcal{L}\left(-\frac{1}{2}t\right)$$

$$= \frac{1}{2s^2} + e^{-6s} \left(\frac{-1}{2s^2}\right) = \frac{1 - e^{-6s}}{2s^2}$$

Now:  $\mathcal{L}(y'' + y) = \mathcal{L}(f(t))$

$$\Rightarrow s^2 Y(s) - \cancel{s}y(0) - y'(0) + Y(s) = \frac{1 - e^{-6s}}{2s^2}$$

$$\Rightarrow (s^2 + 1)Y(s) = \frac{1 - e^{-6s}}{2s^2} + 2$$

$$\therefore Y(s) = \frac{2}{s^2+1} + \frac{1}{2s^2(s^2+1)} - \frac{e^{-6s}}{2s^2(s^2+1)}$$

$$\Rightarrow y(t) = 2 \mathcal{L}^{-1}\left(\frac{1}{s^2+1}\right) + \mathcal{L}^{-1}\left(\frac{1}{2s^2(s^2+1)}\right) - \mathcal{U}_6(t) \mathcal{L}^{-1}\left(\frac{1}{2s^2(s^2+1)}\right)$$

$$\frac{1}{2s^2(s^2+1)} = \frac{1}{2} \left[ \frac{A}{s} + \frac{B}{s^2} + \frac{Cs+D}{s^2+1} \right]$$

$$\Rightarrow 1 = As(s^2+1) + B(s^2+1) + (Cs+D)s^2$$

$$1 = As^3 + As + Bs^2 + B + Cs^3 + Ds^2$$

$$\Rightarrow \left. \begin{array}{l} 0 = A + C \quad (s^3) \\ 0 = B + D \quad (s^2) \\ 0 = A \quad (s^1) \\ 1 = B \quad (s^0) \end{array} \right\} \Rightarrow \begin{array}{l} A = 0 \\ B = 1 \\ C = 0 \\ D = -1 \end{array}$$

$$\therefore \frac{1}{2s^2(s^2+1)} = \frac{1}{2} \left[ \frac{0}{s} + \frac{1}{s^2} + \frac{-1}{s^2+1} \right]$$

Back to (\*\*):

$$\mathcal{L}^{-1}\left(\frac{1}{2s^2(s^2+1)}\right) = \frac{1}{2} \left[ \mathcal{L}^{-1}\left(\frac{1}{s^2}\right) - \mathcal{L}^{-1}\left(\frac{1}{s^2+1}\right) \right]$$

$$= \frac{1}{2} (t - \sin t)$$

$$\begin{aligned} \therefore y(t) &= 2 \sin t + \frac{1}{2} (t - \sin t) - \frac{1}{2} \mathcal{U}_6(t) (t - \sin t) \\ &= 2 \sin t + \frac{1}{2} t - \frac{1}{2} \sin t - \frac{1}{2} \mathcal{U}_6(t) (t - 6 + \sin(t-6)) \\ &= \frac{3}{2} \sin t + \frac{1}{2} t - \frac{1}{2} \mathcal{U}_6(t) (t - 6 + \sin(t-6)) \end{aligned}$$

Home work: Solve Using Laplace Transform the IVP:

①  $2y'' + y' + 2y = g(t)$ ,  $y(0) = y'(0) = 0$ , where

$$g(t) = \begin{cases} 1, & 5 \leq t < 20 \\ 0, & 0 \leq t < 5 \text{ \& } t \geq 20 \end{cases} = \mathcal{U}_5(t) - \mathcal{U}_{20}(t)$$

②  $y'' + 4y' + 3y = h(t)$ ,  $y(0) = y'(0) = 0$ , where

$$h(t) = \begin{cases} 0, & 0 \leq t < 2 \\ -1, & t \geq 2. \end{cases}$$

Good Luck

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