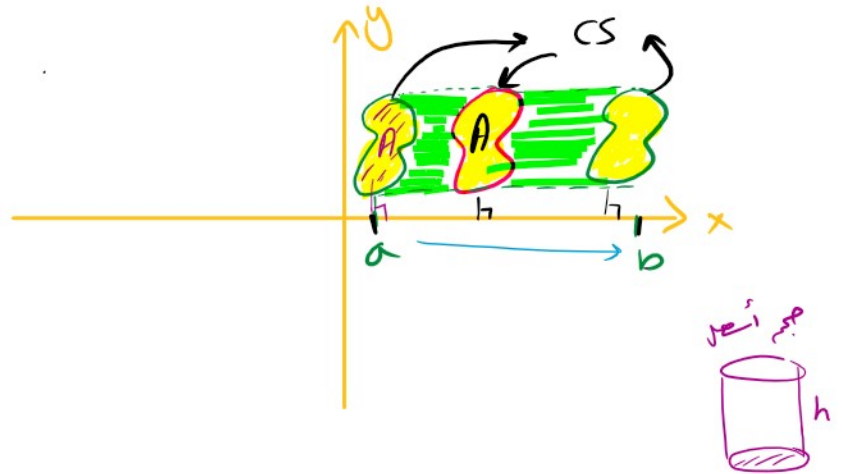


Volumes Using Cross-Sections
 الحجم باستخدام المقاطع العرضية

If CS \perp x-axis
 Then volume is

$V = \int_a^b A(x) dx$
 (Note: 'a' and 'b' are circled in yellow, 'A(x)' is circled in blue, and 'dx' is circled in yellow. A blue arrow points from 'A(x)' to the word 'مساحة CS' (Area of CS).)



If CS \perp y-axis \Rightarrow

$V = \int_c^d A(y) dy$
 (Note: 'c' and 'd' are circled in green, 'A(y)' is circled in blue, and 'dy' is circled in green. A blue arrow points from 'A(y)' to the word 'مساحة CS' (Area of CS).)

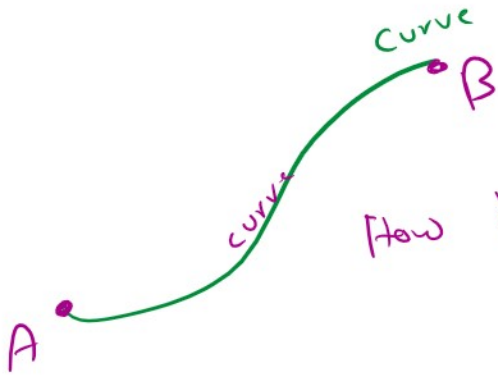


$A(y)$: Area of CS \perp y
 $A(x)$: Area of CS \perp x

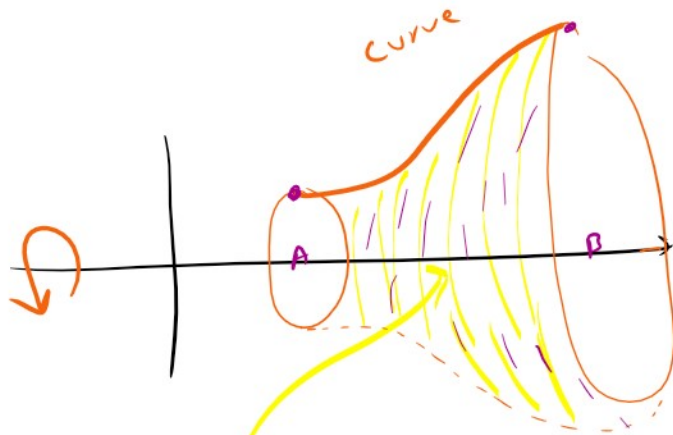
6.1 ✓ Disk Method DM } To find V
 ✓ Washer Method WM } when CS \perp x-axis
 CS \perp y-axis

6.2 Shell Method SM } → To find V
 when CS ⊥ x-axis
 CS ⊥ y-axis

6.3 How to find length curve

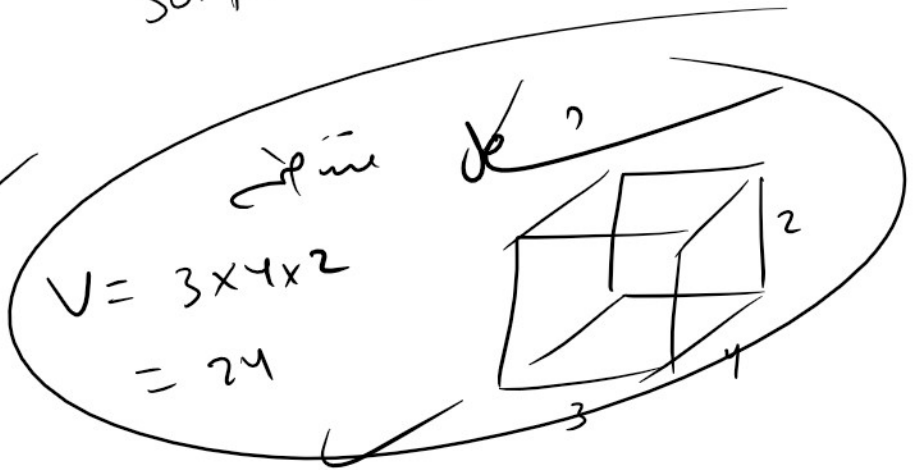


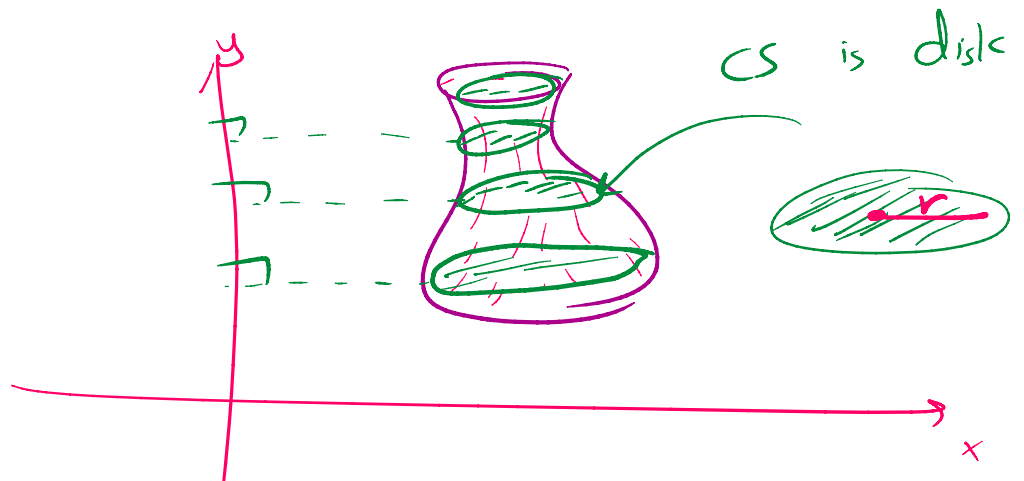
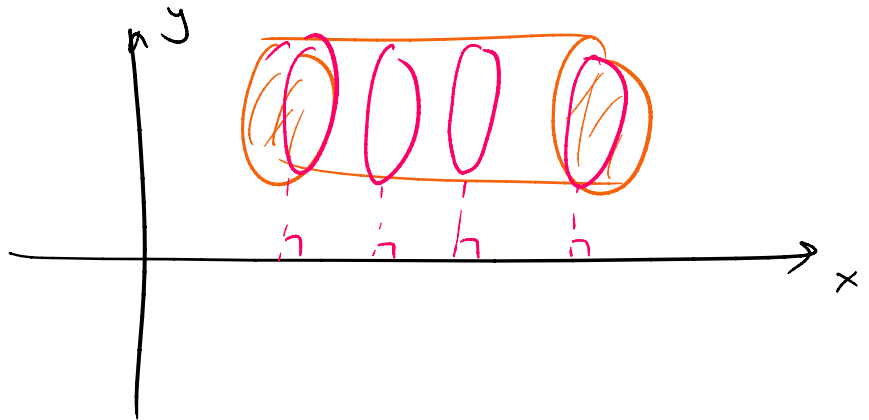
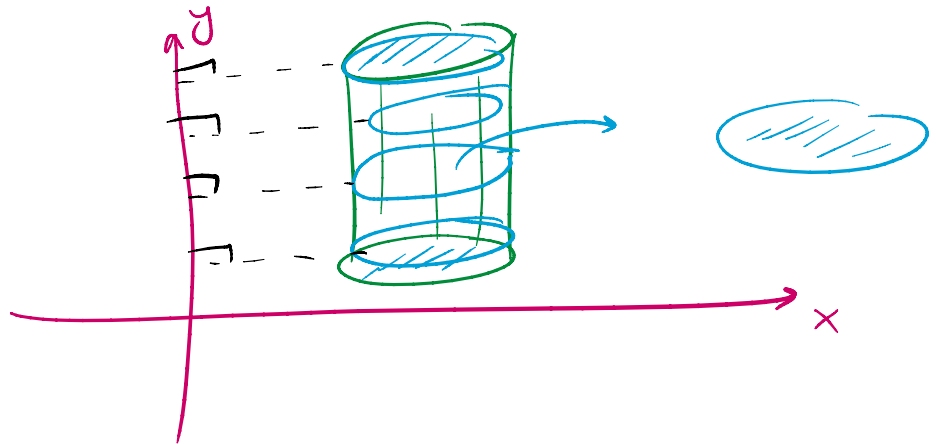
6.4



✓ 20/6 00 SA
 Surface Area.

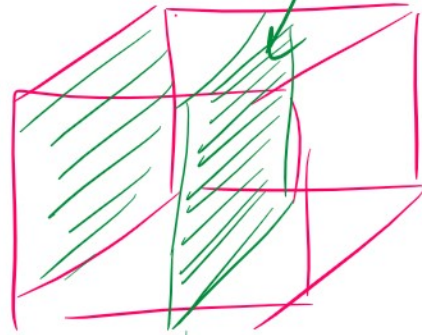
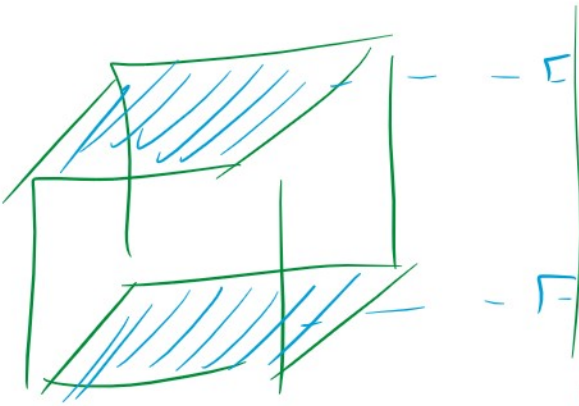
3.4 e i se







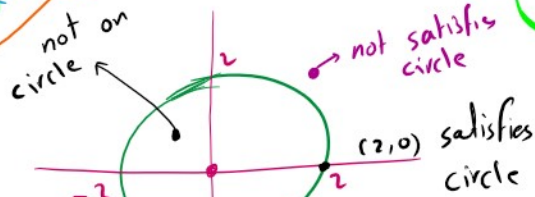
CS is square



6.1

Disk Method

\Rightarrow CS is Disk

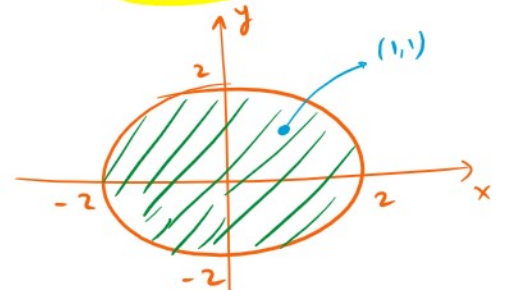


Circle

$$x^2 + y^2 = 4$$

$$2^2 + 0^2 = 4$$

$$4 = 4 \checkmark$$



disk

$$x^2 + y^2 \leq 4$$

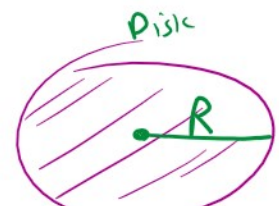
$$1^2 + 1^2 \leq 4$$

$$2 \leq 4 \checkmark$$

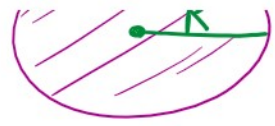
CS \perp x-axis $\Rightarrow V = \int_a^b A(x) dx$

CS \perp y-axis $\Rightarrow V = \int_c^d A(y) dy$

IF CS is disk



CS \perp y-axis $\Rightarrow V = \int_c^d A(y) dy$



$A = R^2 \pi$

If CS is disk and \perp x-axis $\Rightarrow V = \int_a^b A(x) dx = \int_a^b \pi R^2(x) dx$

If CS is disk and \perp y-axis $\Rightarrow V = \int_c^d A(y) dy = \int_c^d \pi R^2(y) dy$

Exp Find the volume of the solid generated by revolving the region bounded by

① the curve $y = \sqrt{x}$, $0 \leq x \leq 4$, x-axis
about x-axis

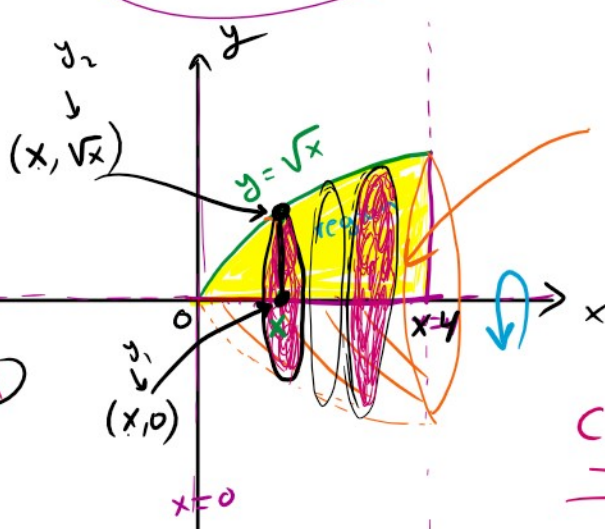
$\rightarrow x$

\Downarrow b

$$V = \int_a^b A(x) dx = \int_a^b \pi R^2(x) dx$$

$$= \int_0^4 \pi (\sqrt{x})^2 dx$$

$$= \int_0^4 \pi x dx$$



V ??

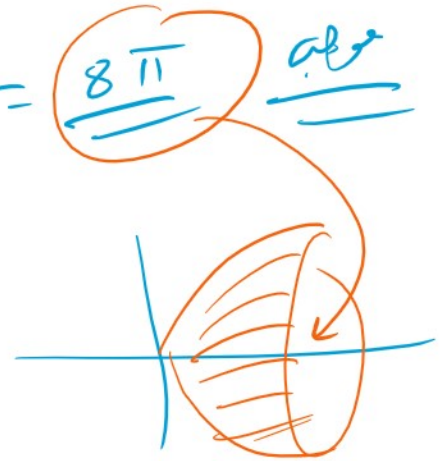
CS is disk
CS \perp x-axis

$$R = \Delta y = y_2 - y_1$$

$$= \sqrt{x} - 0$$

$$= \sqrt{x}$$

$$= \pi \frac{x^2}{2} \Big|_0^4 = \frac{\pi}{2} (4^2 - 0^2) = \frac{\pi}{2} (16) = \frac{8\pi}{1}$$

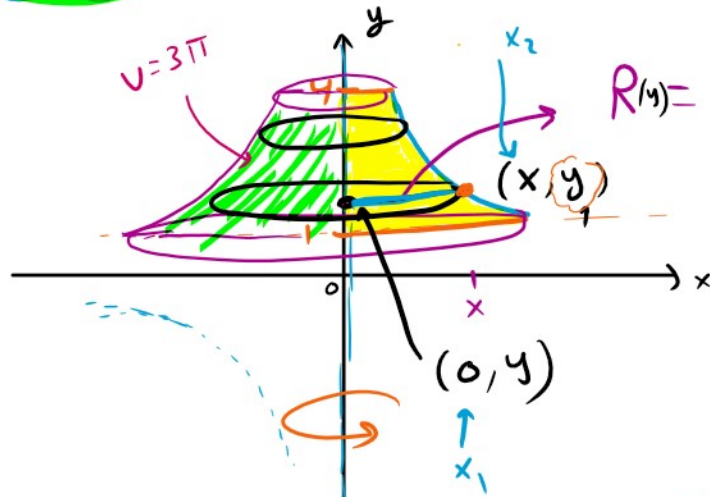


② y -axis, $x = \frac{2}{y}$, $1 \leq y \leq 4$ about y -axis

$$y = \frac{2}{x}$$

CL is disk ✓
CL \perp y -axis

$$V = \int_a^b A(y) dy$$



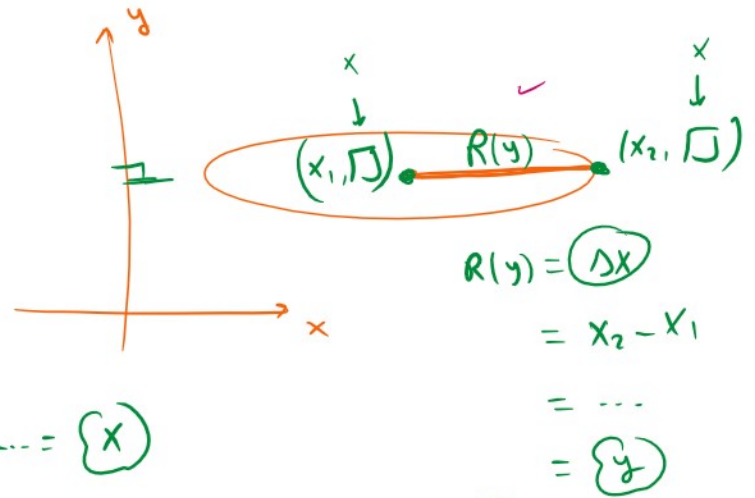
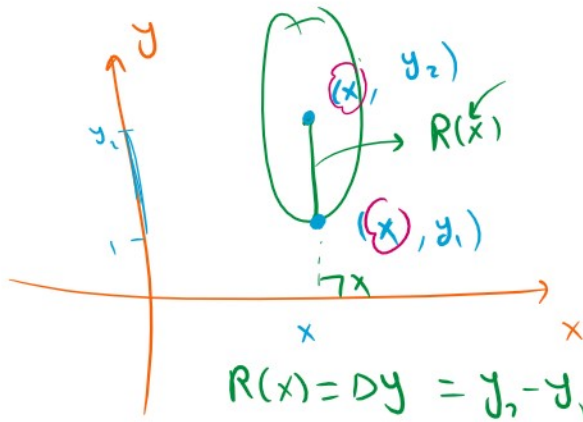
$$\begin{aligned} R(y) = \Delta x &= x_2 - x_1 \\ &= x - 0 \\ &= x \\ &= \frac{2}{y} \end{aligned}$$

$$= \int_1^4 \pi R^2(y) dy = \int_1^4 \pi \left(\frac{2}{y}\right)^2 dy = \pi \int_1^4 \frac{4}{y^2} dy$$

$$= 4\pi \int_1^4 y^{-2} dy = 4\pi \left[-\frac{1}{y} \right]_1^4 = -4\pi \left[\frac{1}{4} - 1 \right]$$

$$= -4\pi \left[\frac{1}{4} - 1 \right] = -4\pi \left(\frac{-3}{4} \right) = 3\pi$$

$$= -4\pi \left[\frac{1}{4} - 1 \right] = -4\pi \left(\frac{-3}{4} \right) = 3\pi$$



$$R(x) = Dy = y_2 - y_1 = \dots = \{x\}$$

$$R(y) = \{Dx\} = x_2 - x_1 = \dots = \{y\}$$

24

3) $y = \sec x$, $y = 0$, $x = -\frac{\pi}{4}$, $x = \frac{\pi}{4}$

about x -axis

$$\sec 0 = \frac{1}{\cos 0} = \frac{1}{1} = 1$$

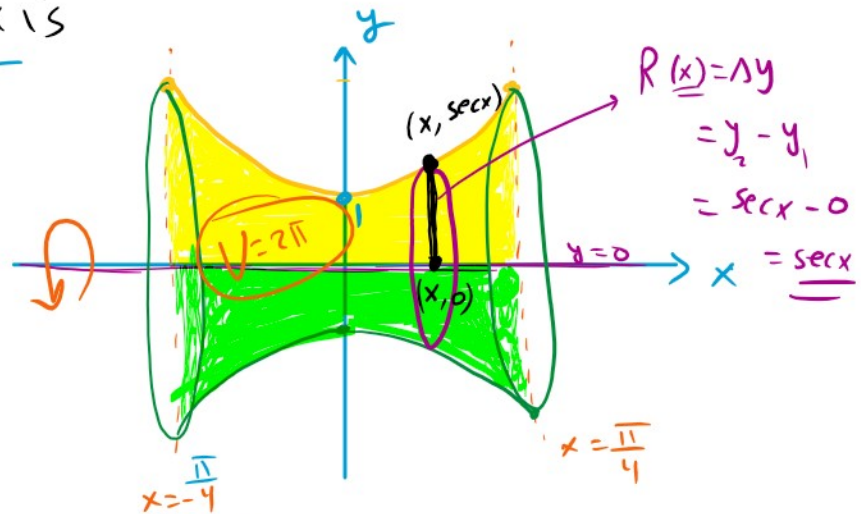
CS is disk

CS \perp x -axis

$$V = \int_a^b A(x) dx = \int_a^b \pi R^2(x) dx = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \pi (\sec x)^2 dx$$

$$= \pi \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sec^2 x dx = \pi \tan x \Big|_{-\frac{\pi}{4}}^{\frac{\pi}{4}} = \pi \left[\tan \frac{\pi}{4} - \tan \left(-\frac{\pi}{4} \right) \right]$$

$\tan \frac{\pi}{4} = 1$
 $\tan \left(-\frac{\pi}{4} \right) = -1$



$$= \pi [1 - -1]$$

$$= \boxed{2\pi}$$

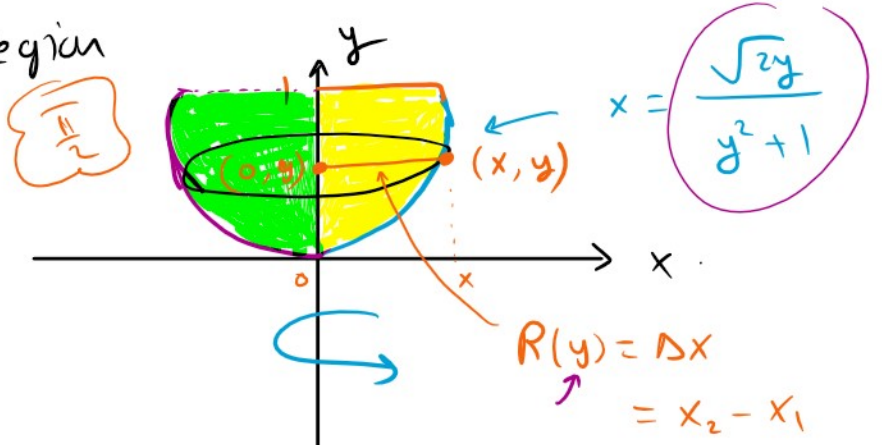
$$\tan^{-x} = -\tan x$$

$$\tan^{-\frac{\pi}{4}} = -\tan \frac{\pi}{4}$$

$$= -1$$

Q32
4

Given the region
Find V



CS is disk
CS \perp y-axis

$$V = \int_c^d \pi R^2(y) dy$$

$$= \int_0^1 \pi \left(\frac{\sqrt{2y}}{y^2+1} \right)^2 dy$$

$$R(y) = \Delta x$$

$$= x_2 - x_1$$

$$= x - 0$$

$$= x$$

$$= \frac{\sqrt{2y}}{y^2+1}$$

$$= \pi \int_0^1 \frac{2y}{(y^2+1)^2} dy$$

$$u = y^2 + 1$$

$$du = 2y dy$$

$$y=0 \Rightarrow u = 0^2 + 1 = 1$$

$$y=1 \Rightarrow u = 1^2 + 1 = 2$$

$$= \pi \int_1^2 \frac{du}{u^2}$$

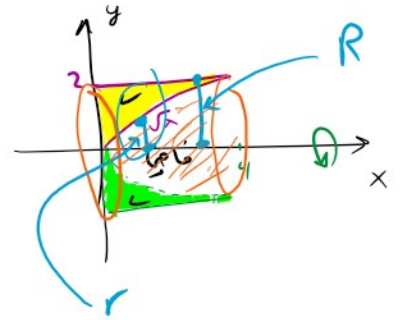
$$= \pi \int_1^2 u^{-2} du = \pi \left[-\frac{1}{u} \right]_1^2 = -\pi \left[\frac{1}{u} \right]_1^2$$

$$= \pi [1 - \frac{1}{2}] = -\pi \left[-\frac{1}{2} \right] = \boxed{\frac{\pi}{2}}$$

$$= -\pi \left[\frac{1}{2} - \frac{1}{1} \right] = -\pi \left[-\frac{1}{2} \right] = \left(\frac{\pi}{2} \right)$$

Washer Method

If CS \perp x-axis with ^{مخرج} outer radius $R(x)$ and inner radius $r(x)$



Then

$$V = \int_a^b A(x) dx = \int_a^b (\pi R^2(x) - \pi r^2(x)) dx$$

هذه الجزء الخارج ←

$$V = \pi \int_a^b (R^2(x) - r^2(x)) dx$$

If CS \perp y-axis with $R(y)$ is outer radius and $r(y)$ is inner radius

Then

$$V = \int_c^d A(y) dy = \int_c^d (\pi R^2(y) - \pi r^2(y)) dy$$

$$= \pi \int_c^d (R^2(y) - r^2(y)) dy$$

Exp Find the volume of the solid generated

$$\pi \int_c^d (R(y) - r(y))^2$$

by rotating the region bounded by

① $y = 2\sqrt{x}$, $y = 2$, $x = 0$
about x -axis

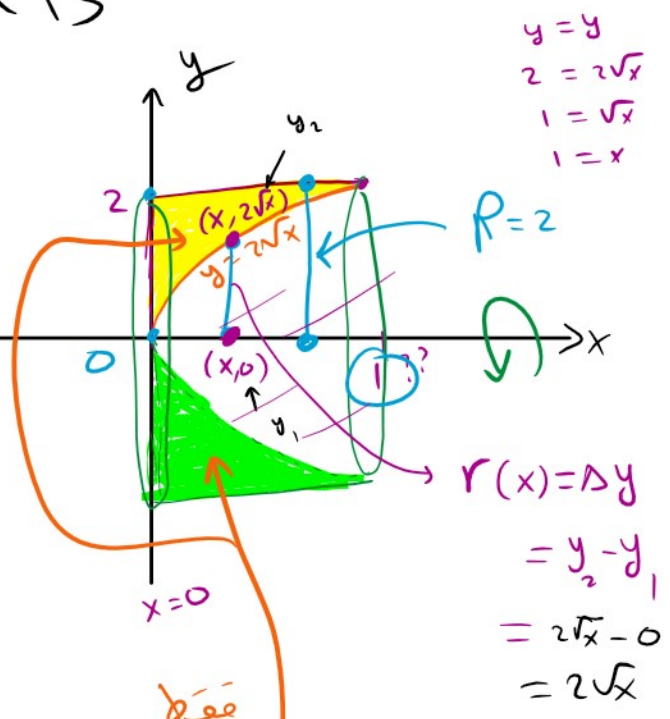


$$V = \pi \int_a^b (R^2(x) - r^2(x)) dx$$

$$= \pi \int_0^1 \left[\frac{2^2}{2} - (2\sqrt{x})^2 \right] dx$$

$$= \pi \int_0^1 (4 - 4x) dx$$

$$= \pi (4x - 2x^2) \Big|_0^1 = \pi (4 - 2) = 2\pi$$



Q44
②

2nd quadrant, left by the circle $x^2 + y^2 = 3$,
right by line $x = \sqrt{3}$

(2) 7 right by line $x = \sqrt{3}$
 above by line $y = \sqrt{3}$

about y-axis

washer



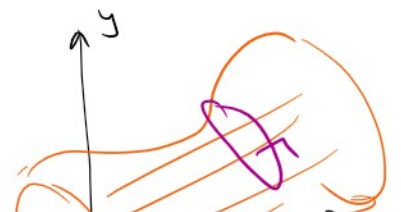
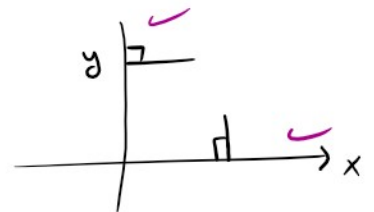
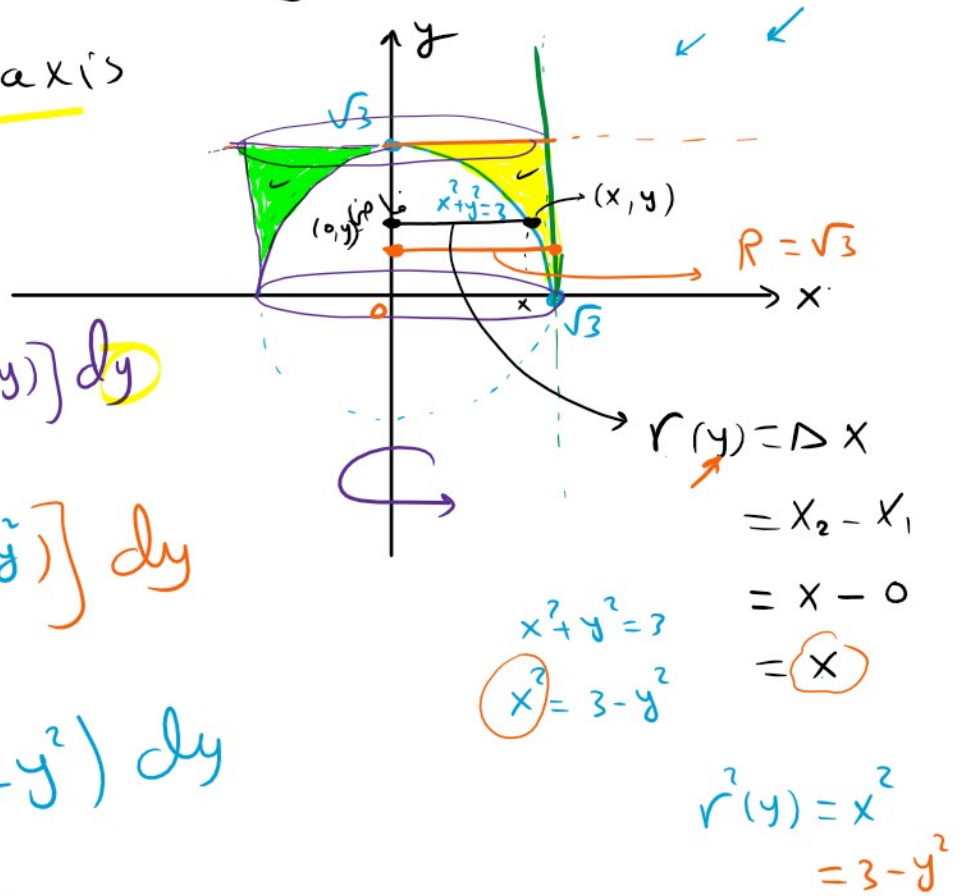
$$V = \pi \int [R^2(y) - r^2(y)] dy$$

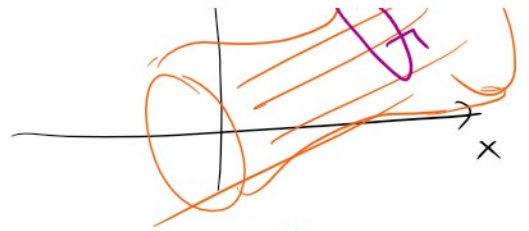
$$= \pi \int_0^{\sqrt{3}} [(\sqrt{3})^2 - (3-y^2)] dy$$

$$= \pi \int_0^{\sqrt{3}} (3 - 3 + y^2) dy$$

$$= \pi \left(\frac{y^3}{3} \right) \Big|_0^{\sqrt{3}} = \frac{\pi}{3} (\sqrt{3} \sqrt{3} \sqrt{3} - 0)$$

$$= \frac{\pi}{3} \cdot 3 \sqrt{3} = \sqrt{3} \pi$$





Remark sometimes the CS is not disk
 ↓
 square
 triangle -

$$CS \perp x\text{-axis} \Rightarrow V = \int_a^b A(x) dx$$

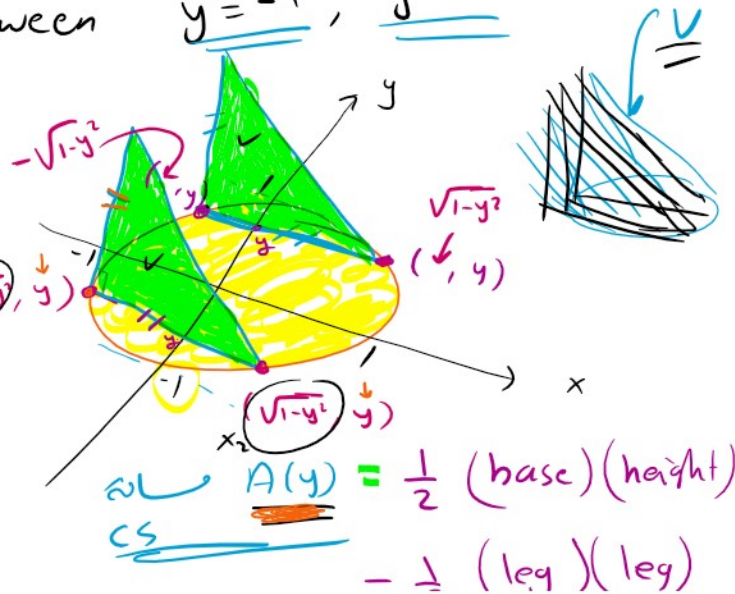
$$CS \perp y\text{-axis} \Rightarrow V = \int_c^d A(y) dy$$

Exp Find the volume of the solid whose base is the disk $x^2 + y^2 \leq 1$, CS's are isosceles right triangles with one leg on the disk.

CS \perp y-axis between $y = -1$, $y = 1$

$$CS \perp y\text{-axis} \Rightarrow V = \int_c^d A(y) dy$$

$$= \int_{-1}^1 2(1-y^2) dy$$



$$x^2 + y^2 = 1$$

$$x^2 + y^2 = 1$$

$$x^2 = 1 - y^2$$

$$x = \pm \sqrt{1 - y^2}$$

$$\text{leg} = \Delta x$$

$$= x_2 - x_1$$

$$= \sqrt{1 - y^2} - (-\sqrt{1 - y^2})$$

$$= 2\sqrt{1 - y^2}$$

CS

$$= \frac{1}{2} (\text{leg})(\text{leg})$$

$$= \frac{1}{2} (\text{leg})^2$$

$$= \frac{1}{2} (2\sqrt{1 - y^2})^2$$

$$= \frac{1}{2} 4^2 (1 - y^2)$$

$$= \underline{\underline{2(1 - y^2)}}$$

$$V = 2 \int_{-1}^1 (1 - y^2) dy$$

$$= 2 \left(y - \frac{y^3}{3} \right) \Big|_{-1}^1$$

$$= 2 \left[\left(1 - \frac{1}{3} \right) - \left(-1 - \frac{-1}{3} \right) \right] = 2 \left[1 - \frac{1}{3} + 1 - \frac{1}{3} \right]$$

$$= 2 \left[2 - \frac{2}{3} \right]$$

$$= 4 - \frac{4}{3}$$

$$= \frac{8}{3}$$

Q2

$$y = \int_{\tan x}^0 \frac{dt}{1 + t^2} \quad \text{Find } y'$$

$u(x)$ is diff

$$y = \int f(t) \underline{dt} \quad \Rightarrow \quad y' = f(u(x)) \cdot \dot{u}(x) - f(a) \rightarrow 0$$

$$y = \int_a f(t) dt \Rightarrow y' = f(u(x)) u'(x) - f(a) \cdot 0$$

$$y = \int_{\tan x}^0 \frac{dt}{1+t^2} \Rightarrow y' = \frac{1}{1+0^2} (0) - \frac{1}{1+\tan^2 x} \sec^2 x$$

$$= 0 - \frac{\sec^2 x}{1+\tan^2 x}$$

$$= - \frac{\sec^2 x}{\sec^2 x}$$

$$= -1$$

$$y = \int_{\tan x}^0 \frac{dt}{1+t^2} = - \int_0^{\tan x} \frac{dt}{1+t^2}$$

$$y' = - \left[\frac{1}{1+\tan^2 x} \right] \sec^2 x = -1$$