Engineering statistics "ENEE2307"

Chapter 2

By: Jibreel Bornat

Notes, questions and forms

Chapter 2
Single random variable and probability distribution

Probability mass function (PMF)

Consider on experiment of flipping three coins. Assume that p(H)= 4/10.
The random variable X represents the number of heads observed find:

a. Probability of observing one head

$$P(HTT) + P(THT) + P(TTH)$$

$$= 3 * P(T)^2 * P(H) \implies 3 * \frac{36}{100} * \frac{4}{10} = \frac{432}{1000}$$

B. $P(X = 0)$

$$P(TTT) = P(T)^3 = \frac{216}{1000}$$

C. $P(X = 2)$

$$P(HHT) + P(HTH) + P(THH)$$

$$= 3 * P(H)^2 * P(T) \implies 3 * \frac{16}{100} * \frac{6}{10} = \frac{288}{1000}$$

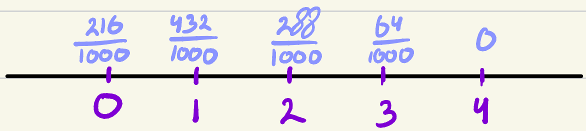
D. $P(X = 3)$

$$P(HHH) = P(H)^3 = \frac{64}{1000}$$

E. $P(X = 4)$

$$P(X = 4) = 0$$

F. Determine the probability mass function (PMF)



$$P(X = x) = \begin{cases} \frac{216}{1000} & x = 0 \\ \frac{432}{1000} & x = 1 \\ \frac{288}{1000} & x = 2 \\ \frac{64}{1000} & x = 3 \\ 0 & \text{otherwise} \end{cases}$$

H. $F_X(0.5)$

$$F_X(0.5) = P(X \leq 0.5)$$

$$= P(X = 0) = 216/1000$$

I. $F_X(2.7)$

$$F_X(2.7) = P(X \leq 2.7)$$
$$= P(X=2) + P(X=1) + P(X=0) \Rightarrow 936/1000$$

J. $F_X(2)$

$$F_X(2) = P(X \leq 2)$$
$$= P(X=2) + P(X=1) + P(X=0) \Rightarrow 936/1000$$

K. $F_X(2^-)$

$$F_X(2^-) = P(X < 2)$$
$$= P(X=1) + P(X=0) \Rightarrow 648/1000$$

L. $F_X(2^+)$

$$F_X(2^+) = F_X(2) = P(X \leq 2)$$
$$= 936/1000$$

Rules:-

- $F_X(\infty) = 1$, $F_X(-\infty) = 0$
- $\sum_{-\infty}^{\infty} P(X=x) = 1$ مجموع جميع الاحتمالات = 1
- if $x_2 > x_1 \Rightarrow F_X(x_2) > F_X(x_1)$
- $F_X(x^+) = F_X(x) = P(X \leq x)$
- $F_X(x^-) = P(X < x)$

Example 2:-

Let X be the random variable for the following CDF

$$F_X(x) = \begin{cases} A & x < -2 \\ 0.2 & -2 \le x < 0 \\ 0.5 & 0 \le x < 3 \\ K & 3 \le x \end{cases}$$

① Find the value of the constants A, K

$F_X(-\infty) = 0 \Rightarrow A = 0$

$F_X(\infty) = 1 \Rightarrow K = 1$

② $F_X(1)$

$P(0 \le X \le 3) = 0.5$ * Just look at the interval

③ $P(x = 0)$

$P(X \le 0) - P(X < 0)$ Rule: $P(X = x) = P(X \le x) - P(X < x)$

$0.5 - 0.2 = 0.3$

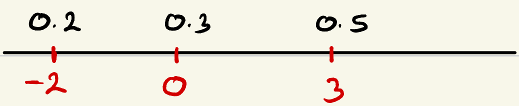
④ $P(X \ge 3)$

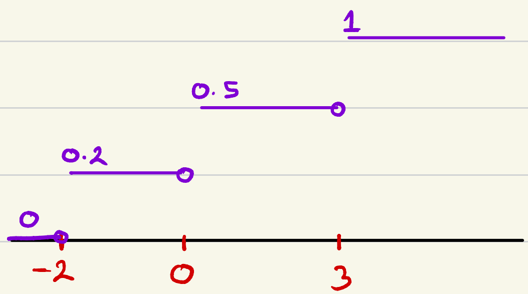
$P(X \ge 3) - P(X > 3)$

$1 - F_X(3^-) \Rightarrow 1 - 0.5 = 0.5$

Comparison

PMF	CDF
$P(X=x) = \begin{cases} 0.2 & x = -2 \\ 0.3 & x = 0 \\ 0.5 & x = 3 \end{cases}$	$F_X(x) = \begin{cases} A & x < -2 \\ 0.2 & -2 \leq x < 0 \\ 0.5 & 0 \leq x < 3 \\ K & 3 \leq x \end{cases}$





Random variable types

① Discrete random variable

it's Countable, we use PMF or CDF

هو عدد معين من الاحتمالات يمكنني عده

Example :-

$X = \begin{cases} 1 & \text{Head} \\ 0 & \text{Tail} \end{cases}$

المقصود انه يوجد عدد محدود و دقيق

من الاحتمالات الذي يمكنني من عده

② Continues random variables

it's NOT Countable, we use PDF or CDF

(interval فترة) موجود لا يمكن حصره

بسبب وجود عدد لا نهائي من الأحتمالات

Example :-

y = The mass of a random animal " $0 < y < 10{,}000$ "

y Can be any Value $\Rightarrow$ $y = 0.1$

$y = 0.2$

$y = 0.12907\ldots$

$y = 70.590017\ldots$

Probability density function (PDF)

it's a method used to determine the Probability under the Curve " The most method used in CH2 "

$$P(A) = \int_a^b f_X(x)\,dx, \quad a \leq x \leq b$$

Example :-

$$P(0 \leq X \leq 2) = \int_0^2 f_X(x)\,dx$$

Properties

① $f_X(x) \geq 0$

② $\int_{-\infty}^{\infty} f_X(x) = 1$

Example

let x be a random variable with the following PDF :-

$$f_X(x) = \begin{cases} 1/4 & -1 \leq x \leq 3 \\ 0 & o.w \end{cases}$$

① Verify that $f_X(x)$ is a Valid PDF

$$\cancelto{0}{\int_{-\infty}^{-1} f(x)} + \int_{-1}^{3} f(x) + \cancelto{0}{\int_{3}^{\infty} f(x)} = 1$$ as rule ①

$$\int_{-1}^{3} \frac{1}{4}\, dx = \frac{1}{4} x \Big|_{-1}^{3} \Rightarrow \frac{3}{4} \overset{+}{-\,-} \frac{1}{4} = 1$$ #

② $P(x \leq 0)$

$$\int_{-\infty}^{0} f(x) = \cancelto{0}{\int_{-\infty}^{-1} f(x)} + \int_{-1}^{0} f(x) \Rightarrow \int_{-1}^{0} \frac{1}{4}\, dx = \frac{1}{4} x \Big|_{-1}^{0}$$

$$\Rightarrow 0 - -\frac{1}{4} = \frac{1}{4}$$ #

③ $F(1)$

$F(1) = P(X \leq 1)$ and it's Solution like the above

④ $P(0.5 \leq X \leq 2.5)$

$$= \int_{0.5}^{2.5} f(x) \Rightarrow \int_{0.5}^{2.5} \frac{1}{4}\,dx \Rightarrow \frac{1}{4}x \Big|_{0.5}^{2.5} \Rightarrow \frac{1}{2} \ \#$$

Expected value of X or The Mean(μ_x)

$E[g(x)] \Rightarrow$ The mean of $g(x) = \mu_{g(x)}$

$E[g(x)] \Longrightarrow$ Discrete :- $\sum_{-\infty}^{\infty} g(x)\, P(X = x)$

Continues :- $\int_{-\infty}^{\infty} g(x)\, f_x(x)\, dx$

Variance (σ_x^2)

$\sigma_x^2 = E[(X - \mu_x)^2] \Rightarrow$ Discrete :- $\sum_{-\infty}^{\infty} (x - \mu_x)^2\, P(X = x)$

Continues :- $\int_{-\infty}^{\infty} (x - \mu_x)^2\, f_x(x)\, dx$

Standard deviation of X (σ_x)

$\sigma_x = \sqrt{\sigma_x^2}$ ⟹ The square root of the variance

وهي تعبر عن مقدار التشتت بين ديار mean

Example :-

Find the mean, variance, standard deviation for the following PMF

$$P(X=x) = \begin{cases} 1/4 & x = -2 \\ 1/4 & x = -1 \\ 1/4 & x = 1 \\ 1/4 & x = 2 \\ 0 & o.w \end{cases}$$

① Mean for X :-

$$\mu_x = \sum_{-\infty}^{\infty} x\, P(X=x) \Rightarrow (-2)(1/4) + (-1)(1/4) + (1)(1/4) + (2)(1/4)$$

$$-1/2 + -1/4 + 1/4 + 1/2 \Rightarrow \boxed{\mu_x = 0}$$

② Variance for x (σ_x^2)

$$\sigma_x^2 = E[(X-\mu_x)^2] \Rightarrow E\{(X-0)^2\} \Rightarrow E\{X^2\}$$

$$= \sum_{-\infty}^{\infty} x^2 P(X=x) = (4)(1/4) + (1)(1/4) + (1)(1/4) + (4)(1/4)$$

$$1 + 1/4 + 1/4 + 1 \Rightarrow \boxed{\sigma_x^2 = 2.5}$$

③ Standard deviation (σ_x)

$$\sigma_x = \sqrt{\sigma_x^2} \Rightarrow \sigma_x = \sqrt{2.5} \Rightarrow \boxed{\sigma_x = 1.581138}$$

Rules for continues and discrete

1. $E\{A\} = A$, A and B are Constants
2. $E\{AX\} = A\,E\{X\}$
3. $E\{AX + B\} = A\,E\{X\} + B$
4. $E\{X^2\} \neq (E\{X\})^2$
5. $\sigma_x^2 = E\{X^2\} - \mu_x^2$

Example :-

Let X be a R.V with mean of (2) and standard deviation of (3)

1. $E\{X^2\}$

$E\{X^2\} = \sigma_x^2 + \mu_x^2 \Longrightarrow E\{X^2\} = 9 + 4 \Longrightarrow \boxed{E\{X^2\} = 13}$

2. $E\{2X^2 - 3X + 5\}$

$E\{2X^2\} - E\{3X\} + E\{5\}$

$2E\{X^2\} - 3E\{X\} + E\{5\}$

$2 * 13 - 3 * 2 + 5$

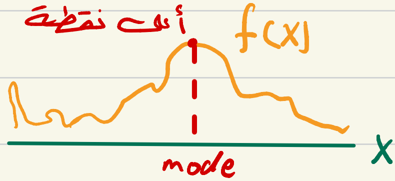
$26 - 6 + 5$

$= \boxed{25}$

Mode and Median

① Mode

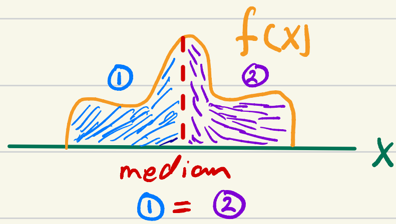
is the value of X at $f_x(x)$ when it's MAX



mode : $f'(x) = 0$, then we find the value of X

② Median

is the value of X that divides the graph into two halfs



median : $\int_{-\infty}^{x_m} f_x(x)\,dx = \frac{1}{2}$, then we find the value of X

Example

$$f_x(x) = \begin{cases} \frac{3}{8}x^2 & 0 \leq x \leq 2 \\ 0 & \text{o.w} \end{cases}$$

① mode

$\Rightarrow f'(x) = 0 \Rightarrow \frac{3}{4}x = 0 \Rightarrow f(0) = 0$ (min) , $f(2) = \frac{3}{4}$ (max)

$\Rightarrow x = 2$ is the mode

② median

$$\Rightarrow \int_{-\infty}^{x_m} f(x) \Rightarrow \int_0^{x_m} \frac{3}{8}x^2 \Rightarrow \left.\frac{x^3}{8}\right|_0^{x_m} = \frac{1}{2} \Rightarrow x_m^3 = 4 \Rightarrow x_m = \sqrt[3]{4}$$

Long question

$$f_x(x) = \begin{cases} K(1+x) & -1 \leq x \leq 0 \\ 0 & o.w \end{cases}$$

Find all of these

① The value of K

$$\int_{-\infty}^{\infty} f_x(x) = 1 \Rightarrow \int_{-1}^{0} K + Kx = 1 \Rightarrow \left[Kx + \frac{Kx^2}{2}\right]\Big|_{-1}^{0} = 1 \Rightarrow 0 - \left(-K + \frac{K}{2}\right) = 1$$

$$\Rightarrow K - \frac{1}{2}K = 1 \Rightarrow \frac{1}{2}K = 1 \Rightarrow \boxed{K = 2}$$

② The mean

$$\mu_x = \int_{-\infty}^{\infty} x f_x(x) \Rightarrow \int_{-1}^{0} x(2+2x) \Rightarrow \int_{-1}^{0} 2x + 2x^2 \Rightarrow \left(x^2 + \frac{2x^3}{3}\right)\Big|_{-1}^{0} \Rightarrow \boxed{\mu_x = -1/3}$$

③ The variance

$$\sigma_x^2 = \int_{-\infty}^{\infty} (x - \mu_x)^2 f_x(x) \Rightarrow \int_{-1}^{0} \left(x + \frac{1}{3}\right)^2 * (2+2x) \Rightarrow \int_{-1}^{0} 2x^3 + \frac{10}{3}x^2 + \frac{14}{9}x + \frac{2}{9}\,dx$$

$$= \left(\frac{x^4}{2} + \frac{10}{9}x^3 + \frac{7}{9}x^2 + \frac{2}{9}x\right)\Big|_{-1}^{0} \Rightarrow -\left(\frac{1}{2} - \frac{10}{9} + \frac{7}{9} - \frac{2}{9}\right) \Rightarrow \boxed{\sigma_x^2 = \frac{1}{18}}$$

OR

Two ways
Same Ans.

$$\sigma_x^2 = E\{x^2\} - E\{x\}^2 \Rightarrow E\{x^2\} = \int_{-1}^{0} 2x^3 + 2x^2 \Rightarrow E\{x^2\} = \frac{1}{6}$$

$$\Rightarrow \sigma_x^2 = \frac{1}{6} - \left(\frac{1}{3}\right)^2 \Rightarrow \sigma_x^2 = \frac{1}{6} - \frac{1}{9} \Rightarrow \boxed{\sigma_x^2 = \frac{1}{18}}$$

④ The standard deviation

$$\sigma_x = \sqrt{\sigma_x^2} \Rightarrow \sigma_x = \sqrt{\frac{1}{18}} \Rightarrow \boxed{\sigma_x = 0.235702}$$

⑤ The mode

$f'(x) = 0 \Rightarrow 2 \neq 0$ then we try a and b because it's linear

$f(-1) = 0$, $f(0) = 2 \Rightarrow \boxed{\text{mode } X = 0}$

min max

⑥ The median

$\int_{-\infty}^{x_m} f(x) = \frac{1}{2} \Rightarrow \int_{-1}^{x_m} (2+2x) = \frac{1}{2} \Rightarrow (2x + x^2)\Big|_{-1}^{x_m} = \frac{1}{2}$

$\Rightarrow (2x_m + x_m^2) - (-2+1) = \frac{1}{2} \Rightarrow x_m^2 + 2x_m + \frac{1}{2} = 0 \Rightarrow 2x_m^2 + 4x_m + 1 = 0$

By general rule :- $x = \frac{b \pm \sqrt{b^2 - 4ac}}{2a} \Rightarrow x = \frac{4 \pm \sqrt{16-8}}{4}$

$$\boxed{x = -1 + \frac{\sqrt{8}}{4} \quad \text{or} \quad x = -1 - \frac{\sqrt{8}}{4}}$$

⑦ $P(X \leq -0.5 \,/\, X \leq -0.2)$

$P(A/B) = \frac{P(A \cap B)}{P(B)} \Rightarrow \frac{P(X \leq -0.5 \cap X \leq -0.2)}{P(X \leq 0.2}$

$\frac{P(X \leq -0.5)}{P(X \leq -0.2)} \Rightarrow \frac{\int_{-1}^{-0.5} (2x+2)}{\int_{-1}^{-0.2} (2x+2)} \Rightarrow \frac{(x^2 + 2x)\Big|_{-1}^{-0.5}}{(x^2 + 2x)\Big|_{-1}^{-0.2}}$

$\frac{(\frac{1}{4} - 1) - (1 - 2)}{(0.04 - 0.4) - (1 - 2)} = \boxed{2.7344}$

8 Suppose $y = 2x + 3$, find mean, variance, standard deviation

- $\mu_y = a\mu_x + b \Rightarrow 2 * -1/3 + 3 \Rightarrow \boxed{\mu_y = \frac{7}{3}}$
- $\sigma_y^2 = a^2 \sigma_x^2 \Rightarrow 4 * 1/18 \Rightarrow \boxed{\sigma_y^2 = \frac{2}{9}}$
- $\sigma_y = \sqrt{\sigma_y^2} \Rightarrow \sigma_y = \sqrt{2/9} \Rightarrow \boxed{\sigma_y = \frac{\sqrt{2}}{3}}$

Common Distributions

① Binomial

It's a random experiment consists of n repeated trials such that:

① *The trials are independent*

② *Each trial results only two possible outcome, a success and failure*

③ *The probability of the success on each trial remains constant*

$$P(X = x) = \binom{n}{x} * p^x * (1-p)^{n-x}, \quad X = 0, 1, 2 \dots n$$

X : *the number of trials that results in a success*

n : *the number of all trials*

$\mu_x = np$ ------------- ①

$\sigma_x^2 = np(1-p)$ --------- ②

Example

Consider an experiment of flipping a coin for 3 times (n), assume $P(H) = 1/4$ (p of success) and $P(T) = 3/4$, determine :

① The probability of getting head for 2 (x) times

$$P(X=x) = \binom{n}{x} p^x (1-p)^{n-x} \Rightarrow P(X=2) = \binom{3}{2} \left(\frac{1}{4}\right)^2 \left(\frac{3}{4}\right)$$

$$\binom{3}{2} = \frac{3!}{2!\,(3-2)!} = 3 \Rightarrow P(X=2) = 3 * \frac{1}{16} * \frac{3}{4} \Rightarrow \boxed{P(X=2) = \frac{9}{64}}$$

② The probability of getting at least one head

عكس

$$P(X \geq 1) = \big(P(X=1) + P(X=2) + P(X=3)\big) \text{ OR } \big(1 - P(X<1)\big)$$

$$= 1 - P(X=0) \Rightarrow 1 - \binom{3}{0} * 1 * \frac{27}{64} \Rightarrow \boxed{P(X \geq 1) = \frac{37}{64}}$$

③ The expected number of heads to be observed in the experiment

$$\mu_x = np \Rightarrow \mu_x = 3 * \frac{1}{4} \Rightarrow \boxed{\mu_x = 3/4}$$

$$\sigma_x^2 = np(1-p) \Rightarrow \sigma_x^2 = 3 * \frac{1}{4} * \frac{3}{4} \Rightarrow \boxed{\sigma_x^2 = \frac{9}{16}}$$

② Geometric

It's a random experiment consists of INFINITY trials such that:

① The trials are independent

② Each trial results only two possible outcome,a success and failure

③ The probability of the success on each trial remains constant

- It's the probability to the first success
- probability of success = p
- probability of failure = 1 – P
- X : the number of times experiment is preformed to the first occurrence of success

$$P(X=x) = (1-P)^{x-1} * P \quad , \quad X = 1, 2, 3, \ldots$$

$$\mu_X = \frac{1}{P} \qquad \sigma_X^2 = \frac{1-P}{P^2}$$

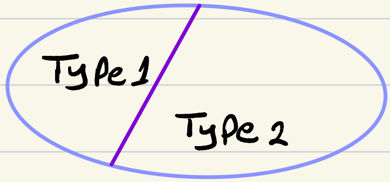
Example:

Let the probability of occurrence of a flood of magnitude greater than a critical magnitude in a given year be 0.02 Assuming that floods occur independently, determine the "return period" defined as the average number of years between floods.

X : Number of years between two floods

$$\mu_X = \frac{1}{P} \Rightarrow \mu_X = \frac{1}{0.02} \Rightarrow \boxed{\mu_X = 50 \text{ years}}$$

③ Hyper Geometric



- Usually Type 1 is smaller than Type 2

$$P(X=x) = \frac{\binom{K}{x}\binom{N-K}{n-x}}{\binom{N}{n}}$$

N : Number of all outcomes

n : number of selected

K : number of Type 1

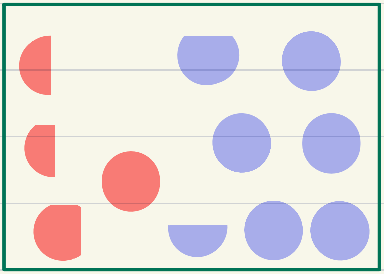
X : number of items selected from Type 1

$$\mu_X = n\frac{K}{N} = np$$

$$\sigma_X^2 = np(1-p)\left(\frac{N-n}{N-1}\right)$$

Example:

① What is the probability of having 2 Red balls in 3 selected balls?



$$P(X=x) = \frac{\binom{K}{x}\binom{N-K}{n-x}}{\binom{N}{n}}$$

N = Type 1 + Type 2 = 5 + 7 = 12

n = number of selected balls = 3

K = number of Type 1 = 5

X = number of wanted red balls = 2

$$P(x=2) = \frac{\binom{5}{2}\binom{7}{1}}{\binom{12}{3}} = \frac{10 * 7}{220} \Rightarrow \boxed{P(x=2) = \frac{70}{220}}$$

+ +

② *what is the probability of getting at least one red balls from 3 selected ones*

$$P(x \geq 1) = 1 - P(x=0) \Rightarrow 1 - \frac{\binom{5}{0}\binom{7}{3}}{\binom{12}{3}} = \boxed{\frac{185}{220}}$$

③ *what is the probability of getting at least 4 red balls from 7 selected ones*

$$P(x \geq 4) = P(x=4) + P(x=5) \Rightarrow \frac{\binom{5}{4}\binom{7}{3}}{\binom{12}{7}} + \frac{\binom{5}{5}\binom{7}{2}}{\binom{12}{7}}$$

$$\frac{5 * 35}{792} + \frac{21}{792} \Rightarrow \boxed{P(x \geq 4) = \frac{196}{792}}$$

④ poisson

$$P(X=x) = e^{-\lambda T}\,\frac{(\lambda T)^x}{x!} \quad , \quad P(X=x)\binom{n}{x}(1-p)^{n-x}(p)^x$$

$$\mu_x = \sigma_x^2 = \lambda T$$

هو عبارة عن عدة أحداث تحصل في
فترة معينة T بمعدل معين λ

حيث يعبر المتغير العشوائي X عن عدد الأحداث التي حصلت في (t_1, t_2)

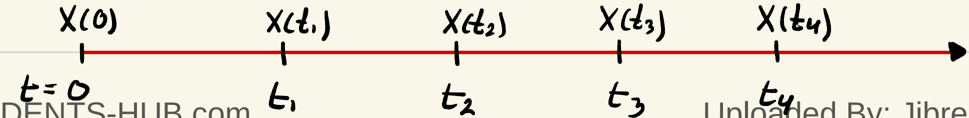
To consider this process as poisson there is some conditions:

① $X(0) = 0$, "we begin Counting from $t = 0$"

② for intervals $(0, t_1)$. (t_2, t_3) $\Rightarrow$ The number of occurrences $\{X(t_1) - X(0)\}$ and $\{X(t_3) - X(t_2)\}$ are Independent

③ Probability of the number of occurrence depends on the length of the interval

④ The Probability of occurrence in small interval Δt is approximately $\lambda \Delta t$



Example

The number of cracks in a section of a highway that are significant enough to require repair is assumed to follow a Poisson distribution with a mean of two cracks per mile.

a- What is the probability that there are no cracks in 5 miles of highway?
b- What is the probability that at least one crack requires repair in ½ miles of highway?
c- What is the probability that at least one crack in 5 miles of highway?

a) $X = 0$, $\lambda = 2$, $T = 5$

$$P(X=x) = e^{-\lambda T} \frac{(\lambda T)^x}{x!} \Rightarrow P(X=0) = e^{-2*5} * \frac{(2*5)^0}{0!}$$

$$\Rightarrow P(X=0) = e^{-10} * 1 \Rightarrow \boxed{P(X=0) = e^{-10}}$$

b) $X \geq 1$, $\lambda = 2$, $T = 0.5$

$$P(X \geq 1) = 1 - P(X<1) = 1 - P(X=0) \Rightarrow P(X \geq 1) = 1 - P(X=0)$$

$$1 - \left(e^{-1} * \frac{1^0}{1}\right) \Rightarrow \boxed{P(X \geq 1) = 1 - e^{-1}}$$

c) $X \geq 1$, $\lambda = 2$, $T = 5$

$$P(X \geq 1) = 1 - P(X=0) \Rightarrow \boxed{P(X \geq 1) = 1 - e^{-10}}$$

Another Common Distributions

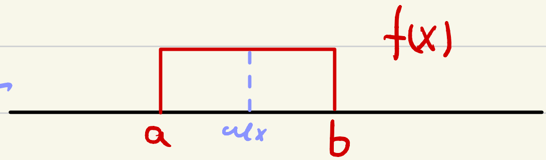
① Uniform Distribution

$$f_X(x) = \begin{cases} \dfrac{1}{b-a} & a \le x \le b \\ 0 & \text{o.w} \end{cases}$$

$$\mu_X = \frac{a+b}{2}$$

$$\sigma_X^2 = \frac{(b-a)^2}{12}$$

يكون متساوي عند جميع النقاط



Example

Let X be a R.V that follows Uniform distribution in the interval [-2, 5], Find :-

① write and Plot the PDF of X

$$f_X(x) = \begin{cases} \dfrac{1}{b-a} & -2 \le x \le 5 \\ 0 & \text{o.w} \end{cases} \Rightarrow \boxed{f_X(x) = \begin{cases} 1/7 & -2 \le x \le 5 \\ 0 & \text{o.w} \end{cases}}$$

② $P(X<0)$

$$= \int_{-2}^{0} f_X(x) \Rightarrow \int_{-2}^{0} 1/7 \, dx \Rightarrow 0 + 2/7 \Rightarrow \boxed{P(X<0) = \frac{2}{7}}$$

③ μ_X and σ_X^2

$$\mu_X = \frac{a+b}{2} \Rightarrow \boxed{\mu_X = \frac{3}{2}}$$

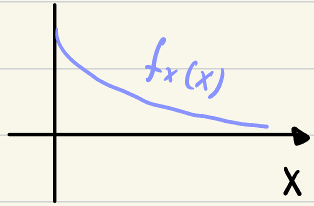
$$\sigma_X^2 = \frac{(b-a)^2}{12} \Rightarrow \boxed{\sigma_X^2 = \frac{49}{12}}$$

② exponential Distribution

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & , x \geq 0 \\ 0 & , o.w \end{cases}$$

$$\mu_X = 1/\lambda$$

$$\sigma_X^2 = 1/\lambda^2$$



Example

Let X be an exponential R.V with mean of 0.2

① Write and Plot the Pdf of X

$$\mu_X = 1/\lambda \Rightarrow \lambda = 1/\mu_X \Rightarrow \lambda = 1/0.2 \Rightarrow \lambda = 5$$

$$f_X(x) = \begin{cases} 5e^{-5x} & x \geq 0 \\ 0 & o.w \end{cases}$$

② $P(X \leq 2)$

$$P(X \leq 2) = \int_{-\infty}^{2} f_X(x) \Rightarrow P(X \leq 2) = \int_{0}^{2} 5e^{-5x} = \left. \frac{5e^{-5x}}{-5} \right|_0^2$$

$$P(X \leq 2) = -e^{-10} + 1 \Rightarrow \boxed{P(X \leq 2) = 1 - e^{-10}}$$

③ Variance

$$\sigma_X^2 = \frac{1}{\lambda^2} \Rightarrow \sigma_X^2 = \frac{1}{25} \Rightarrow \boxed{\sigma_X^2 = 0.04}$$

EXAMPLE (3-24):

Suppose that the depth of water, measured in meters, behind a dam is described by an exponential random variable with pdf:

$$f_X(x) = \begin{cases} \dfrac{1}{13.5}\, e^{\frac{-x}{13.5}} & x > 0 \\ 0 & o.w \end{cases}$$

There is an emergency overflow at the top of the dam that prevents the depth from exceeding 40.6 m. There is a pipe placed 32.0 m below the overflow that feeds water to a hydroelectric generator (turbine).

a- What is the probability that water is wasted though emergency overflow?
b- What is the probability that water will be too low to produce power?
c- Given that water is not wasted in overflow, what is the probability that the generator will have water to derive it?

SOLUTION:

a- P(water wasted through emergency) = $P(X \geq 40.6\text{ m}) = \int_{40.6}^{\infty} \frac{1}{13.5} e^{\frac{-x}{13.5}}\, dx = e^{-3}$

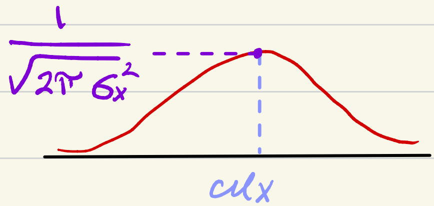
b- P(water too low to produce power) = $P(x < 8.6\text{ m}) = \left(1 - e^{-0.637}\right) = 0.47$

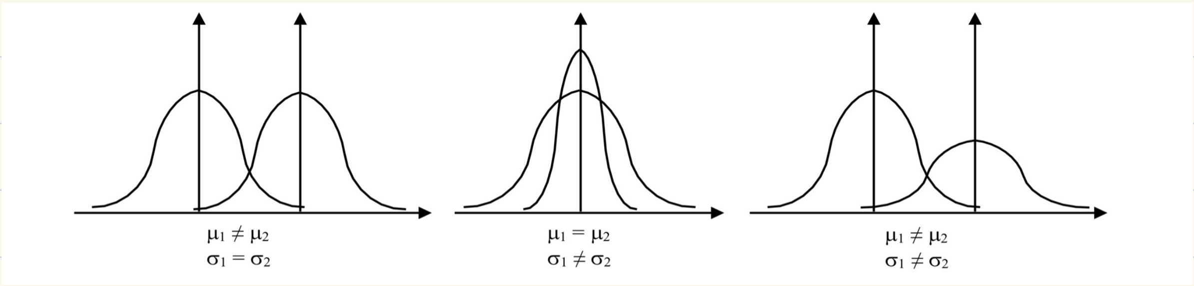
c- P(generator has water to derive it / water is not wasted) = $P(x > 8.6 / x < 40.6)$

$$= \frac{P(x > 8.6 \cap x < 40.6)}{P(x < 40.6)} = \frac{P(8.6 < x < 40.6)}{P(x < 40.6)} = \frac{\int_{8.6}^{40.6} \frac{1}{13.5} e^{\frac{-x}{13.5}}\, dx}{\int_{0}^{40.6} \frac{1}{13.5} e^{\frac{-x}{13.5}}\, dx = e^{-3}} = 0.504$$

③ Gaussian Distribution

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma_x^2}}\, e^{\frac{-(x-\mu_x)^2}{2\sigma_x^2}}$$





- **Two Cases :-**

① when $\mu_x = 0$ and $\sigma_x^2 = 1$ we call it **Standard**
and when it's standard we take the answers from a table

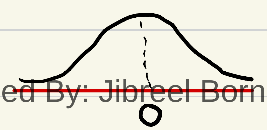
② when $\mu_x \neq 0$ and $\sigma_x^2 \neq 1$ we call it **Non-Standard**
we convert it to standard using $P(X=x) = \phi\left(\frac{x-\mu_x}{\sqrt{\sigma_x^2}}\right)$

Example

Let X be a R.V with mean = 0 and unity Variance

① write and plot the distribution of X

$$f_X(x) = \frac{1}{2\pi}\, e^{\frac{-x^2}{2}}$$



② $P(X \leq 0)$

$P(X \leq g) = \phi(g)$, and if it's standard $(\mu_x = 0, \sigma_x^2 = 1)$ we take $\phi(g)$ from the table they give us

$\Rightarrow$ from the table $\boxed{\phi(0) = 0.5}$

③ $P(X \leq 1.12)$

$P(X \leq 1.12) = \phi(1.12) \Rightarrow$ from the table $\boxed{\phi(1.12) = 0.8686}$

④ $P(X \geq 3.12)$

$P(X \geq 3.12) = 1 - P(X < 3.12) \Rightarrow P(X \geq 3.12) = 1 - \phi(3.12)$

$\phi(3.12) = 0.9991 \Rightarrow P(X \geq 3.12) = 1 - 0.9991 \Rightarrow \boxed{P(X \geq 3.12) = 0.0009}$

⑤ $P(0.5 \leq X \leq 1.7)$

$P(0.5 \leq X \leq 1.7) = P(X \leq 1.7) - P(X < 0.5)$

$\boxed{P(0.5 \leq X \leq 1.7) = \phi(1.7) - \phi(0.5)}$

⑥ $P(X \leq -1.4)$

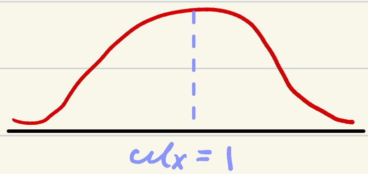
$P(X \leq -1.4) = \phi(-1.4) \Rightarrow \boxed{\phi(-1.4) = 1 - \phi(1.4)}$

Example

Let x be a R.V with $\mu_x = 1$ and $\sigma_x^2 = 9$

① write and Plot the PDF of x

$$f_x(x) = \frac{1}{\sqrt{18\pi}} e^{\frac{-(x-1)^2}{18}}$$



② $P(x \leq 1)$

$P(x \leq 1) = 0.5$ لأن μ_x بقسم Pdf لنصفين

③ $P(x \leq 1.6)$

$$P(x \leq 1.6) = \Phi\left(\frac{x-\mu_x}{\sqrt{\sigma_x^2}}\right) \Rightarrow P(x \leq 1.6) = \Phi(0.2)$$

and after we've transformed it, we go to the table

④ $P(x < -2)$

$$P(x \leq -2) = \Phi\left(\frac{-2-1}{\sqrt{9}}\right) = \Phi(-1) = 1 - \Phi(1)$$

⑤ $P(-2.3 \leq x \leq 5.3)$

$$P(x \leq 5.3) - P(x < -2.3)$$

ومن ثمّ نحول كل واحدة إلى standard و نكمل كأي جدول

Normal approximation for binomial and poisson

Why do we even use it ?

لما يطلب مني مثلا أحسب $P(x<900)$ ، صاض رقم كثير كبير ومحتمل تقدر تحسبه " أنا بقدر لأني مهندس " وقتها بتستخدم طريقة التقريب

① binomial to normal approximation

$$P(x=x) = \varnothing\left(\frac{x-\mu_x}{\sqrt{\sigma_x^2}}\right) \quad , \mu_x = nP$$

$$\sigma_x^2 = nP(1-P)$$

Example

Consider a binomial experiment with n = 1000 and p = 0.2. if X is the number of successes, find the probability at $X \le 240$.

$$\mu_x = nP \Rightarrow \mu_x = 1000 * 0.2 \Rightarrow \mu_x = 200$$

$$\sigma_x^2 = nP(1-P) \Rightarrow \sigma_x^2 = 200 * 0.8 \Rightarrow \sigma_x^2 = 160$$

$$P(x=x) = \varnothing\left(\frac{x-\mu_x}{\sqrt{\sigma_x^2}}\right) \Rightarrow P(x \le 240) = \varnothing\left(\frac{240-200}{\sqrt{160}}\right)$$

$$P(x \le 240) = \varnothing(3.16) \Rightarrow \boxed{P(x \le 240) = 0.9992}$$

② poisson to normal approximation

$$P(x = x) = \phi\left(\frac{x - \mu_x}{\sqrt{\sigma_x^2}}\right)$$

Example

Assume the number of asbestos particles in a cm^3 of dust follow a Poisson distribution with a mean of 1000. If a cm^3 of dust is analyzed, what is the probability that less than 950 particles are found in 1 cm^3?

Remark :-

in Poisson, $\mu_x = \sigma_x^2 = \lambda T$

$\Rightarrow \mu_x = 1000 \quad , \quad \sigma_x^2 = 1000 \quad , \quad T = 1 \quad , \quad \lambda = 1000$

$$P(x = x) = \phi\left(\frac{x - \mu_x}{\sqrt{\sigma_x^2}}\right) \Rightarrow P(x \leq 950) = \phi\left(\frac{950 - 1000}{\sqrt{1000}}\right)$$

$$P(x \leq 950) = \phi(-1.58) \Rightarrow P(x \leq 950) = 1 - \phi(1.58)$$

$$P(x \leq 950) = 1 - 0.943 \Rightarrow \boxed{P(x \leq 950) = 0.057}$$

Transformation of random variables

① Discrete case

Etample

Let X be a Binomial R.V with $n=3$ and $P=3/4$
let $y=2x+3$, find $P(y=y)$?

① First write $P(x=x)$

$$P(x=x)=\begin{cases}\binom{3}{x}\left(\frac{3}{4}\right)^x\left(\frac{1}{4}\right)^{3-x} & , x=0,1,2,3\\ 0 & , o.w\end{cases}$$

② Make a table for all X Probabilities

X	$P(x=x)$	$y=2x+3$	$P(y=y)$
0	$\binom{3}{0}\left(\frac{3}{4}\right)^0\left(\frac{1}{4}\right)^3=\frac{1}{64}$	3	$\frac{1}{64}$
1	$\binom{3}{1}\left(\frac{3}{4}\right)^1\left(\frac{1}{4}\right)^2=\frac{9}{64}$	5	$\frac{9}{64}$
2	$\binom{3}{2}\left(\frac{3}{4}\right)^2\left(\frac{1}{4}\right)^1=\frac{27}{64}$	7	$\frac{27}{64}$
3	$\binom{3}{3}\left(\frac{3}{4}\right)^3\left(\frac{1}{4}\right)^0=\frac{27}{64}$	9	$\frac{27}{64}$

③ write $P(y=y)$

$$P(y=y) = \begin{cases} 1/64 & y=3 \\ 9/64 & y=5 \\ 27/64 & y=7 \\ 27/64 & y=9 \\ 0 & o.w \end{cases}$$

#

Example

let X has the distribution $P(x=x) = \frac{1}{6}$, $x = -3, -2, 0, 1, 2, -1$ and $y = X^2$, Find $P(y=y)$?

$$P(x=x) = \begin{cases} 1/6 & , x = -3, -2, -1, 0, 1, 2 \\ 0 & , o.w \end{cases}$$

x	$P(x=x)$	y	$P(y=y)$
-3	$1/6$	9	$1/6$
-2	$1/6$	4	$1/6$
-1	$1/6$	1	$1/6$
0	$1/6$	0	$1/6$
1	$1/6$	1	$1/6$
2	$1/6$	4	$1/6$

$$P(y=y) = \begin{cases} 1/6 & y=0 \\ 2/6 & y=1 \\ 2/6 & y=4 \\ 1/6 & y=9 \\ 0 & o.w \end{cases}$$

② continues case

Example

Let X be a R.V that follow a Uniform Distribution over the interval $[-2,5]$, $y = 2x+1$, Find Pdf of y ?

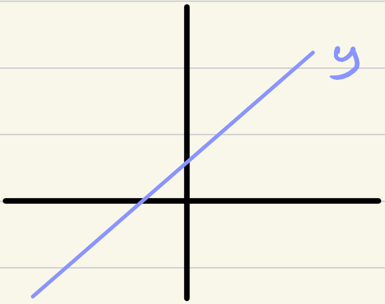
① First : Find the Pdf of X

$$f_x(x) = \begin{cases} \frac{1}{b-a} & b \leq x \leq a \\ 0 & o.w \end{cases} \Rightarrow f_x(x) = \begin{cases} 1/7 & -2 \leq x \leq 5 \\ 0 & o.w \end{cases}$$

② Second : Plot (draw) y

$y = 2x + 1$

2	1	0	-1	-2	x
5	3	1	-1	-3	y



③ Third : Put X with respect to y

$$X = \frac{y-1}{2}$$

④ Fourth : Find $\frac{dy}{dx}$

$\frac{dy}{dx} = 2$

5 Fifth : Find $f_Y(y) = \dfrac{f_X(x)}{\left|\frac{dy}{dx}\right|}$ on each Part of the interval

I $X < -2$

$f_Y(y) = \dfrac{0}{2}$, $X < -2 \Rightarrow \dfrac{y-1}{2} < \dfrac{-2}{1} \Rightarrow y-1 < -4$

$f_Y(y) = 0$ on $y < -3$

II $X > 5$

$f_Y(y) = \dfrac{0}{2}$, $X > 5 \Rightarrow \dfrac{y-1}{2} > 5 \Rightarrow y-1 > 10$

$f_Y(y) = 0$ on $y > 11$

III $-2 < x < 5$

$f_Y(y) = \dfrac{(1/7)}{2}$, $-2 < X < 5 \Rightarrow \times 2 \; -2 < \cancel{\times 2}\dfrac{y-1}{\cancel{2}} \overset{\times 2}{<} 5$

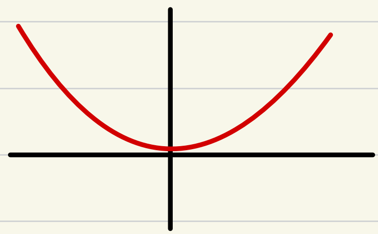
$\underset{+1}{-4} < \underset{+1}{y-1} < \underset{+1}{10} \Rightarrow f_Y(y) = \dfrac{1}{14}$ on $-3<y<11$

$$f_Y(y) = \begin{cases} 1/14 & -3 \leq y \leq 11 \\ 0 & \text{o.w} \end{cases}$$

Assume $R = x^2$, Find Pdf for R

1- $f_x(x) = \begin{cases} 1/7 & -2 \leq x \leq 5 \\ 0 & o.w \end{cases}$

2- $R = x^2$



3- $x = \sqrt{R}$ or $-\sqrt{R}$

4- $\frac{dR}{dx} = 2x$

5- I $x \leq -5$ or $x \leq 5$

$f_R(R) = 0$, $x \leq -5$ and $x \leq 5 \Rightarrow \sqrt{R}^2 \leq -5^2$ or $\sqrt{R}^2 \leq 5^2 \Rightarrow R \leq 25$

II $x \geq -2$ or $x \geq 2$

$f_R = 0$, $x \geq -2$ or $x \geq 2 \Rightarrow \sqrt{R} \geq -2$ or $\sqrt{R} \geq 2 \Rightarrow R \geq 4$

III $-5 \leq x \leq 2$

$f_R = \left.\frac{(1/7)}{|2x|}\right|_{x=+\sqrt{R}} + \left.\frac{(1/7)}{|2x|}\right|_{x=-\sqrt{R}} \Rightarrow f_R = \frac{2}{14\sqrt{R}}$

$f_R(R) = \begin{cases} \frac{1}{7\sqrt{R}} & 4 \leq R \leq 25 \\ 0 & o.w \end{cases}$

By: Jibreel Bornat