7.8 Relative Rate of Growth

$\boxed{2}$ Which of the following functions grow faster than e^x as $x \to \infty$ which grow at the same rate as e^x which grow slower

ⓐ $10x^4 + 30x + 1$

$$\lim_{x \to \infty} \frac{e^x}{10x^4 + 30x + 1} = \frac{\infty}{\infty}$$

$$= \lim_{x \to \infty} \frac{e^x}{40x^3 + 30}$$

$$= \lim_{x \to \infty} \frac{e^x}{120x^2}$$

$$= \lim_{x \to \infty} \frac{e^x}{240x}$$

$$= \lim_{x \to \infty} \frac{e^x}{240} = \infty$$

e^x grows faster than $10x^4 + 30x + 1$ as $x \to \infty$

(b) $x \ln x - x$

$$\lim_{x \to \infty} \frac{e^x}{x \ln x - x} =$$

$$\lim_{x \to \infty} \frac{e^x}{x(\ln x - 1)} = \frac{\infty}{\infty}$$

$$\lim_{x \to \infty} \frac{e^x}{x\left(\frac{1}{x}\right) + (\ln x - 1)} = \lim_{x \to \infty} \frac{e^x}{\ln x}$$

$$= \lim_{x \to \infty} \frac{e^x}{\frac{1}{x}}$$

$$= \lim_{x \to \infty} x e^x = \infty$$

e^x grows faster than $x \ln x - x$

(c) $\sqrt{1 + x^4}$

$$\lim_{x \to \infty} \frac{\sqrt{1 + x^4}}{e^x} = \lim_{x \to \infty} \sqrt{\frac{1 + x^4}{e^{2x}}} = \sqrt{\lim_{x \to \infty} \frac{1 + x^4}{e^{2x}}} \quad \frac{\infty}{\infty}$$

$$= \sqrt{\lim_{x \to \infty} \frac{4x^3}{2e^{2x}}}$$

$$= \sqrt{\lim_{x \to \infty} \frac{12x^2}{4e^{2x}}} = \sqrt{\lim_{x \to \infty} \frac{24x}{8e^{2x}}}$$

$$\lim_{x\to\infty} \sqrt{\frac{1+x^4}{e^{2x}}} = \sqrt{\lim_{x\to\infty} \frac{24}{16\ e^{2x}}}$$

$$= 0$$

$\sqrt{1+x^4}$ grows slower than e^x as $x \to \infty$

ⓓ $\left(\frac{5}{2}\right)^x$

$$\lim_{x\to\infty} \frac{\left(\frac{5}{2}\right)^x}{e^x} = \lim_{x\to\infty} \left(\frac{2.5}{e}\right)^x$$

$$= 0$$

$\left(\frac{5}{2}\right)^x$ grows slower than e^x as $x \to \infty$

ⓔ e^{-x}

$$\lim_{x\to\infty} \frac{e^{-x}}{e^x} = \lim_{x\to\infty} \frac{1}{e^{2x}} = 0$$

e^{-x} grows slower than e^x as $x \to \infty$

ⓕ $x\ e^x$

$$\lim_{x\to\infty} \frac{x e^x}{e^x} = \infty$$

xe^x grows faster than e^x as $x \to \infty$

(g) $e^{\cos x}$

$$\lim_{x\to\infty} \frac{e^{\cos x}}{e^x} = \lim 0 = 0$$

$-1 \le \cos x \le 1$

$e^{-1} \le e^{\cos x} \le e^{1}$

By Sandwich Th.

$$\underbrace{\lim_{x\to\infty} \frac{e^{-1}}{e^x}}_{0} \le \lim_{x\to\infty} \frac{e^{\cos x}}{e^x} \le \underbrace{\lim_{x\to\infty} \frac{e^{1}}{e^x}}_{0}$$

$$\lim_{x\to\infty} \frac{e^{\cos x}}{e^x} = 0$$

Then e^x grows faster than $e^{\cos x}$

(h) e^{x-1}

$$\lim_{x\to\infty} \frac{e^{x-1}}{e^x} = \lim_{x\to\infty} e^{-1} = \frac{1}{e} = \text{constant}$$

Then e^{x-1} and e^x grows at the same rate as $x \to \infty$

Q5 Which of the following functions grow faster than $\ln x$ as $x \to \infty$? which grow slower?

ⓐ $\log_3 x$

$$\lim_{x\to\infty} \frac{\log_3 x}{\ln x} = \lim_{x\to\infty} \frac{\ln x / \ln 3}{\ln x} = \frac{1}{\ln 3}$$

$\log_3 x$ & $\ln x$ grows at the same rate as $x \to \infty$

ⓑ $\ln 2x$

$$\lim_{x\to\infty} \frac{\ln 2x}{\ln x}$$

$$= \lim_{x\to\infty} \frac{\frac{1}{2x}\cdot(2)}{\frac{1}{x}} = 1$$

$\ln 2x$ & $\ln x$ grows at the same rate as $x \to \infty$

ⓒ $\ln \sqrt{x}$

$$\lim_{x\to\infty} \frac{\ln \sqrt{x}}{\ln x} = \lim_{x\to\infty} \frac{\frac{1}{\sqrt{x}}\cdot\frac{1}{2\sqrt{x}}}{\frac{1}{x}}$$

$$= \frac{1}{2}$$

$\ln \sqrt{x}$ & $\ln x$ grows at the same rate as $x \to \infty$

(d) $\sqrt{x}$

$$\lim_{x\to\infty} \frac{\sqrt{x}}{\ln x} = \lim_{x\to\infty} \frac{\frac{1}{2\sqrt{x}}}{\frac{1}{x}} = \lim_{x\to\infty} \frac{\sqrt{x}}{2} = \infty$$

$\sqrt{x}$ grows faster than $\ln x$ as $x \to \infty$

(e) x

$$\lim_{x\to\infty} \frac{x}{\ln x} = \lim_{x\to\infty} \frac{1}{\frac{1}{x}} = \infty$$

x grows faster than $\ln x$ as $x \to \infty$

(f) $5 \ln x$

$$\lim_{x\to\infty} \frac{5 \ln x}{\ln x} = 5$$

$5 \ln x$ & $\ln x$ grows at the same rate as $x \to \infty$

(g) $\frac{1}{x}$

$$\lim_{x\to\infty} \frac{\frac{1}{x}}{\ln x} = \lim_{x\to\infty} \frac{1}{x \ln x} = 0$$

$\frac{1}{x}$ grows slower than $\ln x$ as $x \to \infty$

(h) e^x

$$\lim_{x\to\infty} \frac{e^x}{\ln x} = \lim_{x\to\infty} \frac{e^x}{\frac{1}{x}} = \infty$$

e^x grows faster than $\ln x$ as $x \to \infty$

$\boxed{Q_8}$ Order the following function from <u>slowest</u> growing to <u>fastest</u> growing as $x \to \infty$

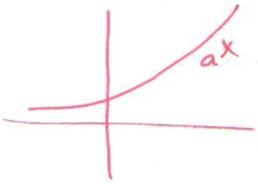
Solution $(\ln 2)^x$, x^2, 2^x, e^x <u>since.</u>

- $$\lim_{x\to\infty} \frac{e^x}{2^x} = \lim_{x\to\infty} \left(\frac{e}{2}\right)^x = \infty$$

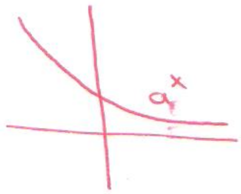
- $$\lim_{x\to\infty} \frac{2^x}{x^2} = \lim_{x\to\infty} \frac{2^x \ln 2}{2x} = \lim_{x\to\infty} \frac{2^x (\ln 2)^2}{2} = \infty$$

<u>Note</u>

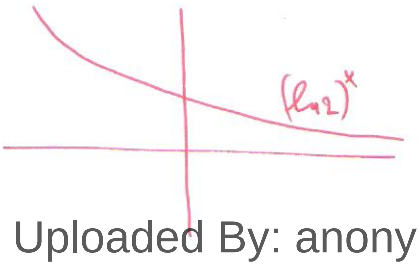
$\lim_{x\to\infty} a^x = \infty, \; a > 1$



$\lim_{x\to\infty} a^x = 0, \; 0 < a < 1$



- $$\lim_{x\to\infty} \frac{(\ln 2)^x}{x^2} = \lim_{x\to\infty} \frac{1}{x^2} \lim_{x\to\infty} (\ln 2)^x, \quad \text{since } 0 < \ln 2 < 1$$



$$= 0 \cdot 0$$

$$= 0$$