10.5
Monday, March 29, 2021 12:57 PM

RT : Ratio Test

$\sum a_n$ < conv / div

$a_n > 0 \quad \forall n$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \rho$$

- If $\rho < 1 \Rightarrow \sum a_n$ conv. ✓
- If $\rho > 1 \Rightarrow \sum a_n$ div ✓
- If $\rho = 1 \Rightarrow$ RT fails

RT $n!$

Exp Check Conv / Div.

① $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{e^n}$ Apply RT $\Rightarrow a_n = \frac{n^2}{e^n} > 0 \quad \forall \; n = 1, 2, \dots$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{(n+1)^2}{e^{n+1}}}{\frac{n^2}{e^n}} \div = \lim_{n\to\infty} \frac{(n+1)^2}{\cancel{e^{n+1}}} \cdot \frac{\cancel{e^n}}{n^2}$$

e

$$= \lim_{n\to\infty} \frac{1n^2 + 2n + 1}{e\, n^2} = \frac{1}{e} \; \boxed{<} \; 1$$

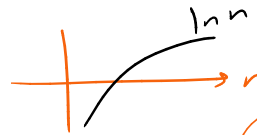
2.718

$\Rightarrow \sum \frac{n^2}{e^n}$ conv. by RT

⑥ $\displaystyle\sum^{\infty} \frac{3^{n+2}}{\ldots}$ Apply RT $\Rightarrow a_n > 0 \quad \forall \; n = 2, 3, \dots$

$\ln n$

⑥ $\sum_{n=2}^{\infty} \frac{3^{n+2}}{\ln n}$ $\to a_n$ Apply RT $\Rightarrow a_n > 0 \ \forall$ ...



$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\cancel{3^{n+3}}^{\,3}}{\ln(n+1)} \cdot \frac{\ln n}{\cancel{3^{n+2}}} = 3 \lim_{n\to\infty} \frac{\ln n}{\ln(n+1)} \quad \frac{\infty}{\infty}$$

$$= 3 \lim_{n\to\infty} \frac{\frac{1}{n}}{\frac{1}{n+1}} = 3 \lim_{n\to\infty} \frac{n+1}{n} = 3(1)$$

$$= 3 \boxed{>} 1$$

Hence, $\sum_{n=2}^{\infty} \frac{3^{n+2}}{\ln n}$ div by RT

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③ $\sum_{n=1}^{\infty} \frac{1}{n}$ "Div" since it is the harmonic series

$\to a_n > 0 \ \forall \ n = 1, 2, \ldots$ $\Rightarrow$ Apply RT

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{1}{n+1}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{n+1} = 1 \boxed{=} 1$$

RT fails to make decision so we try to apply another Test.

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④ $\sum_{n=1}^{\infty} \frac{n!}{e^n}$ apply RT $\Rightarrow a_n = \frac{n!}{e^n} > 0 \ \forall \ n = 1, 2, 3 \ldots$

$$\lim \frac{a_{n+1}}{\ldots} \quad \frac{(n+1)!}{} = (n+1)\cancel{n!} \quad \cancel{e^n} \quad \ldots \quad n+1 \quad = \infty > \boxed{1}$$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)!}{\cancel{e^{n+1}}} \cdot \frac{\cancel{e^{n}}}{\cancel{n!}} = \lim_{n\to\infty} \frac{n+1}{e} = \infty > \boxed{1}$$

Hence, $\sum \frac{n!}{e^n}$ div by RT

$n! = n\,(n-1)(n-2)(n-3)\cdots(3)(2)(1)$

$(n+1)! = (n+1)\underbrace{(n)(n-1)(n-2)\cdots(3)(2)(1)}_{n!}$

$(n+1)! = (n+1)\, n!$ ✓

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(20) (5) $\sum_{n=1}^{\infty} \frac{n!}{10^n}$ $\xrightarrow{a_n}$ Apply RT $\Rightarrow a_n > 0 \quad \forall\ n = 1, 2, \ldots$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} \div = \lim_{n\to\infty} \frac{\cancel{(n+1)!}^{\,n+1}}{\cancel{10^{n+1}}_{\,10}} \cdot \frac{\cancel{10^n}}{\cancel{n!}} = \frac{1}{10} \lim_{n\to\infty} (n+1) = \infty \boxed{>} 1$$

$\sum \frac{n!}{10^n}$ div by RT

$f(n) = 10^n$

$f(n+1) = 10^{n+1}$

$10^n + 1$ ✗

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(7) $\sum^{\infty} \left(\frac{3^n}{n^3}\right) \rightarrow a_n > 0 \quad \forall\ n = 1, 2, \ldots$ Apply Root Test

$\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$

(7) $\sum_{n=1} \left(\frac{3^n}{n^3}\right) \to a_n > 0 \ \forall n$

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[\cancel{n}]{\frac{3^{\cancel{n}}}{n^3}} = \lim_{n\to\infty} \frac{3}{\sqrt[n]{n^3}}$$

$$= 3 \lim_{n\to\infty} \frac{1}{(n^3)^{\frac{1}{n}}} = 3 \lim_{n\to\infty} \frac{1}{n^{\frac{3}{n}}} = \frac{3}{\lim_{n\to\infty} \left(n^{\frac{1}{n}}\right)^3}$$

$$= \frac{3}{\lim_{n\to\infty} \left(\sqrt[n]{n}\right)^3} = \frac{3}{1^3} = 3 \boxed{>} 1$$

$\sqrt[n]{n} \to 1$ " 10.1 "

$\Rightarrow \sum \frac{3^n}{n^3}$ div by Root Test

(15) $\sum_{n=1}^{\infty} \left(1 - \frac{1}{n}\right)^{n^2}$ → Root Test

$a_n = \left(1 - \frac{1}{n}\right)^{n^2}$

$a_1 = (1-1)^{n^2} = 0$ ✓

$a_2 = \left(1 - \frac{1}{2}\right)^2 > 0$

$a_3 = \left(1 - \frac{1}{3}\right)^2 > 0$

$\vdots$

$a_n > 0$ for larg n

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$$\lim_{n\to\infty} \sqrt[n]{a_n}$$

$$\lim_{n\to\infty} \sqrt[n]{\left(1 - \frac{1}{n}\right)^{n^2}}$$

$$\lim_{n\to\infty} \left[\left(1 - \frac{1}{n}\right)^{n^2}\right]^{\frac{1}{n}} = \lim_{n\to\infty} \left(1 - \frac{1}{n}\right)^{n}$$

$$= e^{-1} = \frac{1}{e} \boxed{<} 1$$

$\sum^{\infty} \left(1 - \frac{1}{n}\right)^{n^2}$ Con by Root Test

Hence $\sum_{n=1}^{\infty} \left(1-\frac{1}{n}\right)^{n^2}$ con by Root Test

30) $\sum_{n=1}^{\infty} \underbrace{\left(\frac{1}{n}-\frac{1}{n^2}\right)^n}_{a_n}$ Apply Root Test

$a_1 = (1-1)^1 = 0$

$a_2 = \left(\frac{1}{2}-\frac{1}{4}\right)^2 > 0$

$\vdots$

$a_n > 0$

$\lim_{n\to\infty} \sqrt[n]{a_n}$

$\lim_{n\to\infty} \left(\frac{1}{n}-\frac{1}{n^2}\right) = 0-0 = 0 \boxed{<} 1$

$\Rightarrow \sum \left(\frac{1}{n}-\frac{1}{n^2}\right)^n$ conv. by Root Test

Exp Recursive terms $a_1 = 2$, $a_{n+1} = \frac{2}{n} a_n$

$\frac{a_{n+1}}{a_n} = \frac{2}{n}$

Does $\sum_{n=1}^{\infty} a_n$ conv ?

Apply Ration $\Rightarrow \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{2}{n} = 0 \boxed{<} 1$

Yes it does conv. by RT