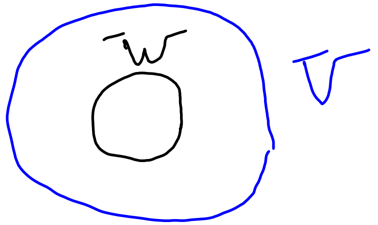
3.2 Subspaces

$W \subset V$



Subspaces

A subset W of a vector space V is called a subspace of V if W is itself a vector space under the addition and scalar multiplication defined on V.

Theorem.

If W is a set of one or more vectors in a vector space V, then W is a subspace of V if and only if the following conditions are satisfied.

(a) $\mathbf{0} \in W$.

(b) If $\mathbf{u}$ and $\mathbf{v}$ are vectors in W, then $\mathbf{u} + \mathbf{v}$ is in W.

(c) If k is a scalar and $\mathbf{u}$ is a vector in W, then $k\mathbf{u}$ is in W.

Remarks

1. In a vector space V, it can be readily verified that $\{\mathbf{0}\}$ and V are subspaces of V. All other subspaces are referred to as *proper subspaces*. We refer to $\{\mathbf{0}\}$ as the *zero subspace*.
2. To show that a subset S of a vector space forms a subspace, we must show that S is nonempty and that the closure properties (i) and (ii) in the definition are satisfied. Since every subspace must contain the zero vector, we can verify that S is nonempty by showing that $\mathbf{0} \in S$.

The vector space M_{mn}. $= \mathbb{R}^{m \times n}$

The set V of all $m \times n$ matrices with the usual matrix operations of addition and scalar multiplication is a vector space. We will denote this vector space by the symbol M_{mn}.

Example. If $V = M_{nn}$ and if $W = \{A \in M_{nn} \mid A^T = A\}$. Show that W is a subspace of the vector space V.

Remark. To show that W is not a subspace of a vector space V:

1) If $\mathbf{0} \notin W$, then $\boldsymbol{W}$ is not a subspace of a vector space $\boldsymbol{V}$.

2) If there exists a vector $\mathbf{u} \in W$ but $-\mathbf{u} \notin W$, then $\boldsymbol{W}$ is not a subspace of a vector space $\boldsymbol{V}$.

3) If there exist two vectors $\mathbf{u}, \mathbf{v} \in W$ but $\mathbf{u} + \mathbf{v} \notin W$, then $\boldsymbol{W}$ is not a subspace of a vector space $\boldsymbol{V}$.

Example. If $V = \mathbb{R}^2$ and if $W = \{(x, y) \in \mathbb{R}^2 \mid x \geq 0, \; y \geq 0\}$. Show that W is not a subspace of the vector space V.

Example. If $V = M_{nn}$ and if $W = \{A \in M_{nn} \ \mid \ A^{-1} \text{ exists}\}$. Determine whether W is a subspace of the vector space V or not.

Example. If $V = M_{22}$ and if $W = \{A \in M_{22} \ \mid \ |A| = 0\}$. Determine whether W is a subspace of the vector space V or not.

EXAMPLE 2 Let $S = \{(x_1, x_2, x_3)^T \mid x_1 = x_2\}$. The set S is nonempty since $\mathbf{0} = (0, 0, 0)^T \in S$. To show that S is a subspace of $\mathbb{R}^3$, we need to verify that the two closure properties hold:

(i) If $\mathbf{x} = (a, a, b)^T$ is any vector in S, then

$$\alpha\mathbf{x} = (\alpha a, \alpha a, \alpha b)^T \in S$$

(ii) If $(a, a, b)^T$ and $(c, c, d)^T$ are arbitrary elements of S, then

$$(a, a, b)^T + (c, c, d)^T = (a + c, a + c, b + d)^T \in S$$

Since S is nonempty and satisfies the two closure conditions, it follows that S is a subspace of $\mathbb{R}^3$. ■

Ex 3

$S' = \{(a, a, 1)^T : a \in \mathbb{R}\} \subset \mathbb{R}^3$

is not a subspace since $\mathbf{0} = (0, 0, 0)^T \notin S'$

EXAMPLE 3 Let

$$S = \left\{ \begin{pmatrix} x \\ 1 \end{pmatrix} \middle| \ x \text{ is a real number} \right\} \subset \mathbb{R}^2$$

$0 = (0,0)^T \notin S$

If either of the two conditions in the definition fails to hold, then S will not be a subspace. In this case the first condition fails since

$$\alpha \begin{pmatrix} x \\ 1 \end{pmatrix} = \begin{pmatrix} \alpha x \\ \alpha \end{pmatrix} \notin S \quad \text{when } \alpha \neq 1$$

Therefore, S is not a subspace. Actually, both conditions fail to hold. S is not closed under addition, since

$$\begin{pmatrix} x \\ 1 \end{pmatrix} + \begin{pmatrix} y \\ 1 \end{pmatrix} = \begin{pmatrix} x+y \\ 2 \end{pmatrix} \notin S$$

■

EXAMPLE 4 Let $S = \{A \in \mathbb{R}^{2\times 2} \mid a_{12} = -a_{21}\}$. The set S is nonempty, since O (the zero matrix) is in S. To show that S is a subspace, we verify that the closure properties are satisfied:

$\mathbb{R}^{2\times 2} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{R} \right\}$

$S \subset \mathbb{R}^{2\times 2}$

(i) If $A \in S$, then A must be of the form

$$A = \begin{bmatrix} a & b \\ -b & c \end{bmatrix}$$

and hence

$$\alpha A = \begin{bmatrix} \alpha a & \alpha b \\ -\alpha b & \alpha c \end{bmatrix}$$

Since the $(2,1)$ entry of αA is the negative of the $(1,2)$ entry, $\alpha A \in S$.

(ii) If $A, B \in S$, then they must be of the form

$$A = \begin{bmatrix} a & b \\ -b & c \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} d & e \\ -e & f \end{bmatrix}$$

It follows that

$$A + B = \begin{bmatrix} a+d & b+e \\ -(b+e) & c+f \end{bmatrix}$$

Hence $A + B \in S$.

EXAMPLE 5 Let S be the set of all polynomials of degree less than n with the property that $p(0) = 0$. The set S is nonempty since it contains the zero polynomial. We claim that S is a subspace of P_n. This follows, because

$S' \subset P_n$

(i) if $p(x) \in S$ and α is a scalar, then

$$\alpha p(0) = \alpha \cdot 0 = 0$$

and hence $\alpha p \in S$; and

(ii) if $p(x)$ and $q(x)$ are elements of S, then

$$(p + q)(0) = p(0) + q(0) = 0 + 0 = 0$$

and hence $p + q \in S$. ■

EX: Let S' be the set of all poly. of degree less than 5 with $P(0) = 1$. $\boxed{S' \subset P_5\ ?}$ S' is not a subspace

EXAMPLE 6 Let $C^{(n)}[a, b]$ be the set of all functions f that have a continuous nth derivative on $[a, b]$. We leave it to the reader to verify that $C^n[a, b]$ is a subspace of $C[a, b]$. ■

Example (Solution Spaces of Homogeneous Systems).

If $V = \mathbb{R}^n$, A is an $m \times n$ matrix, and

$$W = \{\mathbf{x} \in \mathbb{R}^n \mid A\mathbf{x} = \mathbf{0}\}$$

Determine whether W is a subspace of the vector space V or not.

Example. If $V = \mathbb{R}^n$, A is an $m \times n$ matrix, $\mathbf{b} \neq \mathbf{0}$ is an $m \times 1$ vector, and

$$W = \{\mathbf{x} \in \mathbb{R}^n \mid A\mathbf{x} = \mathbf{b}\}$$

Determine whether W is a subspace of the vector space V or not.

The Null Space of a Matrix

Let A be an $m \times n$ matrix. Let $N(A)$ denote the set of all solutions to the homogeneous system $A\mathbf{x} = \mathbf{0}$. Thus,

$$\boxed{N(A) = \{\mathbf{x} \in \mathbb{R}^n \mid A\mathbf{x} = \mathbf{0}\}} \subset \mathbb{R}^n$$

We claim that $N(A)$ is a subspace of $\mathbb{R}^n$. Clearly, $\mathbf{0} \in N(A)$, so $N(A)$ is nonempty. If $\mathbf{x} \in N(A)$ and α is a scalar, then

$$A(\alpha\mathbf{x}) = \alpha A\mathbf{x} = \alpha\mathbf{0} = \mathbf{0}$$

and hence $\alpha\mathbf{x} \in N(A)$. If $\mathbf{x}$ and $\mathbf{y}$ are elements of $N(A)$, then

$$A(\mathbf{x} + \mathbf{y}) = A\mathbf{x} + A\mathbf{y} = \mathbf{0} + \mathbf{0} = \mathbf{0}$$

Therefore, $\mathbf{x} + \mathbf{y} \in N(A)$. It then follows that $N(A)$ is a subspace of $\mathbb{R}^n$. The set of all solutions of the homogeneous system $A\mathbf{x} = \mathbf{0}$ forms a subspace of $\mathbb{R}^n$. The subspace $N(A)$ is called the *null space* of A.

EXAMPLE 9 Determine $N(A)$ if

$$A = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{bmatrix}$$

$Ax = 0$

$[A \mid 0] \xrightarrow{REF} [\quad]$

Solution

Using Gauss–Jordan reduction to solve $A\mathbf{x} = \mathbf{0}$, we obtain

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 0 \\ 2 & 1 & 0 & 1 & 0 \end{array}\right] \xrightarrow{R_2 - 2R_1} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & -2 & 1 & 0 \end{array}\right]$$

$$\xrightarrow{R_1 + R_2} \left[\begin{array}{cccc|c} 1 & 0 & -1 & 1 & 0 \\ 0 & -1 & -2 & 1 & 0 \end{array}\right] \xrightarrow{-R_2} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & \\ \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 2 & -1 & 0 \end{array}\right]$$

The reduced row echelon form involves two free variables, x_3 and x_4.

$$x_1 = x_3 - x_4$$
$$x_2 = -2x_3 + x_4$$

Thus, if we set $x_3 = \alpha$ and $x_4 = \beta$, then

$$\mathbf{x} = \begin{pmatrix} \alpha - \beta \\ -2\alpha + \beta \\ \alpha \\ \beta \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} -1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

is a solution of $A\mathbf{x} = \mathbf{0}$. The vector space $N(A)$ consists of all vectors of the form

$$\alpha \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} -1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

where α and β are scalars. ■

Theorem 3.2.2 *If the linear system* $A\mathbf{x} = \mathbf{b}$ *is consistent and* $\mathbf{x}_0$ *is a particular solution, then a vector* $\mathbf{y}$ *will also be a solution if and only if* $\mathbf{y} = \mathbf{x}_0 + \mathbf{z}$ *where* $\mathbf{z} \in N(A)$.

$Az=0$

Let $A\mathbf{x} = \mathbf{b}$ be a consistent linear system and let $\mathbf{x}_0$ be a particular solution to the system. If there is another solution $\mathbf{x}_1$ to the system, then the difference vector $\mathbf{z} = \mathbf{x}_1 - \mathbf{x}_0$ must be in $N(A)$ since

$$A\mathbf{z} = A\mathbf{x}_1 - A\mathbf{x}_0 = \mathbf{b} - \mathbf{b} = \mathbf{0}$$

Thus if there is a second solution, it must be of the form $\mathbf{x}_1 = \mathbf{x}_0 + \mathbf{z}$ where $\mathbf{z} \in N(A)$.

In general, if $\mathbf{x}_0$ is a particular solution to $A\mathbf{x} = \mathbf{b}$ and $\mathbf{z}$ is any vector in $N(A)$, then setting $\mathbf{y} = \mathbf{x}_0 + \mathbf{z}$, we have

$$A\mathbf{y} = A\mathbf{x}_0 + A\mathbf{z} = \mathbf{b} + \mathbf{0} = \mathbf{b}$$

So $\mathbf{y} = \mathbf{x}_0 + \mathbf{z}$ must also be a solution to the system $A\mathbf{x} = \mathbf{b}$.

($\Leftarrow$) Let $y = x_0 + z$ s.t. $z \in N(A)$

$$Ay = A(x_0 + z) = \underline{Ax_0} + \underline{Az} = b + 0 = b$$

$\therefore$ y is a solution for $Ax = b$.

($\Rightarrow$) Let y be a solution of $Ax = b$

Let $z = y - x_0$

$$Az = A(y - x_0) = Ay - Ax_0 = b - b = 0$$

$\Rightarrow$ $z \in N(A)$.

$Ax=0 \Leftrightarrow x=0$

If A is a 3×3 matrix such that $N(A) = \{0\}$, and $b = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$, then the system $\boxed{Ax = b}$ has exactly one solution. $\rightarrow$ $x = A^{-1}b$

- True (100%)
- False

If A is a 4×3 matrix such that $N(A) = \{0\}$, and $b = \begin{bmatrix} 0 \\ 3 \\ 2 \\ 1 \end{bmatrix}$, then the system $Ax = b$ has exactly one solution.

- True
- False (100%)

Linear Combination

Sometimes we will want to find the "smallest" subspace of a vector space V that contains all of the vectors in some set of interest. The following definition will help us to do that.

Definition.

If $\mathbf{w}$ is a vector in a vector space V, then $\mathbf{w}$ is said to be a linear combination of the vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_r$ in V if $\mathbf{w}$ can be expressed in the form

$$\mathbf{w} = k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots + k_r\mathbf{v}_r$$

where $k_1, k_2, \dots, k_r$ are scalars. These scalars are called the coefficients of the linear combination.

Example. Consider the vectors

$$\mathbf{u} = (1,2,-1) \quad \text{and} \quad \mathbf{v} = (6,4,2)$$

in $\mathbb{R}^3$. Show that $\mathbf{w} = (9,2,7)$ is a linear combination of $\mathbf{u}$ and $\mathbf{v}$.

$$W = K_1 u + K_2 V$$

$$(9,2,7) = (K_1, 2K_1, -K_1) + (6K_2, 4K_2, 2K_2)$$

$$= (K_1 + 6K_2, 2K_1 + 4K_2, -K_1 + 2K_2)$$

$$\Rightarrow \boxed{K_1 + 6K_2 = 9, \quad 2K_1 + 4K_2 = 2, \quad -K_1 + 2K_2 = 7}$$

$$\begin{bmatrix} 1 & 6 & 9 \\ 2 & 4 & 2 \\ -1 & 2 & 7 \end{bmatrix}, \begin{bmatrix} 1 & 6 & 9 \\ 0 & -8 & -16 \\ -1 & 2 & 7 \end{bmatrix}, \begin{bmatrix} 1 & 6 & 9 \\ 0 & -8 & -16 \\ 0 & 8 & 16 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 6 & 9 \\ 0 & 1 & 2 \\ 0 & 8 & 16 \end{bmatrix}, \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 2 \\ 0 & 8 & 16 \end{bmatrix}, \begin{array}{cc} K_1 & K_2 & \\ \end{array} \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow K_1 = -3, \quad K_2 = 2$$

$$W = -3u + 2V$$

Example. Consider the vectors

$$\mathbf{u} = (1,2,-1) \quad \text{and} \quad \mathbf{v} = (6,4,2)$$

in $\mathbb{R}^3$. Show that $\mathbf{w} = (4,-1,8)$ is not a linear combination of $\mathbf{u}$ and $\mathbf{v}$.

$$\begin{bmatrix} 1 & 6 & 4 \\ 2 & 4 & -1 \\ -1 & 2 & 8 \end{bmatrix}, \begin{bmatrix} 1 & 6 & 4 \\ 0 & -8 & -9 \\ -1 & 2 & 8 \end{bmatrix}, \begin{bmatrix} 1 & 6 & 4 \\ 0 & -8 & -9 \\ 0 & 8 & 12 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 6 & 4 \\ 0 & 1 & \frac{9}{8} \\ 0 & 8 & 12 \end{bmatrix}, \begin{array}{c} \begin{array}{cc} x_1 & x_2 \end{array} \\ \begin{bmatrix} 1 & 6 & 4 \\ 0 & 1 & \frac{9}{8} \\ 0 & 0 & 3 \end{bmatrix} \end{array}$$

$0 = 3$!

There is no solution.

Span

$S \subseteq V$

If $S = \{\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_r\}$ is a nonempty set of vectors in a vector space V and W is the set of all possible linear combinations of the vectors in S, then we say that the vectors $\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_r$ span W. We write

$$W = \text{span}\{\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_r\} \quad \text{or} \quad W = \text{span}(S).$$

Remark.

- $\mathbf{u} \in \text{span}\{\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_r\} \iff \mathbf{u} = k_1\mathbf{w}_1 + k_2\mathbf{w}_2 + \cdots + k_r\mathbf{w}_r,\ \ k_1, k_2, \dots, k_r \in \mathbb{R}$
- $W = \text{span}(S)$ is a subspace of V.
- The subspace W is the "smallest" subspace of V that contains all the vectors in S.
- $\text{span}(S)$ is called the subspace of V generated by S.

EXAMPLE 10 In $\mathbb{R}^3$, the span of $\mathbf{e}_1$ and $\mathbf{e}_2$ is the set of all vectors of the form

$$\alpha\mathbf{e}_1 + \beta\mathbf{e}_2 = \begin{bmatrix} \alpha \\ \beta \\ 0 \end{bmatrix}$$

$\text{span}(e_1, e_2) = \left\{ \begin{pmatrix} \alpha \\ \beta \\ 0 \end{pmatrix} : \alpha, \beta \in \mathbb{R} \right\}$

The reader may verify that $\text{Span}(\mathbf{e}_1, \mathbf{e}_2)$ is a subspace of $\mathbb{R}^3$. The subspace can be interpreted geometrically as the set of all vectors in 3-space that lie in the x_1x_2-plane (see Figure 3.2.1). The span of $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ is the set of all vectors of the form

$$\alpha_1\mathbf{e}_1 + \alpha_2\mathbf{e}_2 + \alpha_3\mathbf{e}_3 = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix}$$

Thus, $\text{Span}(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3) = \mathbb{R}^3$.

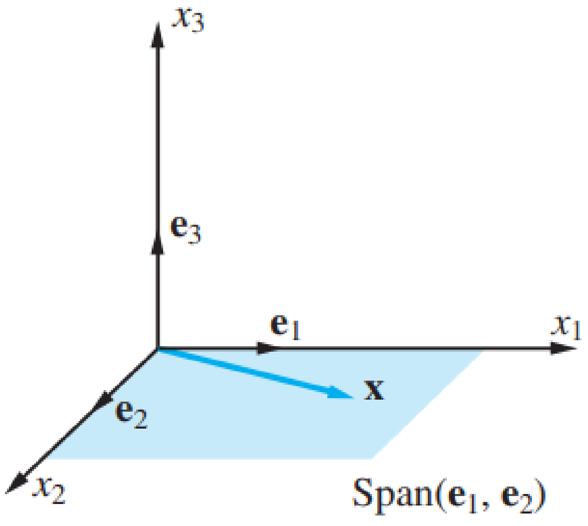


Figure 3.2.1.

Theorem 3.2.1 *If $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are elements of a vector space V, then* $\underbrace{\text{Span}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n)}$ W *is a subspace of V.*

① $\beta \in \mathbb{R}$

$v \in W$

$\Rightarrow \beta v \in W$??

② $v, w \in W$

$\Rightarrow v + w \in W$?

Proof Let β be a scalar and let $\mathbf{v} = \alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_n\mathbf{v}_n$ be an arbitrary element of $\text{Span}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n)$. Since

$$\beta\mathbf{v} = (\beta\alpha_1)\mathbf{v}_1 + (\beta\alpha_2)\mathbf{v}_2 + \cdots + (\beta\alpha_n)\mathbf{v}_n$$

it follows that $\beta\mathbf{v} \in \text{Span}(\mathbf{v}_1, \ldots, \mathbf{v}_n)$. Next, we must show that any sum of elements of $\text{Span}(\mathbf{v}_1, \ldots, \mathbf{v}_n)$ is in $\text{Span}(\mathbf{v}_1, \ldots, \mathbf{v}_n)$. Let $\mathbf{v} = \alpha_1\mathbf{v}_1 + \cdots + \alpha_n\mathbf{v}_n$ and $\mathbf{w} = \beta_1\mathbf{v}_1 + \cdots + \beta_n\mathbf{v}_n$.

$$\mathbf{v} + \mathbf{w} = (\alpha_1 + \beta_1)\mathbf{v}_1 + \cdots + (\alpha_n + \beta_n)\mathbf{v}_n \in \text{Span}(\mathbf{v}_1, \ldots, \mathbf{v}_n)$$

Therefore, $\text{Span}(\mathbf{v}_1, \ldots, \mathbf{v}_n)$ is a subspace of V. ■

Example. Determine whether the vectors

span $(v_1, v_2, v_3) = \mathbb{R}^3$??

$$\mathbf{v}_1 = (1,1,2), \quad \mathbf{v}_2 = (1,0,1), \quad \text{and} \quad \mathbf{v}_3 = (2,1,3)$$

span the vector space $\mathbb{R}^3$.

Let $(a, b, c) \in \mathbb{R}^3$

$(a, b, c) = k_1 v_1 + k_2 v_2 + k_3 v_3$

$$\begin{bmatrix} 1 & 1 & 2 & a \\ 1 & 0 & 1 & b \\ 2 & 1 & 3 & c \end{bmatrix}, \begin{bmatrix} 1 & 1 & 2 & a \\ 0 & -1 & -1 & b-a \\ 2 & 1 & 3 & c \end{bmatrix}, \begin{bmatrix} 1 & 1 & 2 & a \\ 0 & -1 & -1 & b-a \\ 0 & -1 & -1 & c-2a \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 2 & a \\ 0 & 1 & 1 & -b+a \\ 0 & -1 & -1 & c-2a \end{bmatrix}, \begin{array}{c} \begin{array}{ccc} k_1 & k_2 & k_3 \end{array} \\ \begin{bmatrix} 1 & 1 & 2 & a \\ 0 & 1 & 1 & -b+a \\ 0 & 0 & 0 & c-a-b \end{bmatrix} \end{array}$$

For $(a,b,c) \in \mathbb{R}^3$ where $c-a-b \neq 0$, the system has no solution.

$\therefore \quad \text{span}\{v_1, v_2, v_3\} \neq \mathbb{R}^3$.

Example. Determine whether the vectors

$$v_1 = (1,1,0), \quad v_2 = (1,0,1), \quad \text{and} \quad v_3 = (2,1,3)$$

span the vector space $\mathbb{R}^3$.

Let $(a, b, c) \in \mathbb{R}^3$

$(a, b, c) = k_1 v_1 + k_2 v_2 + k_3 v_3$

$$\begin{bmatrix} 1 & 1 & 2 & a \\ 1 & 0 & 1 & b \\ 0 & 1 & 3 & c \end{bmatrix}, \begin{bmatrix} 1 & 1 & 2 & a \\ 0 & -1 & -1 & b-a \\ 0 & 1 & 3 & c \end{bmatrix}, \begin{bmatrix} 1 & 1 & 2 & a \\ 0 & 1 & 1 & -b+a \\ 0 & 1 & 3 & c \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 2 & a \\ 0 & 1 & 1 & -b+a \\ 0 & 0 & 2 & c+b-a \end{bmatrix}, \begin{bmatrix} 1 & 0 & 1 & b \\ 0 & 1 & 1 & -b+a \\ 0 & 0 & 2 & c+b-a \end{bmatrix}, \begin{bmatrix} 1 & 0 & 1 & b \\ 0 & 1 & 1 & -b+a \\ 0 & 0 & 1 & \frac{1}{2}c+\frac{1}{2}b-\frac{1}{2}a \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & \frac{1}{2}b-\frac{1}{2}c+\frac{1}{2}a \\ 0 & 1 & 1 & -b+a \\ 0 & 0 & 1 & \frac{1}{2}c+\frac{1}{2}b-\frac{1}{2}a \end{bmatrix}, \quad \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & \frac{1}{2}b-\frac{1}{2}c+\frac{1}{2}a \\ 0 & 1 & 0 & -\frac{3}{2}b+\frac{3}{2}a-\frac{1}{2}c \\ 0 & 0 & 1 & \frac{1}{2}c+\frac{1}{2}b-\frac{1}{2}a \end{bmatrix} \end{array}$$

$$x_1 = \tfrac{1}{2}(b - c + a)$$
$$x_2 = \tfrac{1}{2}(-3b + 3a - c)$$
$$x_3 = \tfrac{1}{2}(c + b - a)$$

Ex: $w = (1,1,1)$

$= \frac{1}{2}v_1 - \frac{1}{2}v_2 + \frac{1}{2}v_3$

$\therefore \quad \operatorname{span}\{v_1, v_2, v_3\} = \mathbb{R}^3$

Example. Suppose that

$$\mathbf{v}_1 = (1,1,0,2), \quad \mathbf{v}_2 = (1,0,1,-1), \quad \text{and} \quad \mathbf{v}_3 = (1,-1,2,1).$$

Is the vector $\mathbf{w} = (2,2,4,9)$ in span$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

$$(2,2,4,9) = k_1 v_1 + k_2 v_2 + k_3 v_3$$

$$\begin{bmatrix} 1 & 1 & 1 & 2 \\ 1 & 0 & -1 & 2 \\ 0 & 1 & 2 & 4 \\ 2 & -1 & 1 & 9 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & -1 & -2 & 0 \\ 0 & 1 & 2 & 4 \\ 2 & -1 & 1 & 9 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & -1 & -2 & 0 \\ 0 & 1 & 2 & 4 \\ 0 & -3 & -1 & 5 \end{bmatrix}$$

$$
\begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 2 & 4 \\ 0 & -3 & -1 & 5 \end{bmatrix},
\begin{array}{c}
\begin{array}{cccc} K_1 & K_2 & K_3 & \end{array} \\
\begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 4 \\ 0 & -3 & -1 & 5 \end{bmatrix}
\end{array}
$$

$4 = 0$! this is a contradiction.

$\therefore$ there is no solution.

$\therefore \quad w \notin \text{span}\{v_1, v_2, v_3\}$

EXAMPLE 12 The vectors $1-x^2$, $x+2$, and x^2 span P_3. Thus, if ax^2+bx+c is any polynomial in P_3, it is possible to find scalars α_1, α_2, and α_3 such that

$$ax^2+bx+c=\alpha_1(1-x^2)+\alpha_2(x+2)+\alpha_3x^2$$

Indeed,

$$\alpha_1(1-x^2)+\alpha_2(x+2)+\alpha_3x^2=(\alpha_3-\alpha_1)x^2+\alpha_2x+(\alpha_1+2\alpha_2)$$

Setting

$$\left(\begin{array}{ccc|c} -1 & 0 & 1 & a \\ 0 & 1 & 0 & b \\ 1 & 2 & 0 & c \end{array}\right) \xrightarrow{REF}$$

$$\boxed{\begin{aligned} \alpha_3 - \alpha_1 &= a \\ \alpha_2 &= b \\ \alpha_1 + 2\alpha_2 &= c \end{aligned}}$$

and solving, we see that $\alpha_1=c-2b$, $\alpha_2=b$, and $\alpha_3=a+c-2b$. ■

EXERCISES

3. Determine whether the following are subspaces of $\mathbb{R}^{2\times 2}$:

(a) The set of all 2×2 diagonal matrices

(b) The set of all 2×2 triangular matrices

(c) The set of all 2×2 lower triangular matrices

$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} + \begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix} = \begin{pmatrix} a+c & 0 \\ 0 & b+d \end{pmatrix}$

$\begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 2 & 6 \end{pmatrix}$

5. Determine whether the following are subspaces of P_4 (be careful!):

(a) The set of polynomials in P_4 of even degree

$(x^2 + x) + (-x^2) = x$

(b) The set of all polynomials of degree 3

$(x^3 + x) + (-x^3) = x$

(c) The set of all polynomials $p(x)$ in P_4 such that $p(0) = 0$

(d) The set of all polynomials in P_4 having at least one real root

$(x^3 + x^2) + (-x^3 + 1) = x^2 + 1$

23. Let U and V be subspaces of a vector space W. Prove that their intersection $U \cap V$ is also a subspace of W.

24. Let S be the subspace of $\mathbb{R}^2$ spanned by $\mathbf{e}_1$ and let T be the subspace of $\mathbb{R}^2$ spanned by $\mathbf{e}_2$. Is $S \cup T$ a subspace of $\mathbb{R}^2$? Explain.

25. Let U and V be subspaces of a vector space W. Define

$$U + V = \{\mathbf{z} \mid \mathbf{z} = \mathbf{u} + \mathbf{v} \text{ where } \mathbf{u} \in U \text{ and } \mathbf{v} \in V\}$$

Show that $U + V$ is a subspace of W.

23 (i) $0 \in U \cap V$?

$\because 0 \in U, \; 0 \in V$

$\therefore 0 \in U \cap V$

(ii) Let $\beta \in \mathbb{R}, \; v \in U \cap V$

$\Rightarrow v \in U$ and $v \in V$

$\Rightarrow \beta v \in U$ and $\beta v \in V \Rightarrow \beta v \in U \cap V$

(iii) Let $u, v \in U \cap V$

$\Rightarrow u, v \in U$ and $u, v \in V$

$\Rightarrow u + v \in U$ and $u + v \in V$

$\Rightarrow u + v \in U \cap V$

(24) Let $S = \text{span}(e_1)$

and $T = \text{span}(e_2)$

$S \cup T$ is not a subspace of $\mathbb{R}^2$

since $e_1 \in S \cup T$

& $e_2 \in S \cup T$

but $e_1 + e_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \notin S \cup T$