Chapter 2 : sequences in $\mathbb{R}$.

2.1 : limits of sequences.

- An infinite sequenc (briefly, a sequence) is a function whose domain in $\mathbb{N} = \{1, 2, 3, \dots\}$
- A sequence $x_n := f(n)$ will be donated by $x_1, x_2, \dots$ OR $\{x_n\}_{n \in \mathbb{N}}$ OR $\{x_n\}$.

exp:

1. $\{1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots\}$ represents the sequence $\left\{\frac{1}{2^{n-1}}\right\}_{n \in \mathbb{N}}$ or $x_n = \frac{1}{2^{n-1}}$
2. $\{-1, 1, -1, 1, \dots\}$ is the seq. $\{(-1)^n\}_{n \in \mathbb{N}}$.
3. $\{1, 2, 3, 4, \dots\}$ is the seq. $\{n\}_{n \in \mathbb{N}}$.

Important : $\underbrace{\{x_n\}_{n \in \mathbb{N}}}_{\text{seq.}} \neq \underbrace{\{x_n : n \in \mathbb{N}\}}_{\text{set}}$.

exp: $\{(-1)^n\}_{n \in \mathbb{N}} \neq \{(-1)^n, n \in \mathbb{N}\}$

$\underbrace{\{1, -1, 1, -1, \dots\}}_{\text{Seq}} \neq \{-1, 1\}$ only.

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← الترتيب بـ seq مهم أما بـ set مش مهم.

exp: $\{1,2,3,\dots\}$ is different from $\{2,1,3,\dots\}$ as sequences.
But as sets $\{1,2,3,\dots\}$ is identical with $\{2,1,3,\dots\}$.

Def: $\lim_{n\to\infty} x_n = a \in \mathbb{R}$ or $x_n \to a$ as $n \to \infty$.

① A sequence of real numbers $\{x_n\}$ is said to be converges to $a \in \mathbb{R}$ iff $\forall\, \varepsilon > 0$, $\exists$ an $K \in \mathbb{N}$ (in general $K(\varepsilon)$) s.t

$n \geq K \Rightarrow |x_n - a| < \varepsilon$. بدنا نوجد K بدلالة ε

for large K.

توضيح → $\lim_{n\to\infty} \frac{1}{n} = 0$

as a seq. $1, \frac{1}{2}, \frac{1}{3}, \dots$

↳ بعد عدد كبير من n بتوصل صفر.

if $K = 10^6 \rightarrow x_n = 10^{-6} \rightarrow |x_n - a| = |10^{-6} - 0| < \varepsilon$.

كل ما كانت K كبيرة فبصير الفرق $(x_n - a)$ أقل من ε. ←

★ Notations:

a. $\{x_n\}$ converges to a

b. x_n converges to a.

c. $\lim_{n\to\infty} x_n = a$.

d. $x_n \to a$ as $n \to \infty$

e. the limit of $\{x_n\}$ exists and equals a.

كلهم رموز لـ converges بطرق مختلفة

عشان أثبت باستخدام الـ Def1 ← بوخذ ε>0 ونوجد K بحيث $n \geq K$ فبثبت $|x_n - a| < \varepsilon$.

K بتعتمد على ε

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ch1 ← بستخدم أرخميديان

Rmks:

(1) When $x_n \to a$ as $n \to \infty$, you can think of x_n as a seq. of approximations to a and ε as an upper bound for the error. $|x_n - a| < \varepsilon$

n large
$n \geq K$.

(2) The number K in Df ① is choosen so that the error is less than ε when $n \geq K$ In general, the smaller ε gets, the larger K must be.

(3) $x_n \to a$ iff $|x_n - a| \to 0$ as $n \to \infty$, In particular $x_n \to 0$ iff $|x_n| \to 0$ as $n \to \infty$.

(4) K depends on ε $K(\varepsilon)$ CANNot depent on $\boxed{n}$. $K = \frac{1}{\varepsilon} + n$ مستحيل

(5) (summary of Df ①). $x_n \to a \Leftrightarrow |x_n - a|$ is small for large n.

$\forall \varepsilon > 0, \exists K \in \mathbb{N}$

$n \geq K \Rightarrow |x_n - a| < \varepsilon$.

Note: $\frac{1}{n} \to 0$

كيف نختار K؟

كل ما كانت K كبيرة

K large $\to$ $|\frac{1}{n} - 0|$ small $\to$ so error small.

we need to prove !

exp 1: prove that $\frac{1}{n} \to 0$ as $n \to \infty$,

By using the Def's of a limit .

pf: let $\varepsilon > 0$ be given. we need to find $K \in \mathbb{N}$ s.t

$n \geq K \Rightarrow \left|\frac{1}{n} - 0\right| = \frac{1}{n} < \varepsilon$.

Use the Archimedean principle $\exists K \in \mathbb{N}$ s.t $K > \frac{1}{\varepsilon}$ $\quad 1 < K \varepsilon$ $\quad a, b \in \mathbb{R}$ $\exists n \in \mathbb{N}$ $b < na$

Now, $n \geq K \Rightarrow \frac{1}{n} \leq \frac{1}{K} < \varepsilon$. It follows that $\left|\frac{1}{n} - 0\right| = \frac{1}{n}$

$\frac{1}{n} \leq \frac{1}{K} < \varepsilon$, $\forall n \geq K$

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exp: Use the Def , prove that $\lim_{n\to\infty} \frac{2n^2+1}{3n^2} = \frac{2}{3}$.

pf:

let $\varepsilon > 0$ be given . we need to find a $K \in \mathbb{N}$ s.t

$n \geq K \Rightarrow \left|\frac{2n^2+1}{3n^2} - \frac{2}{3}\right| < \varepsilon$

طريقة ايجاد K

بتعتمد على ε مش ع n

لما نحل

$\left|\frac{2n^2+1}{3n^2} - \frac{2}{3}\right| < \varepsilon$

$\frac{1}{3n^2} < \varepsilon$

$3n^2 > \frac{1}{\varepsilon}$

$n > \frac{1}{\sqrt{3\varepsilon}}$

Use the Archimedean principle $\exists K \in \mathbb{N}$ s.t $K > \frac{1}{\sqrt{3\varepsilon}}$

Thus, $n \geq K$ implies $\left|\frac{3n^2+1}{3n^2} - \frac{2}{3}\right| = \left|\frac{6n^2+3-6n^2}{9n^2}\right|$

$$= \frac{1}{3n^2} \leq \frac{1}{3K^2}$$

$$= \frac{1}{3}\left(\frac{1}{K}\right)^2$$

$$< \frac{1}{3}\left(\sqrt{3\varepsilon}\right)^2 = \varepsilon$$

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exp: If $\lim_{n\to\infty} x_n = 2$. prove that $\lim_{n\to\infty}\left(\frac{2x_n+1}{x_n}\right) = \frac{5}{2}$

pf: let $\varepsilon > 0$ be given. since $\lim_{n\to\infty} x_n = 2$, Apply def ①, to this $\varepsilon>0$, $\exists K_1 \in \mathbb{N}$ s.t $n \geq K_1 \Rightarrow |x_n - 2| < \varepsilon$.

with $\varepsilon = 1$ (Next), $\exists K_2 \in \mathbb{N}$ s.t $n \geq K_2 \Rightarrow |x_n - 2| < 1 \rightsquigarrow 1 < x_n < 3$

let $K = \max\{K_1, K_2\}$ and suppose that $n \geq K$, then

$n \geq K_1$, $n \geq K_2$

$$\left|\frac{2x_n+1}{x_n} - \frac{5}{2}\right| = \left|\frac{4x_n + 2 - 5x_n}{2x_n}\right|$$

$$= \left|\frac{-x_n + 2}{2x_n}\right|$$

$$= \frac{|x_n - 2|}{2x_n}$$

$$< \frac{\varepsilon}{2x_n} < \varepsilon$$

$x_n > 1$

$2x_n > 2$

$\frac{1}{2x_n} < \frac{1}{2} < 1$

For all $n \geq K = \max\{K_1, K_2\}$.

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exp: show that the seq. $\{(-1)^n\}_{n\in\mathbb{N}}$ has No limit.

pf: spse that $(-1)^n \to \alpha$ as $n \to \infty$ for some $\alpha \in \mathbb{R}$.
Given $\varepsilon = 1$, $\exists\, K \in \mathbb{N}$ s.t $n \geq K \Rightarrow |(-1)^n - \alpha| < 1$

For n odd this implies $|1+\alpha| = |-1-\alpha| < 1$
and For n even this implies $|1-\alpha| < 1$.

Hence $2 = |1+1| = |1-\alpha+\alpha+1| \underbrace{\leq}_{\text{By Triangle}} |1-\alpha| + |\alpha+1|$

$< 1+1 = 2$

$\therefore$ $2 < 2$, a contradicts

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2.4 Remark: A sequence can have at most one limit.
pf: suppose that $x_n \to \alpha$ and $x_n \to \beta$ as $n \to \infty$ we need to prove that $\alpha = \beta$
since $x_n \to \alpha$ and $x_n \to \beta$, By def, $\forall\, \varepsilon > 0$, $\exists\, K \in \mathbb{N}$ s.t $n \geq K \to |x_n - \alpha| < \frac{\varepsilon}{2}$
and $|x_n - \beta| < \frac{\varepsilon}{2}$.

Now $|\alpha - \beta| = |(\alpha - x_n) + (x_n - \beta)|$
$\leq |\alpha - x_n| + |x_n - \beta|$ خصائص القيمة المطلقة
$= |x_n - \alpha| + |x_n - \beta|$
$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$

$\Rightarrow |\alpha - \beta| < \varepsilon$, $\forall \varepsilon > 0$ (Recall Thm 3 (1.2): $|a| < \varepsilon$, $\forall \varepsilon > 0$ iff $a = 0$).
$\Rightarrow \alpha - \beta = 0$ $\therefore \alpha - \beta = 0$
$\Rightarrow \alpha = \beta$ $\alpha = \beta$ ☒

So its unique ☒

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$=\{x_1, x_2, \dots\}$ $=\{x_{n_1}, x_{n_2}, x_{n_3}, \dots\}$

Def2: A subsequence of a sequence $\{x_n\}_{n\in\mathbb{N}}$ is a sequence of the form $\{x_{n_K}\}_{K\in\mathbb{N}}$ where each $n_K \in \mathbb{N}$ and $n_1 < n_2 < \dots$, Thus, a subsequence $x_{n_1}, x_{n_2}, \dots$ of $x_1, x_2, \dots$ is obtained by deleting from $x_1, x_2, \dots$ all x_n's except those such that $n = n_K$ for some K.

exp: $\{\overset{x_1}{1}, \underbrace{\overset{x_2}{\tfrac{1}{2}}}_{x_{n_1}}, \overset{x_3}{\tfrac{1}{3}}, \underbrace{\overset{x_4}{\tfrac{1}{4}}}_{x_{n_2}}, \overset{x_5}{\tfrac{1}{5}}, \dots\}$ is a seq, $x_n = \frac{1}{n}$

$\{\overset{x_{n_1}}{\tfrac{1}{2}}, \overset{x_{n_2}}{\tfrac{1}{4}}, \overset{x_{n_3}}{\tfrac{1}{6}}, \overset{x_{n_4}}{\tfrac{1}{8}}, \dots\}$ is a subsequence of $\{\overset{x_1}{1}, \overset{x_2}{\tfrac{1}{2}}, \overset{x_3}{\tfrac{1}{3}}, \overset{x_4}{\tfrac{1}{4}}, \overset{x_5}{\tfrac{1}{5}}, \dots\}$

exp: $\{1, 1, 1, 1, \dots\}$ is a subseq. of $\{-1, 1, -1, 1, -1, \dots\}$ by deleting every terms (set $n_K = 2K$) $x_{n_1} = x_2$ سالب

$n_1 = 2$
$n_2 = 4$

RMK: If $\{x_n\}_{n\in\mathbb{N}}$ converges to α and $\{x_{n_K}\}_{K\in\mathbb{N}}$ is any subseq. of $\{x_n\}_{n\in\mathbb{N}}$, then $x_{n_K} \to \alpha$ as $K \to \infty$.

proof: let $\varepsilon > 0$ be given. since $x_n \to \alpha$ as $n \to \infty$

$\exists$ a $K \in \mathbb{N}$ s.t $\boxed{n \geq K \Rightarrow |x_n - \alpha| < \varepsilon}$ $*$

since $n_K \in \mathbb{N}$ and $n_1 < n_2 < n_3 < \dots$, then

By induction, $n_k \geq k$, $\forall k \in \mathbb{N}$ (proof).

Hence, $\underset{\text{مسعول}}{k} \geq \underset{\text{دائبل}}{K} \Rightarrow n_k \geq K$

By $* \Rightarrow |x_{n_k} - \alpha| < \varepsilon$

i.e, $x_{n_k} \to \alpha$ as $k \to \infty$.

$\square$

عامي $\to$ $x_n \to \alpha$

$n \geq K \to |x_n - \alpha| < \varepsilon$

$k \geq K \Rightarrow |x_{n_k} - \alpha| < \varepsilon$.

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Def 3: let $\{x_n\}$ be a sequence of real numbers, Then

(i) $\{x_n\}$ is said to be bounded above iff the set $\{x_n : n \in \mathbb{N}\}$ is bounded above, i.e, iff $\boxed{\exists \text{ an } M \in \mathbb{R} \text{ s.t } x_n \leq M, \forall n \in \mathbb{N}}$.

(ii) $\{x_n\}$ is said to be bounded below iff the set $\{x_n : n \in \mathbb{N}\}$ is bounded below, i.e, iff $\boxed{\exists \text{ an } m \in \mathbb{R} \text{ s.t } x_n \geq m, \forall n \in \mathbb{N}}$.

(iii) $\{x_n\}$ is said to be bounded iff it is bounded both above and below, i.e $\boxed{\exists \text{ a } C > 0 \text{ s.t } |x_n| \leq C, \forall n \in \mathbb{N}}$.

Theorem:

every convergent sequence is bounded but the converse is not true.

proof: let $\{x_n\}$ be a seq. s.t $x_n \to \alpha \in \mathbb{R}$ as $n \to \infty$

let $\varepsilon = 1$ be given, $\exists$ a $K \in \mathbb{N}$ s.t $n \geq K \Rightarrow |x_n - \alpha| < 1$

Hence,
$$
\begin{aligned}
|x_n| &= |x_n - \alpha + \alpha| \\
&\leq |x_n - \alpha| + |\alpha| \\
&< 1 + |\alpha|
\end{aligned}
$$

$$\boxed{|x_n| < 1 + |\alpha| \quad \forall n \geq K}$$

and if $1 \leq n < K$, then

$$\boxed{|x_n| \leq \max\{|x_1|, |x_2|, \dots |x_K|\} := M}$$

$$\therefore |x_n| \leq \max\{M, 1 + |\alpha|\} := C$$

$\therefore$ $|x_n|$ is bounded and dominated by $C := \max\{M, 1+|\alpha|\}$

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RMK: The converse is False.

$$x_n = (-1)^n$$

$|x_n| = |(-1)^n| = 1$ is bdd.

But

$x_n = (-1)^n$ diverges.

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