3.3] Discrete Line spectral Representation

$$x(t) = \sum_{n=-\infty}^{\infty} X_n e^{jn\omega t} = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega t) + \sum_{n=1}^{\infty} b_n \sin(n\omega t)$$

$$X_n = \begin{cases} a_0 & n=0 \\ \dfrac{a_n - jb_n}{2} & n>1 \\ \dfrac{a_{-n} + jb_{-n}}{2} & n<-1 \end{cases}$$

$$\sum_{n=1}^{\infty} C_n \cos(n\omega t + \theta_n)$$

where $C_n = \sqrt{a_n^2 + b_n^2}$

$\theta_n = -\tan^{-1}(b_n / a_n)$

For $n>1$ or $n<-1 \Rightarrow X_n = \underbrace{\frac{1}{2}\sqrt{a_n^2+b_n^2}}_{|X_n|} \angle \underbrace{-\tan^{-1}(b_n/a_n)}_{\theta_{X_n}}$ → Double-sided spectral

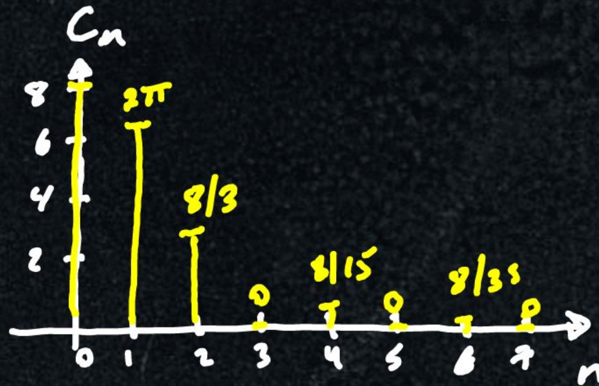
$\Rightarrow \boxed{C_n = 2|X_n| \quad \& \quad \theta_n = \theta_{X_n}}$ → single-sided spectral

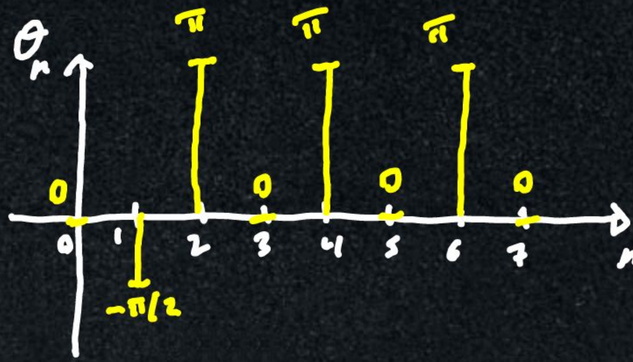
EX :- Determine and plot the single-sided spectral representation of the coefficients of FS defined by :

$$X_n = \begin{cases} \dfrac{4}{1-n^2} & n = 0, \mp 2, \mp 4, \dots \\ 0 & n \text{ odd } \& \ n \neq \mp 1 \\ -jn\pi & n = \mp 1 \end{cases}$$

$$C_n = 2|X_n| = \begin{cases} 8 & n = 0 \\ \dfrac{8}{n^2-1} & n = 2, 4, \dots \\ 2\pi & n = 1 \\ 0 & n = 3, 5, \dots \end{cases}$$

$$\theta_n = \theta_{X_n} = \begin{cases} 0 & n = 0 \\ \pi & n = 2, 4, \dots \\ -\pi/2 & n = 1 \\ 0 & n = 3, 5, \dots \end{cases}$$





3.4] Signal Power using Spectral Representation

Normalized average power:-

$$P = \frac{1}{T_o} \int_{T_o} |x(t)|^2 dt \quad \text{where } x(t) \text{ is periodic signal}$$

$$P = \frac{1}{T_o} \int_{T_o} x(t)\, x^*(t)\, dt = \frac{1}{T_o} \int_{T_o} x(t) \left[\sum_{n=-\infty}^{\infty} X_n^* e^{-jn\omega_o t} \right] dt$$

$$P = \sum_{n=-\infty}^{\infty} X_n^* \left[\frac{1}{T_o} \int_{T_o} x(t) e^{-jn\omega_o t} dt \right] = \sum_{n=-\infty}^{\infty} X_n X_n^* = \sum_{n=-\infty}^{\infty} |X_n|^2$$

Parseval's Theorem

$$X_{-n} = X_n^* \Rightarrow |X_{-n}| = |X_n^*| = |X_n|$$

$$P = \sum_{n=1}^{\infty} |X_{-n}|^2 + |X_o|^2 + \sum_{n=1}^{\infty} |X_n|^2 \Rightarrow \boxed{P = X_o^2 + 2 \sum_{n=1}^{\infty} |X_n|^2}$$

EX :- Calculate the average power of the signal $x(t)$ with the following FS cofficients :-

$$a_0 = a_n = 0 \ , \ b_n = \begin{cases} 0 & n \text{ even} \\ \frac{4A}{\pi n} & n \text{ odd} \end{cases}$$

$$X_n = \frac{a_n - jb_n}{2} = -j\frac{2A}{\pi n} \quad \text{for } n \text{ odd}$$

$$|X_n| = \frac{2A}{\pi n} \quad \text{for } n \text{ odd}$$

$$P = |X_0|^2 + 2\sum_{n=1}^{\infty} |X_n|^2 = 2\sum_{n=1,3,..}^{\infty}\left(\frac{2A}{\pi n}\right)^2 = \frac{8A^2}{\pi^2}\sum_{n=1,3,..}^{\infty}\frac{1}{n^2}$$

$$P = \frac{8A^2}{\pi^2}\sum_{k=0,1,..}^{\infty}\frac{1}{(2k+1)^2} = \frac{8A^2}{\pi^2}\left(\frac{\pi^2}{8}\right) = \boxed{A^2}$$

EX :- Calculate the average power contained in the frequency band $w \in [0, 80]$ and $w_0 = 15$ for $x(t)$ with the following FS coefficients :

$$a_0 = a_n = 0 \ , \ b_n = \begin{cases} 0 & n \text{ even} \\ \frac{4A}{\pi n} & n \text{ odd} \end{cases}$$

Assume $A = \pi/2$

$$n_{min} = \frac{0}{15} = 0 \qquad n_{max} = \text{Floor}\left(\frac{80}{15}\right) = 5$$

$$P = |X_0|^2 + 2\sum_{n=1}^{5} |X_n|^2$$

$$X_n = \frac{a_n - j b_n}{2} = \frac{-j2A}{\pi n} = -j\frac{1}{n} \Rightarrow |X_n| = \frac{1}{n}$$

$$P = 2\sum_{n=1,3,\dots}^{5} \frac{1}{n^2} = 2\left(\frac{1}{(1)^2} + \frac{1}{(3)^2} + \frac{1}{(5)^2}\right) = \boxed{2.3}\ W$$