Chapter 4: Differentiability on $\mathbb{R}$.

4.1: The Derivative.

Def 1: A real function f is said to be differentiable at a point $a \in \mathbb{R}$ iff f is defined on some open interval I containing a and

$$f'(a) := \lim \frac{f(a+h) - f(a)}{h} \quad \text{exsits.} \quad *$$

In this case $f'(a)$ is called the derivative of f at a.

OR $f'(a) = \lim\limits_{x \to a} \dfrac{f(x) - f(a)}{x - a} = L$ exists i.e:

تعتمد

$\forall \varepsilon > 0$, $\exists$ a $\delta(\varepsilon) > 0$ such that if $x \in I$ satisfies $|x - a| < \delta(\varepsilon)$ then

$$\left| \frac{f(x) - f(a)}{x - a} - L \right| < \varepsilon.$$

RMK:

1. the assumption that f be defined on an open interval continiuty a is made so that the quotients in $*$ are defined for all $h \neq 0$ sufficiently small. $a = 1 \to$ h is small. $1-\delta \quad 1 \quad 1+\delta$

هندسيا

2. The graph of $y = f(x)$ has a non-vertical tangent line at $(a, f(a))$ iff $f'(a)$ exists, in this case the slope of the tangent line is $f'(a)$.

let us consider a geometric interpretation of *

spse that f is diffble at a.

a secant line of the graph $y = f(x)$ is aline passing through at least two points on the graph, and a chord is aline segment which runs from one point on the graph to another.

let $x = a + h$.

The slope of the chord passing through $(x, f(x)), (a, f(a))$ is

$\rightsquigarrow \dfrac{f(x) - f(a)}{x - a}$, since $x = a + h$ (* be comes).

$\hookrightarrow h \to 0 \rightsquigarrow x \to a$.

$$\rightsquigarrow f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

Hence, as $x \to a$ the slopes of the chords through $(x, f(x))$ and $(a, f(a))$ approximate the slope of the tangent line of $y = f(x)$ at $x = a$.

Thus, the slope of the tangent line to $y = f(x)$ at $x = a$ is $f'(a)$.

- $y = f(x)$ has a unique tangent line at $(a, f(a))$ iff $f'(a)$ exists,
- If f is diffble at each point in E, then f' is a function on E.

Notations:

- $D_x f = \frac{df}{dx} = f^{(1)}(x) = \dot{f}(x) = \dot{y} = \frac{dy}{dx}$, when $y = f(x)$.

- Higher order derivatives are defined as $f^{(n+1)}(a) := (f^{(n)})'(a)$, $n \in \mathbb{N}$ provided these derivatives exists. $\ddot{f}(x) = (\dot{f}(x))'$.

Notation:

$D_x^n f$, $\frac{d^n f}{dx^n}$, $f^{(n)}$, and $\frac{d^n y}{dx^n}$, $y^{(n)}$ when $y = f(x)$.

exp: let $f(x) = x^2$, using the def. to show that $\dot{f}(a) = 2a$ for all $a \in \mathbb{R}$.

let $\varepsilon > 0$ and set $\delta = \varepsilon$

If $|x-a| < \delta$ then $\left| \frac{f(x) - f(a)}{x-a} - 2a \right|$

$$= \left| \frac{x^2 - a^2}{x-a} - 2a \right|$$

$$= |x + a - 2a|$$

$$= |x - a|$$

$$< \delta$$

$$< \varepsilon$$

$\therefore \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = 2a$ i.e $\dot{f}(a) = 2a$.

بدل من التعريف والعكس صحيح

Thm 1: A real function f is diffble at $x = a \in \mathbb{R}$ iff $\exists$ an open interval I and a function $F: I \to \mathbb{R}$ such that $a \in I$, f is defined on I, F is continuous at a and $f(x) = F(x)(x-a) + f(a)$ holds $\forall x \in I$, in which case $F(a) = \bar{f}(a)$.

$\hookrightarrow F(x) = \lim \dfrac{f(x) - f(a)}{x - a} = \bar{f}(a)$.

Proof:

$\Rightarrow$ suppose that f is diffble at a, Then f is defined on some open interval I containing a and $\bar{f}(a) = \lim\limits_{x \to a} \dfrac{f(x) - f(a)}{x - a}$ exists.

Define

$$F(x) = \begin{cases} \dfrac{f(x) - f(a)}{x - a}, & x \neq a \\ \bar{f}(a), & x = a \end{cases}$$

Then $f(x) = F(x)(x-a) + f(a)$, $\forall x \in I$

Moreover, F is contin. at a and $F(a) = \bar{f}(a)$ since

$$\lim_{x \to a} F(x) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \bar{f}(a) = F(a).$$

$\Leftarrow$ conversly, suppose that $\exists$ an open interval I and $F: I \to \mathbb{R}$ s.t $a \in I$, f is defined on I, F is cont. at a and $f(x) = F(x)(x-a) + f(a)$, $\forall x \in I$.

Then $F(x) = \dfrac{f(x) - f(a)}{x - a}$, $x \neq a$

continuity of F — def n of differtiabls.

$$F(a) = \lim_{x \to a} F(x) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \bar{f}(a).$$

$\therefore$ f is diffble at a and $\bar{f}(a) = F(a)$

$\to T(x) = \bar{f}(a)x$

والعكس صحيح **Thm2:** A real function f is diffble at $x=a$ iff $\exists$ a function T of the form $T(x) := mx$ such that $\lim_{h\to 0} \frac{f(a+h)-f(a)-T(h)}{h} = 0$.

$T(h) = mh$

الميل

Proof:

for $h\neq 0$ $\Rightarrow$ suppose that f is diffble at a. Define T as $T(x) = mx$ where $m = \bar{f}(a)$. Then $\frac{f(a+h)-f(a)-T(h)}{h} = \frac{f(a+h)-f(a)-\bar{f}(a)h}{h}$

$$= \frac{f(a+h)-f(a)}{h} - \bar{f}(a)$$

$\longrightarrow 0$ as $h\to 0$.

$\Leftarrow$ conversly, suppose that $\exists$ a function T of the form $T(x) = mx$ s.t $\lim \frac{f(a+h)-f(a)-T(h)}{h} = 0$,

Then for $h\neq 0$, $\frac{f(a+h)-f(a)}{h} = \frac{f(a+h)-f(x) - T(h)}{h} + \frac{T(h)}{h}$

$$= \frac{f(a+h)-f(a)-T(h)}{h} + \frac{m\cancel{h}}{\cancel{h}}$$

$$\lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = 0 + m = m$$

$\bar{f}(a) = m$

that is $\bar{f}(a)$ exists and equals m.

$\therefore f$ is diffble at $x=a$ ∎

يستخدم لاثبات conti وانه اثبت Not diff.

Thm 3: If f is diffble at a, then f is continuous at a.

proof: العكس غير صحيح ($f(x) = |x|$.) (exp:

suppose that f is diffble at a.

By Thm 1, $\exists$ an open interval I and a function F continuous at a s.t $f(x) = F(x)(x-a) + f(a)$, $\forall x \in I$.

Taking the limit as $x \to a$, we see:

$$\lim_{x\to a} f(x) = \left(\lim_{x\to a} F(x)\right) \lim_{x\to a} (x-a) + \lim_{x\to a} f(a).$$

$$= \underbrace{F(a)}_{\text{exist}} \cdot 0 + f(a).$$

$$= 0 + f(a)$$

$$= f(a).$$

Inparticular, $f(x) \to f(a)$ as $x \to a$, i.e f is cont. at a

TO

Not diff يستخدم لاثبات

exp: $\begin{cases} 1, & x > 0 \\ -1, & x < 0 \end{cases}$

Not diff becaus is Not cont.

RMK: The converse of thm 3 is False.

exp: $f(x) = |x|$ is continuous at 0 but it is not diffble at 0.

proof: since $x \to 0$, $|x| \to 0$, f is continuous at 0.

on the otherhand,

اليمين $f'_+(0) = \lim_{h\to 0^+} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0^+} \frac{|h| - |0|}{h} = \lim_{x\to 0^+} \frac{h}{h} = 1$.

اليسار $f'_-(0) = \lim_{h\to 0^-} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0^-} \frac{|h| - 0}{h} = \lim_{x\to 0^-} \frac{-h}{h} = -1$

since $f'_+(0) \neq f'_-(0)$, then $f'(0)$ does not exist,

There for f is not diffble at 0.

Def 2: let I be a nondegenerate interval

i. A function $f: I \to \mathbb{R}$ is said to be diffble on I iff

extended $\quad f'_I(a) := \lim\limits_{\substack{x \to a \\ x \in I}} \dfrac{f(x) - f(a)}{x - a}$ exists and is finite $\forall a \in I$. | اذا a كانت طرفية |

$[c, a]$ / $[a, \infty) \rightsquigarrow x \to a^+$ / $f(x) = x^2$ is diffble on $[1, 2]$?

$f'(a)^-$ $\qquad f'^+(1) \qquad f'(2)^-$

ii. f is said to be continuously diffble on I iff f'_I exist and is continuous on I. والعكس صحيح

$f(x) = x^2$ is cont. diff on $\mathbb{R}$?

$f'(x) = 2x$ is cont. on $\mathbb{R}$ and exists so $f(x)$ is cont. diffb. ✓

RMK: When a is not an endpoint of I, $f'_I(a)$ is the same as $f'(a)$.

- If f is diffble on $[a, b]$, Then

$$f'(a) = \lim_{h \to 0^+} \frac{f(a+h) - f(a)}{h} \quad \text{and} \quad f'(b) = \lim_{h \to 0^-} \frac{f(b+h) - f(b)}{h}.$$

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exp: show that $f(x) = x^{\frac{3}{2}}$ is diffble on $[0, \infty)$ and $f'(x) = \frac{3\sqrt{x}}{2}$, $\forall x \in [0, \infty)$.

proof: By The power Rule, $f'(x) = \frac{3}{2} x^{\frac{1}{2}} = \frac{3}{2}\sqrt{x}$, $\forall x \in (0, \infty)$.

مهم Remark $\quad f'(0) = \lim\limits_{h \to 0^+} \dfrac{f(0+h) - f(0)}{h} = \lim\limits_{h \to 0^+} \dfrac{h^{\frac{3}{2}} - 0}{h} = \lim\limits_{h \to 0^+} \sqrt{h} = 0.$

$\therefore f'(x) = \frac{3}{2}\sqrt{x} \quad \forall x \in [0, \infty).$

Notation : $C^n(I)$

let I be a nondegenerate interval , For $n \in \mathbb{N}$ we define the collection of functions $C^n(I)$ By :

$$C^n(I) := \{ f : f : I \to \mathbb{R} \text{ and } f^{(n)} \text{ exists and its continuous on } I \} .$$

- When $f \in C^n(I)$, $\forall n \in \mathbb{N}$, we shall denote it by $f \in C^\infty(I)$.

- Notice that $C'(I)$ is precisely the collection of real functions which are continuously diffble on I .

- $C^n([a,b]) = C^n[a,b]$.

- $C^\infty(I) \subset C^m(I) \subset C^n(I)$ for integer $m > n > 0$. exp : $C^3(I) \subseteq C^2(I)$.

- Not every function which is diffble on $\mathbb{R}$ belongs to $C'(\mathbb{R})$. Counterexample →

→

← C^2 موجودة في C^3 ؟؟ C^2 تحوي C^3

لم المشتقة الثالثة بس تكون موجودة ومتصلة اذاً أكيد الثانية موجودة ومتصلة .

لم بس لو المشتقة الثانية موجودة مش شرط الثالثة تكون موجودة .

exp: $f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & , x \neq 0 \\ 0 & , x = 0 \end{cases}$ is diffble on $\mathbb{R}$ but not continuously diffble on any interval contains the origin.

Sol:

$$f'(0) = \lim_{h \to 0} \frac{f(h+0) - f(0)}{h} = \lim_{h \to 0} \frac{h^2 \sin\left(\frac{1}{h}\right) - 0}{h}$$

$$= \lim_{h \to 0} h \sin \frac{1}{h} = 0 \quad \text{(By squeez Thm)}$$

$\therefore f'(0) = 0$ exists $\Rightarrow f$ is diffble at $x = 0$.

مشان نثبت انو cont

لازم نثبت :

$\lim_{x \to 0} f'(x) = \underbrace{f'(0)}_{=0 \text{ defined}}$

ONE

But $f'(x) = 2x \sin\left(\frac{1}{x}\right) - \cos \frac{1}{x}$, $x \neq 0$.

So not cont

$\lim_{x \to 0} f'(x)$ does Not exists so f' is not continuous on any interval containing the origin.

$\therefore f$ is diffble on $\mathbb{R}$ But it is Not continuously diffble on $\mathbb{R}$.

RMK: A function which is diffble on two sets is not necessarily diffble on their union. diff on A and diff on B But not necessary to diff on $A \cup B$

counterexp on RMK

exp: $\rightarrow$ f diffble on $[0,1]$, f diffble on $[-1,0]$

But not necessary diffb on $[0,1] \cup [-1,0]$.

exp: $f(x) = |x|$ is diffble on $[0,1]$ and on $[-1,0]$, But Not on $[-1,1]$.

proof:

$$f(x) = \begin{cases} x & , 0 \leq x \leq 1 \\ -x & , -1 \leq x < 0 \end{cases}$$

→ clear f is diffble on $[-1,0) \cup (0,1]$. على الفترة عدا الصفر

$$\left.\begin{aligned} \underset{[-1,0)}{f'(0)} &= \lim_{h \to 0^-} \frac{|h|}{h} = -1 \\ \underset{[0,1]}{f'(0)} &= \lim_{h \to 0^+} \frac{|h|}{h} = 1 \end{aligned}\right\} \neq$$

∴ f is diffble on $[0,1]$ and on $[-1,0]$ But Not on $[-1,1]$.