10.7 Power Series

Power Series is an infinite sum of Polynomials.

Def: A power series about $x=0$ is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n + \cdots.$$

Def: A power series about $x=a$ is a series of the form:

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_n(x-a)^n + \cdots$$

in which the center a and the coefficients $c_0, c_1, c_2, \ldots, c_n, \ldots$ are constants.

Example: If $c_0 = c_1 = \cdots = c_n = \cdots = 1$ and $a=0$

then $\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots$

first term in G.S

Which is Geometric series with $a = 1$ & $r = x$

$$\Rightarrow \sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \cdot \text{ for } -1 < x < 1.$$

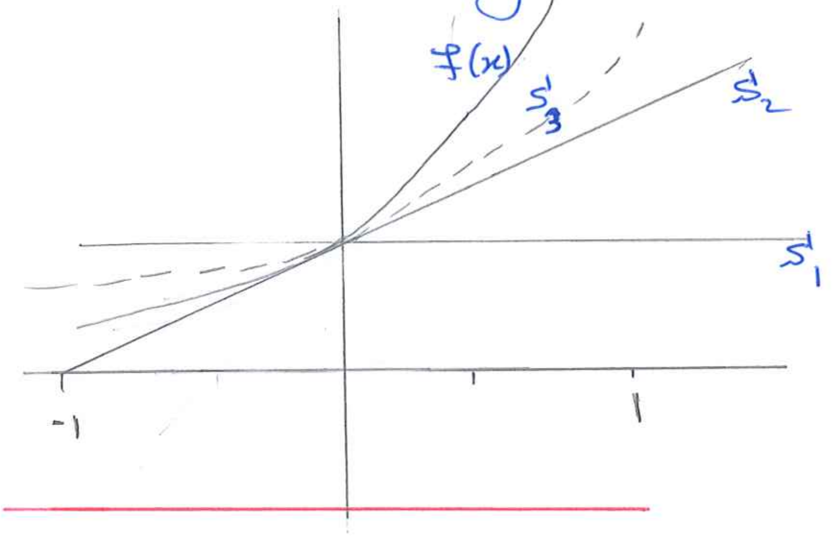
(Reciporcal Power Series).

Remark: If $f(x) = \frac{1}{1-x}$, $-1 < x < 1$ Then we can approximate $f(x)$ Using the nth partial sum: $S_1 = 1$

$S_2 = 1 + x$

$S_3 = 1 + x + x^2$

$\vdots$



Example: $\sum_{n=0}^{\infty} \left(\frac{-1}{2}\right)^n (x-2)^n$

$$= 1 - \frac{1}{2}(x-2) + \frac{1}{4}(x-2)^2 + \cdots$$

is a power series with $c_n = \left(-\frac{1}{2}\right)^n$ coefficients and center $= 2$.

Also, this series is Geometric Series with $a = 1$ and $r = -\frac{1}{2}(x-2)$. The Series Converges if

$$|r| < 1 \iff \left|-\frac{1}{2}(x-2)\right| < 1 \iff \frac{1}{2}|x-2| < 1$$

$$\iff |x-2| < 2 \iff -2 < x-2 < 2 \iff 0 < x < 4$$

$$\therefore \sum_{n=0}^{\infty} \left(\frac{-1}{2}\right)^n (x-2)^n = \frac{1}{1+\frac{1}{2}(x-2)} = \frac{2}{x} \, , \; 0 < x < 4.$$

Remark: The power Series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$$

Converges on an Interval around a ; $\underbrace{(a-R, a+R)}_{\Updownarrow \; |x-a|<R.}$;

and diverges outside the Interval.

- R is called Radius of Convergence.
- $(a-R, a+R)$ is called Interval of Convergence.
- At $x = a-R$ and $x = a+R$, we need to check for the Convergence.

Example: Find the Radius of Convergence and the Interval of Convergence for the following

$\boxed{1}$ $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$.

Applying Ratio Test with nonnegative terms:
(or Root Test).

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{n+1}\cdot\frac{n}{x^n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{n x}{n+1}\right| = |x| \lim_{n\to\infty}\left|\frac{n}{n+1}\right| = |x| < 1.$$

Thus : R (Radius of Convergence) = 1

center = 0 $\Rightarrow$ Interval of Convergence $(-1, 1)$. ??? ???

(The series Converges Absolutely on $(-1, 1)$).

At the end points :

If $\boxed{x = 1}$ $\Rightarrow \sum_{n=1}^{\infty} (-1)^{n-1} \cdot \frac{1}{n}$ which is Conditionally Convergent

Since its Alternating harmonic Series.

If $\boxed{x = -1}$ $\Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^{2n-1}}{n} = -1 \sum_{n=1}^{\infty} \frac{1}{n}$

which is Harmonic series (Divergent).

Thus , the Series Converges $-1 < x \leq 1$

$$= \begin{cases} \text{Abs. Converges} & (-1, 1) \\ \text{Converges} & (-1, 1] \\ \text{Conditionaly Conv.} & x = 1 \end{cases}$$

Example: (a) Find the Radius of Convergence and the Interval of Convergence for the following series:

(b) For what values of x does the series Converge absolutely ?

(c) For what values of x does the series Converges Conditionally ?

$\boxed{1}$ $\displaystyle\sum_{n=1}^{\infty} \frac{(3x-2)^n}{n}$, center $= \frac{2}{3}$

Q4

(a) Apply Ratio Test :

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(3x-2)^{n+1}}{(n+1)} \cdot \frac{n}{(3x-2)^n}\right|$$

$$= \lim_{n\to\infty} \left|\frac{(3x-2)\,n}{(n+1)}\right| = |3x-2| \lim_{n\to\infty}\left(\frac{n}{n+1}\right) = |3x-2| < 1$$

$$\Rightarrow \quad -1 < 3x-2 < 1 \quad \Leftrightarrow \quad \frac{1}{3} < x < 1$$ converges Absol.

At the end points:

If $\boxed{x = \frac{1}{3}} \Rightarrow \displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges by A.S.T.

If $\boxed{x = 1} \Rightarrow \sum_{n=1}^{\infty} \frac{1}{n}$ diverges Harmonic series

$\therefore$ Interval of Convergence is $\left[\frac{1}{3}, 1\right)$, $\left[\frac{1-\frac{1}{3}}{2}\right]$ $\boxed{R = \frac{1}{3}}$

(b) The Series Converges Absolutely : $\left(\frac{1}{3}, 1\right)$

(c) The Series Converges Conditionally : $x = \frac{1}{3}$.

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[2] $\displaystyle \sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$.

Using Ratio Test: $\displaystyle \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n}\right|$

$\displaystyle = \lim_{n\to\infty} \left|\frac{x}{n+1}\right| = |x| \lim_{n\to\infty} \frac{1}{n+1} = |x| \cdot 0 = 0 < 1$

Thus, $R = \infty$, (i.e) The series Converges absolutely for all x.

$\therefore$ Interval of Convergence : $(-\infty, \infty) = \mathbb{R}$.

$\boxed{3}$ $\sum_{n=0}^{\infty} n!\, x^n$

Apply Ratio Test :

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(n+1)!\; x^{n+1}}{n!\; x^n} \right| = |x| \lim_{n\to\infty} (n+1)$$

$$= \infty > 1$$

Thus, $R = 0$ . .

The series diverges $\forall x$ except when $\boxed{x = 0}$

$\boxed{4}$ $\sum_{n=0}^{\infty} \frac{(x-2)^n}{10^n}$

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(x-2)^{n+1}}{10^{n+1}} \cdot \frac{10^n}{(x-2)^n} \right| = \lim_{n\to\infty} \left| \frac{(x-2)}{10} \right|$$

$$= \left| \frac{x-2}{10} \right| < 1 \quad \Longleftrightarrow \quad |x-2| < 10$$

$\therefore$ $R = 10$ & The series Converges Absolutly : $(-8, 12)$

At $\boxed{x = -8}$: $\sum_{n=0}^{\infty} \frac{(-10)^n}{10^n} = \sum_{n=0}^{\infty} (-1)^n$ Diverges by

nth term test.

At $x = 12 \Rightarrow \sum_{n=0}^{\infty} \frac{10^n}{10^n} = \sum_{n=0}^{\infty} 1$

which is diverges by nth term test.

$\boxed{5}$ $\sum_{n=1}^{\infty} \frac{(-1)^n (4x-1)^n}{3^n \sqrt{n+1}}$

$$\lim_{n\to\infty} \left| \frac{(-1)^{n+1} (4x-1)^{n+1}}{3^{n+1} \sqrt{n+2}} \cdot \frac{3^n \sqrt{n+1}}{(-1)^n (4x-1)^n} \right|$$

$$= \lim_{n\to\infty} \frac{|4x-1|}{3} \frac{\sqrt{n+1}}{\sqrt{n+2}} = \frac{|4x-1|}{3} < 1$$

$$\Leftrightarrow -3 < 4x-1 < 3 \Leftrightarrow -\frac{1}{2} < x < 1.$$

Radius of Convergence $= \frac{1+\frac{1}{2}}{2} = \boxed{\frac{3}{4}} = R$

At $\boxed{x = -\frac{1}{2}} \Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^n (-3)^n}{3^n \sqrt{n+1}} = \sum_{n=1}^{\infty} \frac{(-1)^{2n}}{\sqrt{n+1}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1}}$

By L. C. T with $b_n = \frac{1}{\sqrt{n}}$, $\sum b_n$ diverges

& $\lim_{n\to\infty} \frac{1}{\sqrt{n+1}} \cdot \frac{\sqrt{n}}{1} = 1$

$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1}}$ Diverges.

At $\boxed{x=1} \Rightarrow \sum \dfrac{(-1)^n (3^n)}{3^n \sqrt{n+1}} = \sum_{n=1}^{\infty} \dfrac{(-1)^n}{\sqrt{n+1}}$

So, By A.S.T with $u_n = \dfrac{1}{\sqrt{n+1}}$

(i) $u_n > 0, \forall n$, (ii) $u_n' < 0$, (iii) $u_n \xrightarrow[n\to\infty]{} 0$

Therefore, at $x=1$, the series is Conditionally Convergent.

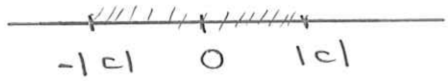
$\therefore$ Interval of Convergence is $\left(-\frac{1}{2}, 1\right]$

Theorem: The Convergence theorem for power series:

If $\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots$ Converges

at $x = c \neq 0$, then it converges absolutely

for all x with $|x| < |c|$



If the series diverges at $x = d$, then it diverges

for all x with $|x| > |d|$

Remark: In example $\boxed{2}$, the series Converges at $x=3$

$\Rightarrow$ it Converges absolutely for $|x| < 3$.

In example $\boxed{4}$, the series Converges at $x=3$,

$\Rightarrow$ it Converges absolutely for $|x-2| < 3$.

Corollary to the previous Theorem:

For a given power series $\sum_{n=0}^{\infty} c_n (x-a)^n$, there are only three posibilities:

1) There is a $R > 0$, such that the series diverges for x with $|x-a| > R$, but Converges absolutely for x with $|x-a| < R$.

The series May not Converges at the end points $x = a - R$ and $x = a + R$.

2) The series Converges absolutely for all $x \in (-\infty, \infty)$. In this case $\boxed{R = \infty}$.

3) The series Converges at $x = a$ and

diverges else where, in this case $\boxed{R = 0}$.

Operations on power Series.

Theorem: The Series Multiplication Theorem for Power Series: If $A(x) = \sum_{n=0}^{\infty} a_n x^n$ and $B(x) = \sum_{n=0}^{\infty} b_n x^n$ Converge absolutely for $|x| < R$, and

$$C_n = a_0 b_n + a_1 b_{n-1} + a_2 b_{n-2} + \cdots + a_{n-1} b_1 + a_n b_0$$

$$= \sum_{k=0}^{\infty} a_k b_{n-k},$$

then $\sum_{n=0}^{\infty} C_n x^n$ Converges Absolutely to $A(x)B(x)$ for $|x| < R$. That is:

$$\left(\sum_{n=0}^{\infty} a_n x^n \right) \cdot \left(\sum_{n=0}^{\infty} b_n x^n \right) = \sum_{n=0}^{\infty} c_n x^n$$

Example: Find the first four nonzero terms of

$$\left(\sum_{n=0}^{\infty} x^n\right)\cdot\left(\sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1}\right)$$

$$= \left(1+x+x^2+x^3+\cdots\right)\left(x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots\right)$$

$$= \left(x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots\right)+\left(x^2-\frac{x^3}{2}+\frac{x^4}{3}-\frac{x^5}{5}+\cdots\right)$$

$$+\left(x^3-\frac{x^4}{2}+\frac{x^5}{3}-\frac{x^6}{4}+\cdots\right)+\cdots$$

$$= x+\frac{x^2}{2}+\frac{5x^3}{6}-\frac{x^4}{6}+\cdots.$$

Theorem: If $\sum_{n=0}^{\infty} a_n x^n$ converges Absolutely for $|x|<R$, then:

$\sum_{n=0}^{\infty} a_n \left(f(x)\right)^n$ converges Absolutely for any continuous function f on $|f(x)|<R$.

Example: $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ Converges Absolutely

for $|x| < 1$. Then:

- $\frac{1}{1-4x^2} = \sum_{n=0}^{\infty} (4x^2)^n$ converges Absolutely

for $|4x^2| < 1 \iff |x| < \frac{1}{2}$.

- $\frac{1}{1-\ln x} = \sum_{n=0}^{\infty} (\ln x)^n$ Converges Absolutely

for $|\ln x| < 1 \iff -1 < \ln x < 1 \iff \frac{1}{e} < x < e$.

Theorem: The term by term Differentiation Theorem:

If $\sum c_n (x-a)^n$ has radius of Convergence $R > 0$,

it defines a function

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n \quad \text{on} \quad a-R < x < a+R.$$

This function f has derivatives of all orders

inside the interval such that:

$$f'(x) = \sum_{n=1}^{\infty} n c_n (x-a)^{n-1},$$

$$f''(x) = \sum_{n=2}^{\infty} n(n-1) c_n (x-a)^{n-2}$$

$$\vdots$$

Each of these derived series converges on $|x-a| < R$.

Example: find $f'(x)$ and $f''(x)$ if

$$f(x) = \frac{1}{1-x} = 1 + x + x^2 + \cdots = \sum_{n=0}^{\infty} x^n, \quad |x| < 1.$$

$$f'(x) = \frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + \cdots = \sum_{n=1}^{\infty} n x^{n-1}, \quad |x| < 1$$

$$f''(x) = \frac{2}{(1-x^3)} = 2 + 6x + 12x^2 + \cdots = \sum_{n=2}^{\infty} n(n-1) x^{n-2}, \quad |x| < 1$$

Example: Find the sum $\sum_{n=1}^{\infty} \frac{n}{2^{n-1}}$

$$\sum_{n=1}^{\infty} \frac{n}{2^{n-1}} = \sum_{n=1}^{\infty} n \left(\frac{1}{2}\right)^{n-1}$$

Using $\sum_{n=1}^{\infty} n x^{n-1} = \frac{1}{(1-x)^2}, \quad |x| < 1$

substitute $x = \frac{1}{2}$, then

$$\sum_{n=1}^{\infty} n\left(\frac{1}{2}\right)^{n-1} = \frac{1}{\left(1-\frac{1}{2}\right)^2} = \boxed{4}.$$

Caution: Term by term differentiation theorem might not work for other kind of series.

For example: $f(x) = \sum_{n=1}^{\infty} \frac{\sin(n!\,x)}{n^2}$ Converges $\forall x$.

But $f'(x) = \sum_{n=1}^{\infty} \frac{n!\,\cos(n!\,x)}{n^2}$ diverges $\forall x$.

Theorem: The Term by Term Integration Theorem:

Suppose that $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$ converges for $|x-a| < R$, $(R > 0)$. Then

$\sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}$ converges for $|x-a| < R$

and $\int f(x)\,dx = \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1} + C$

for $|x-a| < R$.

Example: Find a power series for $f(x) = \ln(1+x)$.

We know that:

$$\ln(1+x) = \int_0^x \frac{1}{1+t}\,dt.$$

But $\frac{1}{1+t} = 1 - t + t^2 - t^3 + \cdots$, Converges $|t| < 1$.

$$\Rightarrow \ln(1+x) = \int_0^x \frac{1}{1+t}\,dt = \int_0^x (1 - t + t^2 - t^3 + \cdots)\,dt$$

$$= t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + \cdots \Big]_0^x$$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

تحفظ

Therefore: $\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n}$, $|x| < 1$

Example: Find a power series for $f(x) = \tan^{-1} x$.

We know that:

$$\tan^{-1} x \underset{+C}{} = \int \frac{1}{1+x^2}\,dx$$

$$= \int \sum_{n=0}^{\infty} (-x^2)^n \, dx \quad , \quad |-x^2| < 1$$

$$= \int \sum_{n=0}^{\infty} (-1)^n x^{2n} \, dx \quad , \quad |x| < 1$$

$$\Rightarrow \quad \tan^{-1} x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} + C \, , \quad |x| < 1$$

If $x = 0$, then $\tan^{-1} 0 = 0 + C \Rightarrow C = 0$

ثبتها $\therefore$

$$\tan^{-1} x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} \, , \quad |x| < 1 .$$

Example: Identify the function

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} + \cdots$$

Notice that :

$$f'(x) = 1 - x^2 + x^4 - x^6 + \cdots \quad , \quad |x| < 1$$

which is a Geometric series $\Rightarrow f'(x) = \dfrac{1}{1+x^2}$.

Hence $f(x) = \int f'(x) \, dx = \int \frac{1}{1+x^2} \, dx = \tan^{-1} x + C$

Notice that $f(0) = 0 \Rightarrow \boxed{C = 0}$

$\therefore \; f(x) = \tan^{-1} x$.

Example: Find $\sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{\sqrt{3}}{2}\right)^{2n+1}}{2n+1}$.

$\Rightarrow$ sum $= \tan^{-1}\left(\frac{\sqrt{3}}{2}\right)$

Example: Find $\sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1}$

$\Rightarrow$ sum $= \tan^{-1}(1) = \frac{\pi}{4}$.

Example: Write $\ln\left(\frac{3}{2}\right)$ as an infinite series.

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n}, \quad |x| < 1$$

substitute $\boxed{x = \frac{1}{2}}$ $\Rightarrow \ln\left(1+\frac{1}{2}\right) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}\left(\frac{1}{2}\right)^n}{n}$

Example: Represent the following functions as power series.

$\boxed{1}$ $f(x) = \frac{x^2}{1+x} = x^2 \sum_{n=0}^{\infty} (-1) x^n, \quad |x| < 1$

$$= \sum_{n=0}^{\infty} (-1)^n x^{n+2}, \quad |x| < 1.$$

$\boxed{2}$ $g(x) = \dfrac{x}{(1-x)^2}$

We know $\dfrac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad , |x|<1$

$$\left(\frac{1}{1-x}\right)' = \frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} n\, x^{n-1}$$

$\therefore g(x) = x \cdot \sum_{n=0}^{\infty} n\, x^{n-1} = \sum_{n=1}^{\infty} n\, x^n \quad , |x|<1$

$\boxed{3}$ $h(x) = \dfrac{1}{x}$

$$\frac{1}{x} = \frac{1}{1-(1-x)} = \sum_{n=0}^{\infty} (1-x)^n \quad , \quad |1-x|<1$$

$$= \sum_{n=0}^{\infty} (1-x)^n \quad , \quad 0 < x < 2$$

Example: Find the Radius and the Interval of Convergence of the following series:

1. $$\sum_{n=1}^{\infty} \frac{(-1)^n \, n \, (3x-1)^n}{n^3+1}$$

Using Ratio test:

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(-1)^{n+1}(n+1)(3x-1)^{n+1}}{\left[(n+1)^3+1\right]} \cdot \frac{(n^3+1)}{(-1)^n n (3x-1)^n}\right|$$

$$= \lim_{n\to\infty} |3x-1| \left(\frac{n+1}{n}\right)\cdot\left(\frac{n^3+1}{(n+1)^3+1}\right) = |3x-1| < 1$$

$$\Rightarrow \quad -1 < 3x-1 < 1 \quad \Leftrightarrow \quad 0 < x < \frac{2}{3}$$

$\therefore$ Radius of Convergence: $\left(\frac{2}{3}-0\right)/2 = \boxed{\frac{1}{3}} = R$.

At $\boxed{x=0}$ $\Rightarrow \sum_{n=1}^{\infty} \frac{n}{n^3+1}$ (Converges).

Using L.C.T with $b_n = \frac{1}{n^2}$, $\Rightarrow \lim_{n\to\infty} \frac{a_n}{b_n} = 1$

Since $\sum_{n=1}^{\infty} b_n$ Converges, then $\sum_{n=1}^{\infty} \frac{n}{n^3+1}$ Converges.

At $\boxed{x=\frac{2}{3}}$ $\Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^n n}{n^3+1}$ (Converges)

Since the Absolute values of the series Converges

$\Rightarrow$ Interval of Convergence $= \left[0, \frac{2}{3}\right]$.

(2) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{\sqrt{n+1}} (e-x)^n$

The series has the form:

$$\sum_{n=1}^{\infty} \frac{(-1)^n \, n}{\sqrt{n+1}} (x-e)^n$$ (so its power series).

Using Ratio Test:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}(n+1)\cdot(x-e)^{n+1}}{\sqrt{n+2}} \cdot \frac{\sqrt{n+1}}{(-1)^n \, n \, (x-e)^n}\right|$$

$$= |x-e| < 1 \Rightarrow e-1 < x < e+1$$

$\therefore$ Radius of Convergence $= \dfrac{e+1-(e-1)}{2} = \boxed{1}$

At $\boxed{x=e-1}$: $\Rightarrow \displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n \, n}{\sqrt{n+1}} (-1)^n = \sum_{n=1}^{\infty} \frac{n}{\sqrt{n+1}}$

which is divergent by nth term test ($\lim_{n\to\infty} a_n = \infty$).

At $x = e+1 \Rightarrow \displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n \, n}{\sqrt{n+1}}$, which is Alternating

series with $u_n = \dfrac{n}{\sqrt{n+1}} \longrightarrow \infty \neq 0$

so $\displaystyle\lim_{n\to\infty} \frac{(-1)^n n}{\sqrt{n+1}}$ DNE, so the series diverges.

$\therefore$ Interval of Convergence $(e-1, e+1)$.

Lecture Problems:

Q36 Find the radius and Interval of Convergence

for $\sum_{n=1}^{\infty} (\sqrt{n+1} - \sqrt{n})(x-3)^n$.

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$$= \sum_{n=1}^{\infty} \left(\frac{1}{\sqrt{n+1} + \sqrt{n}} \right) (x-3)^n$$

$$\lim_{n \to \infty} \left| \left(\frac{(x-3)^{n+1}}{\sqrt{n+2} + \sqrt{n+1}} \right) \cdot \left(\frac{\sqrt{n+1} + \sqrt{n}}{(x-3)^n} \right) \right|$$

$$= |x-3| \lim_{n \to \infty} \frac{\sqrt{n+1} + \sqrt{n}}{\sqrt{n+2} + \sqrt{n+1}} = |x-3| < 1$$

$$\Rightarrow \quad 2 < x < 4$$

At $\boxed{x=2} \Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n+1} + \sqrt{n}}$ which is Conditionally Converged check!!!

At $\boxed{x=4} \Rightarrow \sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1} + \sqrt{n}}$ which is divergent.

$\therefore$ Radius of Convergence $= \frac{4-2}{2} = 1$

Interval of Convergence $= [2, 4)$.

Q40 $\sum_{n=1}^{\infty}\left(\frac{n}{n+1}\right)^{n^2}x^n$. Find the Radius of Convergence.

$$\lim_{n\to\infty}\sqrt[n]{a_n}<1 \iff \lim_{n\to\infty}\sqrt[n]{\left(\frac{n}{n+1}\right)^{n^2}\cdot x^n}<1$$

$$=|x|\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n<1 \quad \ldots(*)$$

Now, Let us find $\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n$:

$$\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n=\lim_{n\to\infty}e^{n\ln\left(\frac{n}{1+n}\right)}=\lim_{n\to\infty}e^{\frac{\ln n-\ln(n+1)}{\frac{1}{n}}}$$

$$\overset{L.H}{=}\lim_{n\to\infty}e^{\frac{\frac{1}{n}-\frac{1}{n+1}}{-\frac{1}{n^2}}}=\lim_{n\to\infty}e^{-\frac{n^2}{n(n+1)}}=\boxed{e^{-1}} \text{ (Converges)}$$

Back to $(*)$ $|x|e^{-1}<1 \iff |x|<e$

$\therefore$ $-e<x<e$, with radius of Convergence $=e$

Recall : $\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n=e^x$

If $x=1 \Rightarrow \lim_{n\to\infty}\left(\frac{n+1}{n}\right)^n=e$

$$\Rightarrow \lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n=e^{-1}$$

10.8 Taylor and Maclaurin Series:

Question: If a function $f(x)$ has derivatives of all orders on an Interval I, can it be expressed as a power series on I?

If it can, what will its coefficients be?

The following definition answer the question.

Def: Let f be a function with derivatives of all orders throughout some interval containing a as an interior point. Then the Taylor series generated by f at $x = a$ is

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k = f(a) + f'(a)(x-a)$$

$$+ \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots .$$

Def: The Maclaurin series generated by f is

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k = f(0) + f'(0)x + \frac{f''(0)x^2}{2!} + \cdots$$

$$\cdots + \frac{f^{(n)}(0)}{n!} x^n + \cdots .$$

Which is Taylor series of f at $x = 0$.

Def: Let f be a function with derivatives of order k for $k = 1, 2, \ldots, N$ in some interval Containing a as an interior point. Then for any integer n from 0 through N, the Taylor polynomial of order n generated by f at $x = a$ is the polynomial:

$$P_n(x) = f(a) + f'(a)(x-a) + \cdots + \frac{f^{(k)}(a)}{k!}(x-a)^k + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n .$$

Remark: $f(x)$ Can be approximated using polynomials:

That is: $P_0(x) = f(a)$.

$P_1(x) = f(a) + f'(a)(x-a)$. "Linearization"

$P_2(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2$.

$\vdots$

Example: $\boxed{1}$ Find the Taylor series of $f(x) = e^x$

at $x = 0$? "Maclaurin Series"

$f(x) = e^x$, $f'(x) = e^x$, $f''(x) = e^x$, ---

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)x^3}{3!} + \cdots$$

$$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$= \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

$\boxed{2}$ Find Taylor polynomial of order : 0, 1, 2, 3.

$P_0(x) = 1$.

$P_1(x) = 1 + x$.

$P_2(x) = 1 + x + \frac{x^2}{2}$.

$P_3(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} \approx e^x$

Notice that More terms is more accurate approx.

We say of order n, not of degree n, Since $f^{(n)}(a)$ maybe 0

Example: Find the Taylor Series of $f(x) = \frac{1}{x}$ at $x = 2$? Does the series Converge to $\frac{1}{x}$?

Sol: We need to find: $f(2)$, $f'(2)$, $f''(2)$, ...

$f(x) = x^{-1}$, $f'(x) = -x^{-2}$, $f''(x) = 2x^{-3}$, --

$\Rightarrow f(2) = \frac{1}{2}$, $f'(2) = \frac{-1}{4} = \frac{-1}{2^2}$, $f''(2) = \frac{2}{8} = \frac{1}{2^3}$ ...

$$\Rightarrow \sum_{k=0}^{\infty} \frac{f^{(k)}(2)}{k!}(x-2)^k = \frac{1}{2} - \frac{(x-2)}{2^2} + \frac{(x-2)^2}{2^3} + \cdots$$

Which is a geometric series, with $\overset{\text{First term}}{a} = \frac{1}{2} \neq 0$

and $r = \frac{-(x-2)}{2}$.

It converges absolutely if $|r| < 1 \iff |x-2| < 2$

$\iff 0 < x < 4$

and its sum is: $\frac{\frac{1}{2}}{1 + \frac{(x-2)}{2}} = \frac{1}{2\left(\frac{2+x-2}{2}\right)} = \frac{1}{x}$.

$\therefore$ Taylor series generated by $f(x) = \frac{1}{x}$ at $x = 2$

Converges to $\frac{1}{x}$, for $0 < x < 4$.

Example: Find Taylor Series generated by

$f(x) = \cos x$ at $x = 0$? "Maclaurin Series"

sol: $f(x) = \cos x$, $f'(x) = -\sin x$, $f''(x) = -\cos x$, ...

$\Rightarrow f(0) = 1$, $f'(0) = 0$, $f''(0) = -1$, $f'''(0) = 0$, ...

$$\therefore \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$= \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} .$$

(converges to $\cos x$, $\forall x$
Next section)

Remark: If we want to find First order Taylor polynomial generated by $f(x) = \cos x$ at $x = 0$

$P_1(x) = 1$, which has degree 0 not 1.

Remark: Taylor polynomials for $f(x) = \cos x$ of order $2n$ and

$2n+1$ are Identical , since :

$$f^{(2n)}(0) = (-1)^n \quad \text{while} \quad f^{(2n+1)}(0) = 0$$

$$\therefore P_{2n}(x) = P_{2n+1}(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + \frac{(-1)^n x^{2n}}{(2n)!} .$$

Remark: Maclaurin Series for $f(x) = \sin x$

is : $1 - x + \frac{x^3}{3!} - \frac{x^5}{5!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$. (check)

Converges to $\sin x$, $\forall x$ as $n \to \infty$

Example: Find Maclaurin Series for $\cosh x$.

Recall that $\cosh x = \frac{e^x + e^{-x}}{2}$

$$\Rightarrow \frac{1}{2}\left[\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right) + \left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots\right)\right]$$

$$= \frac{1}{2}\left[2 + \frac{2x^2}{2!} + 2\frac{x^4}{4!} + 2\frac{x^6}{6!} + \cdots\right]$$

$$= 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$$

Example : Find Maclaurin Series for $f(x) = \begin{cases} 0 & , x = 0 \\ e^{-\frac{1}{x^2}} & , x \neq 0 \end{cases}$

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k = 0 + 0.x + 0.x^2 + 0.x^3 + \cdots + 0.x^n + \cdots$$

So the series converges for every x (its sum $= 0$).

But converges to $f(x)$ only at $x = 0$.

Thus, Taylor series generated by $f(x)$ is not equal to $f(x)$ itself.

Example: Find Maclaurin Series for $f(x) = \ln\left(\frac{1+x}{1-x}\right)$

Notice that $f'(x) = \dfrac{2}{1-x^2} = \dfrac{a}{1-r}$

$\Rightarrow$ Geometric series :

$$\sum_{k=0}^{\infty} 2x^{2k} = 2 + 2x^2 + 2x^4 + 2x^6 + \cdots$$

$$\therefore \ln\left(\frac{1+x}{1-x}\right) = \int_0^x \frac{2}{1-t^2}\,dt$$

$$= 2x + \frac{2}{3}x^3 + \frac{2}{5}x^5 + \cdots = \sum 2\frac{x^{2k+1}}{2k+1}.$$

Example: Maclaurin series for $f(x) = x\sin x$.

we know that Maclaurin series for $\sin x$ is

$$1 - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$\Rightarrow$ Maclaurin series for $f(x)$ is

$$x - \frac{x^4}{3!} + \frac{x^6}{5!} - \frac{x^8}{7!} + \cdots + = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+2}}{(2k+1)!}.$$