Birzeit University
Mathematics Department
Math3331
Quiz#1

Instructor: Dr. Ala Talahmeh | **First Semester 2020/2021**
Name:... | **Number:**..............
Time: 70 minutes | **Date: 1/10/2020**

Exercise#1 [5 marks]. Solve the inequality

$$1 < x^2 < 4.$$

Exercise#2 [5 marks]. Let $x, y, a,$ and b be positive real numbers not equal to 0. Assume that $\frac{x}{y} < \frac{a}{b}$.

Show that

$$\frac{x}{y} < \frac{x+a}{y+b} < \frac{a}{b}.$$

Exercise#3 [5 marks]. Suppose that $\beta = \inf A < \infty$. Let $\varepsilon > 0$ be given. Prove that there is an $x \in A$ such that $\beta + \varepsilon > x$.

Exercise#4 [5 marks]. For each $n \in \mathbb{N}$, let $I_n = \left(-\frac{1}{n}, \frac{1}{n}\right)$. Show that

$$\bigcap_{n=1}^{\infty} I_n = \{0\}.$$

Exercise#5 [10 marks]. Consider the set

$$A = \left\{\frac{(-1)^n}{n} : n \in \mathbb{N}\right\}.$$

a. Show that A is bounded from above. Find the supremum. Is this supremum a maximum of A?

b. Show that A is bounded from below. Find the infimum. Is this infimum a minimum of A?

Good Luck

Quiz #1 (key)

Exercise #1. Solve $1 < x^2 < 4$.

Solution. $1 < x^2 < 4 = \{x : x^2 > 1\} \cap \{x : x^2 < 4\}$

$= \{x : |x| > 1\} \cap \{x : |x| < 2\}$

$= \underbrace{\{x : x < -1 \text{ or } x > 1\}} \cap \{x : -2 < x < 2\}$

$= (-\infty, -1) \cup (1, \infty) \quad \cap (-2, 2)$

$= \quad (-2, -1) \cup (1, 2)$

Exercise #2. First, let us prove $\frac{x}{y} < \frac{x+a}{y+b}$.

We have $x(y+b) - y(x+a) = xb - ya = yb\left(\frac{x}{y} - \frac{a}{b}\right) < 0$

Since $\frac{x}{y} < \frac{a}{b}$ and all numbers are positive. So,

$x(y+b) < y(x+a)$ or $\frac{x}{y} < \frac{x+a}{y+b}$.

The other inequality follows from the same ideas. $\blacksquare$

Exercise #3. Suppose not. That is, $\beta+\varepsilon \le x, \ \forall x \in A$. In this case, $\beta+\varepsilon$ is a lower bound of A. Thus, we must have $\beta+\varepsilon \le \beta$ (since $\beta = \inf A$ is the greatest lower bound). Thus, $\varepsilon \le 0$ which is impossible. $\blacksquare$

Exercise #4. $\bigcap_{n=1}^{\infty} I_n = \{0\}$, where $I_n = (-\frac{1}{n}, \frac{1}{n})$.

Proof. Clearly, $0 \in \bigcap_{n=1}^{\infty} I_n$. Suppose that $x \in \bigcap_{n=1}^{\infty} I_n$ with $x \neq 0$.

Case 1. $x > 0$. by Archimedean Principle $\exists$ a positive integer n such that $x > \frac{1}{n}$. But this implies $x \notin (-\frac{1}{n}, \frac{1}{n})$ and therefore $x \notin \bigcap_{n=1}^{\infty} I_n$.

Case 2 $x < 0$. Then $-x > 0$ and by Archimedean principle, $\exists$ a positive integer n such that $-x > \frac{1}{n}$ so that $x < -\frac{1}{n}$. Again this implies that $x \notin \bigcap_{n=1}^{\infty} I_n$.

Hence $\bigcap_{n=1}^{\infty} I_n = \{0\}$ $\blacksquare$

Exercise #5. $A = \left\{ \frac{(-1)^n}{n} : n \in \mathbb{N} \right\}$.

(a) Clearly, $\frac{1}{2}$ is an upper bound of A. Let $M > 0$ be an upper bound of A. We will show that $\frac{1}{2} \leq M$.

Suppose the contrary, i.e, $M < \frac{1}{2}$. Since M is an upper bound of A, we have $\frac{(-1)^n}{n} \leq M$, $\forall n \in \mathbb{N}$.

In particular, letting $n = 2$, we obtain $\frac{1}{2} \leq M < \frac{1}{2}$ which is impossible. Thus, $\frac{1}{2} \leq M$ so that $\sup A = \frac{1}{2}$.

Since $\sup A$ is an element of A, we conclude that

$\max A = \sup A = \frac{1}{2}$.

(b) Clearly, -1 is a lower bound of A. Let m be a lower bound of A. We will show that $m \leq -1$.

Suppose not, i.e, $m > -1$. Letting $n = 1$, we find that

$-1 = \frac{(-1)^1}{1} \geq m > -1$ which is impossible. Therefore, we must have $m \leq -1$. This proves that $\inf A = -1$.

Since $-1 \in A$, then $-1 = \min A$. $\blacksquare$