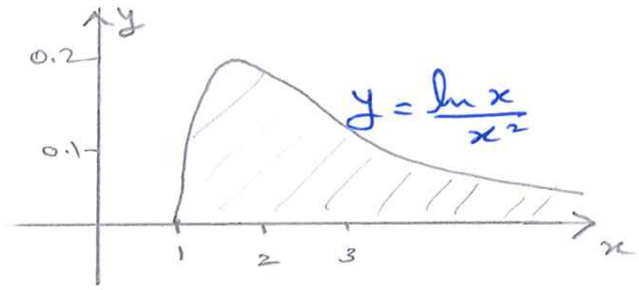
8.7 Improper Integrals:

We studied before the definite Integral such that the domain of integration $[a,b]$ is finite and the range of the Integral is finite.

But this is Not all the Cases. For example:

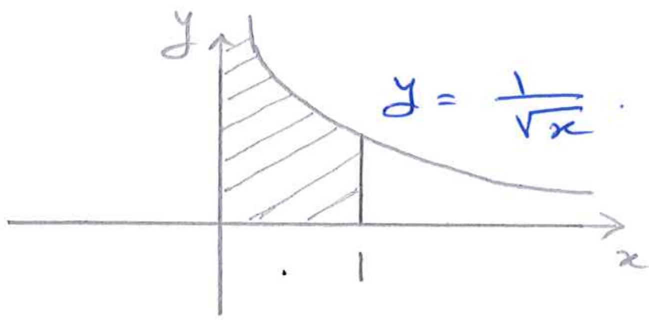
1) $y = \dfrac{\ln x}{x^2}$, $1 \leq x < \infty$ has Infinite domain

$$\text{Area} = \int_1^\infty \frac{\ln x}{x^2}\,dx$$



2) $y = \dfrac{1}{\sqrt{x}}$, $0 \leq x \leq 1$. The range of the Integral

$$\int_0^1 \frac{1}{\sqrt{x}}\,dx$$ is Infinite.



In each Case, the Integrals are said to be Improper

Def: Improper integrals of Type "I" are Integrals with infinite limits of integration.

(1). If $f(x)$ is Continuous on $[a, \infty)$, then

$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx$$

(2) If $f(x)$ is Continuous on $(-\infty, b]$, then

$$\int_{-\infty}^b f(x)\,dx = \lim_{a\to-\infty} \int_a^b f(x)\,dx$$

(3) If $f(x)$ is Continuous on $(-\infty, \infty)$, then

$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^c f(x)\,dx + \int_c^\infty f(x)\,dx$$

where $c \in \mathbb{R}$.

In each case, if the limit is finite, we say that the improper integral Converges, and that the limit is the value of the improper integral. "Area" $f(x) \geq 0$

If the Limit fails to exist, the Improper Integral diverges. "Infinite area" if $f(x) \geq 0$

Example: Is the area under the Curve

$y = \frac{\ln x}{x^2}$ from $x=1$ to $x=\infty$ finite ?

If so, what is its value ?

$$\text{Area} = \int_1^{\infty} \frac{\ln x}{x^2}\,dx \overset{\text{Type I}}{\underset{(1)}{=}} \lim_{b\to\infty} \int_1^{b} \frac{\ln x}{x^2}\,dx \quad \dots(*)$$

Let $u = \ln x$, $dv = \frac{1}{x^2}dx$

$du = \frac{1}{x}dx$, $v = \frac{-1}{x}$] By parts.

$$(*) = \lim_{b\to\infty}\left[-\frac{\ln x}{x}\Big|_1^b + \int_1^b \frac{dx}{x^2}\right]$$

$$= \lim_{b\to\infty}\left[-\frac{\ln b}{b}\right] + \lim_{b\to\infty}\left[-\frac{1}{x}\Big|_1^b\right]$$

$$= \lim_{b\to\infty}\left[-\frac{\ln b}{b}\right] + \lim_{b\to\infty}\left[-\frac{1}{b}+1\right]$$

$$\overset{L.H}{=} \lim_{b\to\infty}\left[\frac{-\frac{1}{b}}{1}\right] + 1$$

$$= 0 + 1 = \boxed{1} \quad (\text{finite})$$

So, the Improper Integral Converges and the area $= 1$.

Example : $\displaystyle\int_0^\infty \frac{dx}{x^2+1} = \lim_{b\to\infty} \int_0^b \frac{dx}{x^2+1}$

$$= \lim_{b\to\infty} \tan^{-1}x \Big|_0^b = \lim_{b\to\infty} \left[\tan^{-1}b - \tan^{-1}0\right]$$

$$= \lim_{b\to\infty} \left(\tan^{-1} b\right) = \boxed{\frac{\pi}{2}}$$

Example : $\displaystyle\int_{-\infty}^\infty \frac{dx}{1+x^2} = \int_{-\infty}^0 \frac{dx}{1+x^2} + \int_0^\infty \frac{dx}{1+x^2}$

$$= \frac{\pi}{2} + \frac{\pi}{2} = \boxed{\pi}$$

Example : $\displaystyle\int_0^\infty \frac{16\tan^{-1}x}{1+x^2}\,dx = \lim_{b\to\infty} \int_0^b \frac{16\tan^{-1}x}{1+x^2}\,dx$ ...(**)

Let $u = \tan^{-1}x$, then $du = \frac{1}{1+x^2}\,dx$

$$(**) = \lim_{b\to\infty} \int_0^b 16u\,du = \lim_{b\to\infty} \left[8u^2\right]_0^{\tan^{-1}b}$$

$$= \lim_{b\to\infty} 8\left[\left(\tan^{-1}b\right)^2 - \left(\tan^{-1}0\right)^2\right]$$

$$= 8\left(\frac{\pi}{2}\right)^2 = \boxed{2\pi^2}$$

Example: For what values of p does the Integral

$$\int_1^\infty \frac{dx}{x^p}$$ converge? When the Integral does converge, what is its Value?

sol: Assume $p \neq 1$, then

$$\int_1^\infty \frac{dx}{x^p} = \lim_{b\to\infty} \int_1^b x^{-p}\,dx = \lim_{b\to\infty} \frac{x^{-p+1}}{-p+1}\Big|_1^b$$

$$= \lim_{b\to\infty}\left[\frac{b^{-p+1}}{-p+1} - \frac{1}{-p+1}\right] = \frac{1}{1-p}\lim_{b\to\infty}\left[b^{1-p} - 1\right]$$

$$= \begin{cases} \dfrac{1}{p-1}, & p>1 \\ \infty, & p<1 \end{cases}$$

Conclusion: The Improper integral Converges to $\frac{1}{p-1}$ if $p>1$ and diverges if $p<1$

Now, if $p=1$, then

Diverges.

$$\int_1^\infty \frac{dx}{x} = \lim_{b\to\infty}\int_1^b \frac{dx}{x} = \lim_{b\to\infty} \ln|x|\Big|_1^b = \lim_{b\to\infty} \ln b = \infty$$

Summary for the previous example:

$$\int_1^{\infty} \frac{dx}{x^p} = \begin{cases} \text{divergent}, & \text{if } p \le 1 \\ \frac{1}{p-1}, & \text{if } p > 1 \end{cases}$$

We call this integral "p- integral" test

Example: $\int_1^{\infty} \frac{dx}{x^{2024}} = \frac{1}{2024-1} = \frac{1}{2023}$. Converges

But: $\int_1^{\infty} \frac{dx}{x^{0.9}}$ diverges.

Integrals with Vertical Asymptotes:

Def: Integrals of functions that become infinite at a point within the interval of integration are improper integrals of Type II.

1) If $f(x)$ is continuous on $(a, b]$ and discontinuous at a, then

$$\int_a^b f(x)\,dx = \lim_{c \to a^+} \int_c^b f(x)\,dx.$$

2). If $f(x)$ is continuous on $[a, b)$ and discontinuous at b, then $\displaystyle\int_a^b f(x)\,dx = \lim_{c \to b^-} \int_a^c f(x)\,dx$.

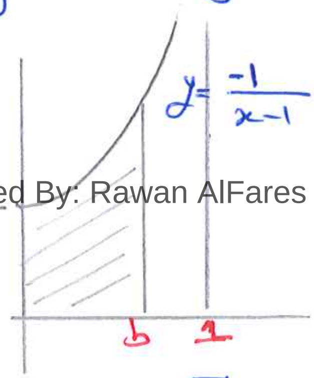
3) If $f(x)$ is discontinuous at c, where $a < c < b$ and continuous on $[a, c) \cup (c, b]$, then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

In each case, if the limit is finite, we say the improper integral converges, and that the limit is the value of the improper integral.

If the limit does not exist, the integral diverges.

Example: $\displaystyle\int_0^1 \frac{dx}{1-x} = \lim_{c \to 1^-} \int_0^c \frac{dx}{1-x}$



$$= \lim_{c \to 1^-} \Big[-\ln|1-x|\Big]_0^c = \lim_{c \to 1^-} \Big[-\ln|1-c| + \ln|1|\Big]$$

$$= \lim_{c \to 1^-} \left(-\ln|1-c|\right) = -(-\infty) = \infty \text{ (Diverges).}$$

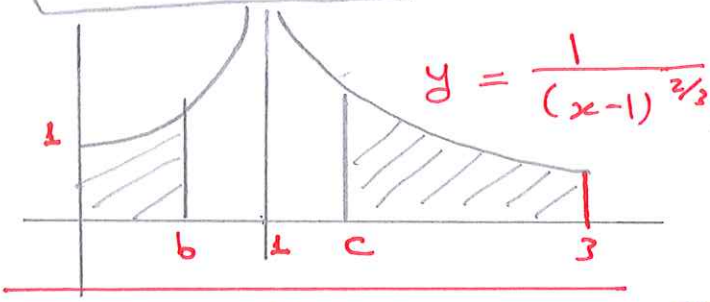
<u>Example</u>: $\displaystyle\int_0^3 \frac{dx}{(x-1)^{2/3}}$

$$\int_0^3 \frac{dx}{(x-1)^{2/3}} = \int_0^1 \frac{dx}{(x-1)^{2/3}} + \int_1^3 \frac{dx}{(x-1)^{2/3}}$$

$$= \lim_{b\to 1^-}\Big[3(x-1)^{\frac{1}{3}}\Big]_0^b + \lim_{c\to 1^+}\Big[3(x-1)^{\frac{1}{3}}\Big]_c^3$$

$$= \lim_{b\to 1^-}\Big[3(b-1)^{\frac{1}{3}}+3\Big] + \lim_{c\to 1^+}\Big[3(2)^{\frac{1}{3}} - 3(c-1)^{\frac{1}{3}}\Big]$$

$$= \boxed{3 + 3\sqrt[3]{2}}$$



<u>Example</u>: $$\int_1^\infty \frac{dx}{x\sqrt{x^2-1}} = \int_1^2 \frac{dx}{x\sqrt{x^2-1}} + \int_2^\infty \frac{dx}{x\sqrt{x^2-1}}$$

$$= \lim_{b\to 1^+}\int_b^2 \frac{dx}{x\sqrt{x^2-1}} + \lim_{c\to\infty}\int_2^c \frac{dx}{x\sqrt{x^2-1}}$$

$$= \lim_{b\to 1^+}\Big[\sec^{-1}|x|\Big]_b^2 + \lim_{c\to\infty}\Big[\sec^{-1}|x|\Big]_2^c$$

$$= \lim_{b\to 1^+}\Big[\cancel{\sec^{-1}2} - \sec^{-1}b\Big] + \lim_{c\to\infty}\Big[\sec^{-1}c - \cancel{\sec^{-1}2}\Big]$$

$$= \lim_{b\to 1^+}(-\sec^{-1}b) + \lim_{c\to\infty}\sec^{-1}c = 0 + \frac{\pi}{2} = \boxed{\frac{\pi}{2}}$$

Tests for Convergence and Divergence.

When we can't evaluate the Improper Integrals directly, we try to determine whether they are Convergence or divergence.

Using one of the following tests:

1) Direct Comparison Test (DCT)
2) Limit Comparison Test (LCT)

<u>Illustration:</u> $\int_1^{\infty} e^{-x^2}\,dx$. can't be evaluated directly, but for $x \geq 1$, we have

$$x \leq x^2 \Rightarrow -x \geq -x^2 \quad \cdots (*)$$

Now since e^x is Increasing function then

$$\cdots(*) \Rightarrow e^{-x} \geq e^{-x^2}, \quad \forall x \geq 1$$

$$\Rightarrow \int_1^{\infty} e^{-x^2}\,dx \leq \int_1^{\infty} e^{-x}\,dx \approx 0.368. \quad (201)$$

$\lim_{b\to\infty}\int_1^b$ Using substitution

Thm: Direct Comparison Test:

Let $f(x)$ and $g(x)$ be continuous on $[a,\infty)$

with $0 \leq f(x) \leq g(x)$, $\forall x \geq a$

1) If $\int_a^\infty g(x)\,dx$ Converges, then $\int_a^\infty f(x)\,dx$ Converges.

2) If $\int_a^\infty f(x)\,dx$ diverges, then $\int_a^\infty g(x)\,dx$ diverges.

Examples on Test the Convergence

1) $\displaystyle\int_1^\infty \frac{\sin^2 x}{x^2}\,dx$.

we know : $0 \leq \sin^2 x \leq 1$

$$\Rightarrow \quad 0 \leq \underbrace{\frac{\sin^2 x}{x^2}}_{f(x)} \leq \underbrace{\frac{1}{x^2}}_{g(x)} , \forall x \geq 1$$

But by p-test : $\displaystyle\int_1^\infty \frac{dx}{x^p}\,dx = \begin{cases} \text{divergent} , & P \leq 1 \\ \frac{1}{p-1} , & P > 1 \end{cases}$

we have $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx = 1$ (converges)

$\Rightarrow \displaystyle\int_1^\infty \frac{\sin^2 x}{x^2}\,dx$ Converges by DCT.

2) $$\int_1^\infty \frac{dx}{\sqrt{x^2-0.1}}$$

For $x \geq 1$, $x^2 - 0.1 \leq x^2$

$$\Rightarrow \sqrt{x^2-0.1} \leq \sqrt{x^2} = |x| = x$$

$$\Rightarrow \overset{g(x)}{\frac{1}{\sqrt{x^2-0.1}}} \geq \overset{f(x)}{\frac{1}{x}}, \quad \forall x \geq 1$$

Since $\int_1^\infty \frac{dx}{x}$ diverges by p-test, then

Using DCT $\int_1^\infty \frac{dx}{\sqrt{x^2-0.1}}$ diverges.

---

3) $$\int_1^\infty \frac{dx}{x\sqrt{x^2-0.1}}$$

For $x \geq 1$, $x^2 - 0.1 \geq x^2 - \frac{x^2}{2} = \frac{x^2}{2}$

$$\Rightarrow \sqrt{x^2-0.1} \geq \sqrt{\frac{x^2}{2}} = \frac{|x|}{\sqrt{2}} = \frac{x}{\sqrt{2}}$$

Thus $x\sqrt{x^2-0.1} \geq \frac{x^2}{\sqrt{2}}$

$$\Rightarrow \frac{1}{x\sqrt{x^2-0.1}} \leq \frac{\sqrt{2}}{x^2}$$

Now $\int_1^\infty \frac{\sqrt{2}}{x^2}\,dx$ Converges by p-test

then $\int_1^\infty \frac{dx}{x\sqrt{x^2-0.1}}$ also Converges by DCT

4) $\int_0^\pi \frac{dt}{\sqrt{t}+\sin t}$

Note that $\sin t \geq 0$ on $[0,\pi]$

$\Rightarrow \quad \sqrt{t}+\sin t \geq \sqrt{t} \geq 0$

$\Rightarrow \quad \frac{1}{\sqrt{t}+\sin t} \leq \frac{1}{\sqrt{t}}$ on $[0,\pi]$

Now $\int_0^\pi \frac{dt}{\sqrt{t}} = \lim_{a\to 0^+} \int_a^\pi t^{-\frac{1}{2}}\,dt = \lim_{a\to 0^+} 2\sqrt{t}\Big|_a^\pi$

$= \lim_{a\to 0^+}\left[2\sqrt{\pi} - 2\sqrt{a}\right] = 2\sqrt{\pi}$

$\therefore$ Since $\int_0^\pi \frac{1}{\sqrt{t}}\,dt$ Converges, then

$\int_0^\pi \frac{dt}{\sqrt{t}+\sin t}$ also converges by DCT

Thm: Limit Comparison Test: (LCT)

If the positive functions f and g are continuous on $[a, \infty)$ and if

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = L \quad , \quad 0 < L < \infty$$

Cal I, they grow at the same rate.

then $\int_a^\infty f(x)\,dx$ and $\int_a^\infty g(x)\,dx$

both Converge or both diverge.

Note: ① When we use LCT, if both functions Converge, this does not mean that they have the same Values.

② DCT can be used on $[a, b]$ or $[a, \infty)$

but LCT can be used only on $[a, \infty)$

Examples on Test the Convergence.

Back to example (# 3)

5) $\displaystyle\int_1^\infty \frac{dx}{x\sqrt{x^2-0.1}}$

positive $[1,\infty)$

Let $f(x) = \dfrac{1}{x\sqrt{x^2-0.1}}$ and $g(x) = \dfrac{1}{x\sqrt{x^2}} = \dfrac{1}{x^2}$

Now, by p-test $\displaystyle\int_1^\infty g(x)\,dx = \int_1^\infty \frac{1}{x^2}\,dx$ Conv.

Now: $\displaystyle\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{1\cdot x^2}{x\sqrt{x^2-0.1}}$

$$= \lim_{x\to\infty} \frac{x}{\sqrt{x^2\left(1-\frac{0.1}{x^2}\right)}} = \lim_{x\to\infty} \frac{x}{x\sqrt{1-\frac{0.1}{x^2}}} = 1$$

so by LCT, since $\displaystyle\int_1^\infty g(x)\,dx$ converges

then $\displaystyle\int_1^\infty \frac{dx}{x\sqrt{x^2-0.1}}$ also converges.

6) $\displaystyle\int_1^{\infty} \frac{dx}{1+x^2}$

Let $f(x) = \dfrac{1}{x^2}$ & $g(x) = \dfrac{1}{1+x^2}$

f and g are positive on $[1, \infty)$

$\displaystyle\int_1^{\infty} f(x)\,dx = \int_1^{\infty} \frac{1}{x^2}\,dx$ converges by p-test.

$$\Rightarrow \lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{\frac{1}{x^2}}{\frac{1}{x^2+1}} = \lim_{x\to 1} \frac{x^2+1}{x^2} = 1$$

$\therefore$ by LCT : $\displaystyle\int_1^{\infty} \frac{dx}{1+x^2}$ converges.

---

7) $\displaystyle\int_1^{\infty} \frac{1-e^{-x}}{x}\,dx$ $\left(\dfrac{1-e^{-x}}{x} < \dfrac{1}{x} \to \text{div.} \text{ we can't use DCT} \right)$

Let $f(x) = \dfrac{1-e^{-x}}{x}$ and $g(x) = \dfrac{1}{x}$

we know that $\displaystyle\int_1^{\infty} \frac{1}{x}\,dx$ diverges by p-test

$$\Rightarrow \lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{\frac{1-e^{-x}}{x}}{\frac{1}{x}} = \lim_{x\to\infty} 1-e^{-x} = 1$$

$\Rightarrow$ by LCT $\displaystyle\int_1^{\infty} \frac{1-e^{-x}}{x}\,dx$ diverges.

(8) $\displaystyle\int_0^1 \frac{dt}{t-\sin t}$, Improper Integral of type II

Note that the Interval is $[0,1]$ so, we can't use LCT. Besides we can't evaluate the Integral so, we will use DCT.

For $t \in (0,1]$, $\sin t > 0$

$$\Rightarrow \quad -\sin t < 0$$

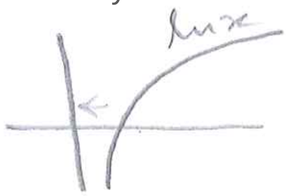
$$\Rightarrow \quad t - \sin t < t.$$

$$\Rightarrow \quad \underbrace{\frac{1}{t-\sin t}}_{f(t)} > \underbrace{\frac{1}{t}}_{g(t)}$$

Consider $\displaystyle\int_0^1 \frac{1}{t}\,dt = \lim_{b\to 0^+}\int_b^1 \frac{1}{t}\,dt$ (Not p-test since $(0,1]$ Not $[1,\infty)$)

$$= \lim_{b\to 0^+} \ln|t|\Big|_b^1 = \lim_{b\to 0^+}\left(\ln 1 - \ln b\right)$$

$$= -\lim_{b\to 0^+} \ln b = \infty \quad (\text{Div.})$$



$x > 0$

so by DCT, $\displaystyle\int_0^1 \frac{1}{t-\sin t}\,dt$ diverges.

(9) $\displaystyle\int_{\boxed{0}}^{\infty} \frac{x \, dx}{\sqrt{1+x^6}}$

Take $\displaystyle g(x) = \frac{x}{\sqrt{x^6}} = \frac{1}{x^2}$

Note $\displaystyle\int_{\boxed{1}}^{\infty} g(x)\, dx = \int_{1}^{\infty} \frac{1}{x^2}\, dx$ Converges (p-test)

Now, $\displaystyle\lim_{x \to \infty} \frac{f(x)}{g(x)} = \lim_{x \to \infty} \frac{x^3}{\sqrt{1+x^6}} = 1$

So by LCT: $\displaystyle\int_{\boxed{1}}^{\infty} \frac{x \, dx}{\sqrt{1+x^6}}$ also Converges.

But $$\int_{0}^{\infty} \frac{x \, dx}{\sqrt{1+x^6}} = \underbrace{\int_{0}^{1} \frac{x}{\sqrt{1+x^6}}\, dx}_{\text{تكامل عادي}} + \underbrace{\int_{1}^{\infty} \frac{x}{\sqrt{1+x^6}}\, dx}_{\text{Improper which Converges}}$$

$\therefore \displaystyle\int_{0}^{\infty} \frac{x \, dx}{\sqrt{1+x^6}}$ converges,

10) $$\int_{\pi}^{\infty} \frac{2+\cos x}{\sqrt{x^2+1}}\, dx$$

Note : $-1 < \cos x$

$\Rightarrow 1 = 2-1 < 2+\cos x \quad \cdots (1)$

and $\quad x^2+1 < x^2+x^2 = 2x^2 \;, \quad x \geq \pi.$

$\Rightarrow \sqrt{x^2+1} < \sqrt{2}\, x \quad$ on $[\pi, \infty)$

$\Rightarrow \dfrac{1}{\sqrt{x^2+1}} > \dfrac{1}{\sqrt{2}\, x} \quad \cdots (2)$

$$1 \;\&\; 2 \left\{ \begin{array}{l} 1 < 2+\cos x \\ \dfrac{1}{\sqrt{2}\,x} < \dfrac{1}{\sqrt{x^2+1}} \end{array} \right.$$

$$\frac{1}{\sqrt{2}\, x} < \frac{2+\cos x}{\sqrt{x^2+1}}$$

Since $\displaystyle\int_{\pi}^{\infty} \frac{dx}{\sqrt{2}\, x}$ is divergent, so by

DCT $\displaystyle\int_{\pi}^{\infty} \frac{2+\cos x}{\sqrt{x^2+1}}\, dx$ is divergent.