Chapter 4 :

4.1 : The Derivative .

Def 1 : A real function f is said to be differentible at a point $a \in R$ iff f is defined on some open interval I containing a and $\bar{f}(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$ exists .

oR $\bar{f}(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = L$.

$\hookrightarrow$ $\forall \varepsilon > 0$, $\exists$ a $\delta(\varepsilon) > 0$ s.t if $x \in I$ satisfies $|x-a| < \delta(\varepsilon)$ then $\left| \frac{f(x) - f(a)}{x - a} - L \right| < \varepsilon$.

RMK : on Note book .

Thm 1 : A real function f is diffble at $x = a \in \mathbb{R}$ iff $\exists$ an open interval I and a function $F : I \to R$ such that $a \in I$, f is defined on I , F is continuous at a and $f(x) = F(x)(x-a) + f(a)$ holds $\forall x \in I$ in which case $F(a) = \bar{f}(a)$.

Thm 2 : A real function f is diffble at $x = a$ iff $\exists$ a function T of the form $T(x) = mx$ such that $\lim_{h \to 0} \frac{f(a+h) - f(a) - T(h)}{h} = 0$.

$m = \bar{f}(a)$.

Thm 3 : If f is diffble at a , then f is continuous at a .

RMK : The converse of thm 3 is False .

Def 2 : let I be a nondegenerate interval

i. A function $f : I \to R$ is said to be diffble on I iff $\bar{f}_I(a) = \lim_{\substack{x \to a \\ x \in I}} \frac{f(x) - f(a)}{x - a}$ exist and finite .

ii. f is said to be continuously diffble on I iff $\bar{f}_I$ exist and continuous on I .

RMK :

- When a is not an end point of I, $f'_I(a)$ is the same as $f'(a)$.

- If f is diffble on $[a,b]$ then, $f'(a) = \lim_{h\to 0^+} \frac{f(a+h)-f(a)}{h}$ and $f'(b) = \lim_{h\to 0^-} \frac{f(b+h)-f(b)}{h}$.

Notations : $C^n(I)$: let I be an degenerate interval, For $n\in\mathbb{N}$ we defined the collection of functions $C^n(I)$ By : $C^n(I) = \{ f : I \to \mathbb{R}$ and f^n exists and continuous on $I \}$.

- when $f \in C^n(I)$, $\forall n\in\mathbb{N}$ we shall denote it by $f\in C^\infty(I)$.

- $C^1(I)$ is precisely the collection of real function which are continuously diffble on I.

- $C^n([a,b]) = C^n[a,b]$.

- $C^\infty(I) \subset C^m(I) \subset C^n(I)$ for integers $m > n > 0$.

- Not every function which is diffble on $\mathbb{R}$ belongs to $C^1(\mathbb{R})$.

Note : A function which is diffble on two sets is not necessarily diffble on their union.

4.2: Differentiability Theorems:

Thm 4: let $I \subset R$ be an interval. let $a \in I$, $\alpha \in R$ and let $f: I \to R$, $g: I \to R$ be functions that diffble at a, then $f+g$, αf, $f.g$, $\frac{f}{g}$ are all diffble at a. in Fact

1. $(f+g)'(a) = f'(a) + g'(a)$.
2. $(\alpha f)'(a) = \alpha f'(a)$.
3. $(f.g)'(a) = f'(a)\, g(a) + g'(a)\, f(a)$.
4. $\left(\frac{f}{g}\right)'(a) = \dfrac{g(a) f'(a) - f(a) g'(a)}{g^2(a)}$.

Thm 5: let f and g be real functions If f is diffble at a and g is diffble at $f(a)$ then $g \circ f$ is diffble at a with $(g \circ f)'(a) = g'(f(a))\, f'(a)$.

4.3: Mean Value Theorem:

lemma: Rolle's Thm: suppose that $a, b \in R$ with $a < b$. If f is continuous on $[a,b]$ diffble on (a,b) and $f(a) = f(b)$ then $f'(c) = 0$ for some $c \in (a,b)$.

Thm 6: suppose that $a, b \in R$ with $a < b$

1. Generalized Mean Value Theorem:

If f, g are continuous on $[a,b]$ and diffble on (a,b) then there is a $c \in (a,b)$ s.t

$$g'(c)\,(f(b) - f(a)) = f'(c)\,(g(b) - g(a)).$$

2. Mean Value Theorem:

If f is continuous on $[a,b]$ and diffble on (a,b) then there is a $c \in (a,b)$ s.t

$$f(b) - f(a) = f'(c)\,(b-a) \quad = \quad f'(c) = \frac{f(b) - f(a)}{b-a}.$$

RMK: ON Note Book

Def: let E be a nonempty subset of R and $f: E \to R$

1. f is said to be increasing (resp. strictly increasing) on E iff $x_1, x_2 \in E$ and $x_1 < x_2 \Rightarrow f(x_1) \le f(x_2)$ (resp. $f(x_1) < f(x_2)$).

2. f is said to be decreasing (resp. strictly decreasing) on E iff $x_1, x_2 \in E$ and $x_1 < x_2 \Rightarrow f(x_1) \ge f(x_2)$ (resp. $f(x_1) > f(x_2)$).

3. f is said to be Monotone (resp. strictly Monotone) on E iff f is either decreasing or increasing (resp. either strictly decreasing or strictly increasing) on E.

Thm 7: suppose that $a, b \in R$ with $a < b$ that f is continuous on $[a,b]$ and that f is diffble on (a,b)

1. If $f'(x) > 0$ (resp. $f'(x) < 0$) $\forall x \in (a,b)$ then f is strictly increasing (resp. strictly decreasing) on $[a,b]$.

2. If $f'(x) = 0$, $\forall x \in (a,b)$ then f is constant on $[a,b]$.

3. If g is continuous on $[a,b]$ and diffble on (a,b) and if $f'(x) = g'(x)$ $\forall x \in (a,b)$ then $f-g$ is constant on $[a,b]$.

Thm 8: suppose that f is increasing on $[a,b]$:

1. If $c \in [a,b)$ then $f(c^+)$ exists and $f(c) \le f(c^+) = \lim_{x \to c^+} f(x)$.

2. If $c \in (a,b]$ then $f(c^-)$ exists and $f(c^-) = \lim_{x \to c^-} f(x) \le f(c)$.

Thm 9: If f is Monotone on an interval I, then f has at ~~least~~ most countably many points of discontinuity on I.

Thm 10 : Bernoulli's inequality

let α be a positive real number. If $0 < \alpha \leq 1$ then $(1+x)^{\alpha} \leq 1 + \alpha x$ for all $x \in [-1, \infty)$.

And if $\alpha \geq 1$ then $(1+x)^{\alpha} \geq 1 + \alpha x$ for all $x \in [-1, \infty)$.

Thm 11 : Intermediate Value Theorem For derivatives.

suppose that f is diffable on $[a,b]$ with $f'(a) \neq f'(b)$. If y_0 is a real number which lies between $f'(a)$ and $f'(b)$ then there is an $x_0 \in (a,b)$ s.t $f'(x_0) = y_0$.