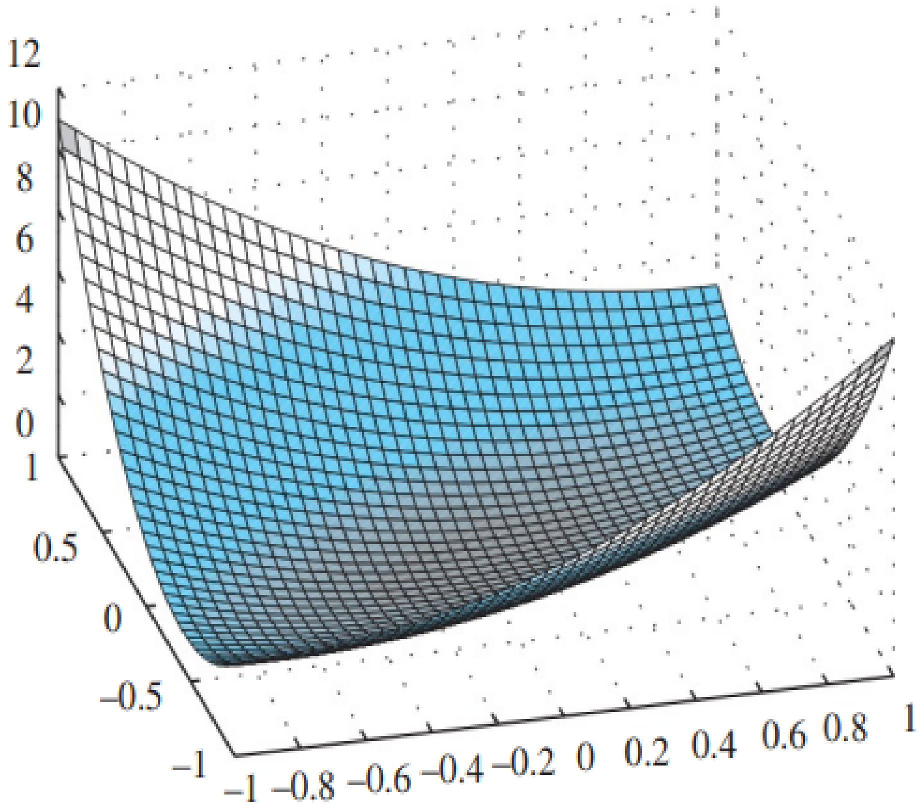
CHAPTER

6



Eigenvalues

6.1 Eigenvalues and Eigenvectors

Definition Let A be an $n \times n$ matrix. A scalar λ is said to be an **eigenvalue** or a **characteristic value** of A if there exists a nonzero vector $\mathbf{x}$ such that $A\mathbf{x} = \lambda\mathbf{x}$. The vector $\mathbf{x}$ is said to be an **eigenvector** or a **characteristic vector** belonging to λ.

EXAMPLE 2 Let

$$A = \begin{pmatrix} 4 & -2 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{x} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

Since

$$A\mathbf{x} = \begin{pmatrix} 4 & -2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ 3 \end{pmatrix} = 3 \begin{pmatrix} 2 \\ 1 \end{pmatrix} = 3\mathbf{x}$$

it follows that $\lambda = 3$ is an eigenvalue of A and $\mathbf{x} = (2, 1)^T$ is an eigenvector belonging to λ. Actually, any nonzero multiple of $\mathbf{x}$ will be an eigenvector, because

$$A(\alpha\mathbf{x}) = \alpha A\mathbf{x} = \alpha\lambda\mathbf{x} = \lambda(\alpha\mathbf{x})$$

Let A be an $n \times n$ matrix and λ be a scalar. The following statements are equivalent:

(a) λ is an eigenvalue of A.

(b) $(A - \lambda I)\mathbf{x} = \mathbf{0}$ has a nontrivial solution.

(c) $N(A - \lambda I) \neq \{\mathbf{0}\}$ (This is called the eigenspace corresponding to the eigenvalue)

(d) $A - \lambda I$ is singular.

(e) $\det(A - \lambda I) = 0$ (This is called the characteristic equation)

We will now use statement **(e)** to determine the eigenvalues in a number of examples.

$Ax = \lambda x \iff Ax - \lambda x = 0 \iff (A - \lambda I)x = 0$ (1)

$\iff (A - \lambda I)$ singular

$\iff |A - \lambda I| = 0$ (2)

EXAMPLE 3 Find the eigenvalues and the corresponding eigenvectors of the matrix

$$A = \begin{bmatrix} 3 & 2 \\ 3 & -2 \end{bmatrix}$$

(i) $|A - \lambda I| = 0 \Longrightarrow \begin{vmatrix} 3-\lambda & 2 \\ 3 & -2-\lambda \end{vmatrix} = 0$

$\Rightarrow (3-\lambda)(-2-\lambda) - 6 = 0 \Longrightarrow -6 + \lambda^2 - \lambda - 6 = 0 \Longrightarrow \lambda^2 - \lambda - 12 = 0$

$\Rightarrow (\lambda - 4)(\lambda + 3) = 0 \Longrightarrow \lambda_1 = -3, \lambda_2 = 4$

(ii) (a) $\lambda_1 = -3$:

$\begin{pmatrix} 6 & 2 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Longrightarrow \begin{pmatrix} 6 & 2 & 0 \\ 3 & 1 & 0 \end{pmatrix} \xrightarrow{\frac{1}{6}R_1} \begin{pmatrix} 1 & \frac{1}{3} & 0 \\ 3 & 1 & 0 \end{pmatrix}$

$$\xrightarrow{R_2 - 3R_1} \begin{array}{ccc} x_1 & x_2 & \\ \end{array}\begin{pmatrix} 1 & \frac{1}{3} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$x_2 = \alpha \ , \ x_1 = -\frac{1}{3}\alpha \Longrightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{3}\alpha \\ \alpha \end{pmatrix} = \alpha \begin{pmatrix} -\frac{1}{3} \\ 1 \end{pmatrix}$

Let $\alpha = 3$, then $v_1 = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$

(b) $\lambda_2 = 4$:

$$\begin{pmatrix} -1 & 2 \\ 3 & -6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} -1 & 2 & 0 \\ 3 & -6 & 0 \end{pmatrix} \xrightarrow{-R_1} \begin{pmatrix} 1 & -2 & 0 \\ 3 & -6 & 0 \end{pmatrix}$$

$$\xrightarrow{R_2 - 3R_1} \begin{array}{ccc} x_1 & x_2 & \\ \end{array}$$
$$\begin{pmatrix} 1 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$x_2 = \alpha$, $x_1 = 2\alpha$ $\Rightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2\alpha \\ \alpha \end{pmatrix} = \alpha \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

Let $\alpha = 1$, then $V_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

Solution

The characteristic equation is

$$\begin{vmatrix} 3-\lambda & 2 \\ 3 & -2-\lambda \end{vmatrix} = 0 \qquad \text{or} \qquad \lambda^2 - \lambda - 12 = 0$$

Thus, the eigenvalues of A are $\lambda_1 = 4$ and $\lambda_2 = -3$. To find the eigenvectors belonging to $\lambda_1 = 4$, we must determine the null space of $A - 4I$.

$$A - 4I = \begin{bmatrix} -1 & 2 \\ 3 & -6 \end{bmatrix}$$

Solving $(A - 4I)\mathbf{x} = \mathbf{0}$, we get

$$\mathbf{x} = (2x_2, x_2)^T$$

Hence, any nonzero multiple of $(2, 1)^T$ is an eigenvector belonging to λ_1, and $\{(2, 1)^T\}$ is a basis for the eigenspace corresponding to λ_1. Similarly, to find the eigenvectors for λ_2, we must solve

$$(A + 3I)\mathbf{x} = \mathbf{0}$$

In this case, $\{(-1, 3)^T\}$ is a basis for $N(A + 3I)$ and any nonzero multiple of $(-1, 3)^T$ is an eigenvector belonging to λ_2.

EXAMPLE 4 Let

$$A = \begin{bmatrix} 2 & -3 & 1 \\ 1 & -2 & 1 \\ 1 & -3 & 2 \end{bmatrix}_{3 \times 3}$$

Find the eigenvalues and the corresponding eigenspaces.

① $|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} 2-\lambda & -3 & 1 \\ 1 & -2-\lambda & 1 \\ 1 & -3 & 2-\lambda \end{vmatrix} = 0$

$(2-\lambda)\begin{vmatrix} -2-\lambda & 1 \\ -3 & 2-\lambda \end{vmatrix} + 3\begin{vmatrix} 1 & 1 \\ 1 & 2-\lambda \end{vmatrix} + \begin{vmatrix} 1 & -2-\lambda \\ 1 & -3 \end{vmatrix} = 0$

$(2-\lambda)\left[(-2-\lambda)(2-\lambda)+3\right] + 3\left[(2-\lambda)-1\right] + \left[-3-(-2-\lambda)\right] = 0$

$(2-\lambda)\left[\lambda^2-1\right] + 3(1-\lambda) + (\lambda-1) = 0$

$(2-\lambda)(\lambda-1)(\lambda+1) + 3(1-\lambda) + (\lambda-1) = 0$

$(\lambda-1)\left[(2-\lambda)(\lambda+1) - 3 + 1\right] = 0$

$$(\lambda-1)\left[2\lambda+2-\lambda^2-\lambda-3+1\right]=0$$

$$(\lambda-1)\left[-\lambda^2+\lambda\right]=0$$

$$\lambda(\lambda-1)(-\lambda+1)=0$$

$$\Rightarrow \lambda_1=0\ ,\ \lambda_2=\lambda_3=1$$

ii) a) $\lambda_1=0$:

$$\begin{pmatrix} 2 & -3 & 1 \\ 1 & -2 & 1 \\ 1 & -3 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 2 & -3 & 1 \\ 1 & -2 & 1 \\ 1 & -3 & 2 \end{pmatrix} \xrightarrow{R_{12}} \begin{pmatrix} 1 & -2 & 1 \\ 2 & -3 & 1 \\ 1 & -3 & 2 \end{pmatrix}$$

$$\xrightarrow{R_2-2R_1} \begin{pmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ 1 & -3 & 2 \end{pmatrix} \xrightarrow{R_3-R_1} \begin{pmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix}$$

$$\xrightarrow{R_3+R_2} \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \begin{pmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \end{array} \Rightarrow \begin{aligned} x_3 &= \alpha \\ x_2 &= \alpha \\ x_1 &= 2\alpha-\alpha=\alpha \end{aligned} \longrightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}=\alpha\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

(b) $\lambda_2 = \lambda_3 = 1$:

$$\begin{pmatrix} 1 & -3 & 1 \\ 1 & -3 & 1 \\ 1 & -3 & 1 \end{pmatrix} \xrightarrow[R_3 - R_1]{R_2 - R_1} \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \begin{pmatrix} 1 & -3 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{array}$$

$x_2 = \alpha$, $x_3 = \beta$, $x_1 = 3\alpha - \beta$

$$\Rightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3\alpha - \beta \\ \alpha \\ \beta \end{pmatrix} = \alpha \underbrace{\begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}}_{V_1} + \beta \underbrace{\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}}_{V_2}$$

$$V_1 = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} , \quad V_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

Solution

$$\begin{vmatrix} 2-\lambda & -3 & 1 \\ 1 & -2-\lambda & 1 \\ 1 & -3 & 2-\lambda \end{vmatrix} = -\lambda(\lambda-1)^2$$

Thus, the characteristic polynomial has roots $\lambda_1 = 0$, $\lambda_2 = \lambda_3 = 1$. The eigenspace corresponding to $\lambda_1 = 0$ is $N(A)$, which we determine in the usual manner:

$$\left[\begin{array}{ccc|c} 2 & -3 & 1 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & -3 & 2 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Setting $x_3 = \alpha$, we find that $x_1 = x_2 = x_3 = \alpha$. Consequently, the eigenspace corresponding to $\lambda_1 = 0$ consists of all vectors of the form $\alpha(1, 1, 1)^T$. To find the eigenspace corresponding to $\lambda = 1$, we solve the system $(A - I)\mathbf{x} = \mathbf{0}$:

$$\left[\begin{array}{ccc|c} 1 & -3 & 1 & 0 \\ 1 & -3 & 1 & 0 \\ 1 & -3 & 1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & -3 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Setting $x_2 = \alpha$ and $x_3 = \beta$, we get $x_1 = 3\alpha - \beta$. Thus, the eigenspace corresponding to $\lambda = 1$ consists of all vectors of the form

$$\begin{bmatrix} 3\alpha - \beta \\ \alpha \\ \beta \end{bmatrix} = \alpha \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

■

Complex Eigenvalues

Not only do the complex eigenvalues of a real matrix occur in conjugate pairs, but so do the eigenvectors. Indeed, if λ is a complex eigenvalue of a real $n \times n$ matrix A and $\mathbf{z}$ is an eigenvector belonging to λ, then

$$A\bar{\mathbf{z}} = \bar{A}\,\bar{\mathbf{z}} = \overline{A\mathbf{z}} = \overline{\lambda\mathbf{z}} = \bar{\lambda}\,\bar{\mathbf{z}}$$

Thus, $\bar{\mathbf{z}}$ is an eigenvector of A belonging to $\bar{\lambda}$. In Example 5, the eigenvector computed for the eigenvalue $\boxed{\lambda = 1 + 2i}$ was $\mathbf{z} = (1, i)^T$, and the eigenvector computed for $\boxed{\bar{\lambda} = 1 - 2i}$ was $\bar{\mathbf{z}} = (1, -i)^T$.

$\lambda = 1 + 2i \Rightarrow Z = \begin{pmatrix} 1 \\ i \end{pmatrix}$

$\bar{\lambda} = 1 - 2i \Rightarrow \bar{Z} = \begin{pmatrix} 1 \\ -i \end{pmatrix}$

EXAMPLE 5 Let

$$A = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$$

Compute the eigenvalues of A and find bases for the corresponding eigenspaces.

i) $|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} 1-\lambda & 2 \\ -2 & 1-\lambda \end{vmatrix} = 0 \Rightarrow (1-\lambda)^2 + 4 = 0$

$i = \sqrt{-1}$

$\Rightarrow 1 - 2\lambda + \lambda^2 + 4 = 0 \Rightarrow \lambda^2 - 2\lambda + 5 = 0$

$\Rightarrow \lambda_{1,2} = \dfrac{-B \pm \sqrt{B^2 - 4AC}}{2A} = \dfrac{2 \pm \sqrt{4 - 4(1)(5)}}{2} = 1 \pm \sqrt{-4}$

$= 1 \pm 2i$

$\Rightarrow \lambda_1 = 1 + 2i \quad , \quad \lambda_2 = \bar{\lambda}_1 = 1 - 2i$

ii) a) $\lambda_1 = 1 + 2i$:

$$\begin{pmatrix} 1-\lambda_1 & 2 \\ -2 & 1-\lambda_1 \end{pmatrix} = \begin{pmatrix} -2i & 2 \\ -2 & -2i \end{pmatrix} \xrightarrow{R_{12}} \begin{pmatrix} -2 & -2i \\ -2i & 2 \end{pmatrix}$$

$$\xrightarrow{-\frac{1}{2}R_1} \begin{pmatrix} 1 & i \\ -2i & 2 \end{pmatrix} \xrightarrow{R_2+2iR_1} \begin{array}{cc} x_1 & x_2 \\ \multicolumn{2}{c}{\begin{pmatrix} 1 & i \\ 0 & 0 \end{pmatrix}} \end{array}$$

$x_2 = \alpha$, $x_1 = -i\alpha$

$$\Rightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -i\alpha \\ \alpha \end{pmatrix} = \alpha \begin{pmatrix} -i \\ 1 \end{pmatrix}$$

Let $\alpha = 1$, then $V_1 = \begin{pmatrix} -i \\ 1 \end{pmatrix}$

(b) $\lambda_2 = \bar{\lambda}_1 = 1-2i$: $V_2 = \bar{V}_1 = \begin{pmatrix} i \\ 1 \end{pmatrix}$

Solution

$$\begin{vmatrix} 1-\lambda & 2 \\ -2 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 + 4$$

The roots of the characteristic polynomial are $\lambda_1 = 1 + 2i$, $\lambda_2 = 1 - 2i$.

$$A - \lambda_1 I = \begin{pmatrix} -2i & 2 \\ -2 & -2i \end{pmatrix} = -2\begin{pmatrix} i & -1 \\ 1 & i \end{pmatrix}$$

It follows that $\{(1, i)^T\}$ is a basis for the eigenspace corresponding to $\lambda_1 = 1 + 2i$. Similarly,

$$A - \lambda_2 I = \begin{pmatrix} 2i & 2 \\ -2 & 2i \end{pmatrix} = 2\begin{pmatrix} i & 1 \\ -1 & i \end{pmatrix}$$

and $\{(1, -i)^T\}$ is a basis for $N(A - \lambda_2 I)$.

The Product and Sum of the Eigenvalues

It is easy to determine the product and sum of the eigenvalues of an $n \times n$ matrix A. If $p(\lambda)$ is the characteristic polynomial of A, then

$$p(\lambda) = \det(A - \lambda I) = \begin{vmatrix} a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - \lambda & & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & & a_{nn} - \lambda \end{vmatrix} \tag{4}$$

Thus, the lead coefficient of $p(\lambda)$ is $(-1)^n$, and hence if $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of A, then

$$\begin{aligned} p(\lambda) &= (-1)^n(\lambda - \lambda_1)(\lambda - \lambda_2)\cdots(\lambda - \lambda_n) \\ &= (\lambda_1 - \lambda)(\lambda_2 - \lambda)\cdots(\lambda_n - \lambda) \end{aligned} \tag{6}$$

It follows from (4) and (6) that

$$\boxed{\lambda_1 \cdot \lambda_2 \cdots \lambda_n = p(0) = \det(A)}$$

we also see that the coefficient of $(-\lambda)^{n-1}$ is $\sum_{i=1}^{n} a_{ii}$. If we use (6) to determine this same coefficient, we obtain $\sum_{i=1}^{n} \lambda_i$. It follows that

$$\boxed{\sum_{i=1}^{n} \lambda_i = \sum_{i=1}^{n} a_{ii}}$$

The sum of the diagonal elements of A is called the *trace* of A and is denoted by $\operatorname{tr}(A)$.

EXAMPLE 6 If

$$A = \begin{pmatrix} 5 & -18 \\ 1 & -1 \end{pmatrix}$$

then

$$\det(A) = -5 + 18 = 13 \qquad \text{and} \qquad \operatorname{tr}(A) = 5 - 1 = 4$$

The characteristic polynomial of A is given by

$$\begin{vmatrix} 5-\lambda & -18 \\ 1 & -1-\lambda \end{vmatrix} = \lambda^2 - 4\lambda + 13$$

and hence the eigenvalues of A are $\lambda_1 = 2 + 3i$ and $\lambda_2 = 2 - 3i$. Note that

$$\lambda_1 + \lambda_2 = 4 = \operatorname{tr}(A)$$

and

$$\lambda_1 \lambda_2 = 13 = \det(A)$$

Similar Matrices

We close this section with an important result about the eigenvalues of similar matrices. Recall that a matrix B is said to be *similar* to a matrix A if there exists a nonsingular matrix S such that $B = S^{-1}AS$.

Theorem 6.1.1 *Let A and B be $n \times n$ matrices. If B is similar to A, then the two matrices have the same characteristic polynomial and, consequently, the same eigenvalues.*

Proof Let $p_A(x)$ and $p_B(x)$ denote the characteristic polynomials of A and B, respectively. If B is similar to A, then there exists a nonsingular matrix S such that $B = S^{-1}AS$. Thus,

$$\begin{aligned} p_B(\lambda) &= \det(B - \lambda I) \\ &= \det(S^{-1}AS - \lambda I) \\ &= \det(S^{-1}(A - \lambda I)S) \\ &= \det(S^{-1})\det(A - \lambda I)\det(S) \\ &= p_A(\lambda) \end{aligned}$$

The eigenvalues of a matrix are the roots of the characteristic polynomial. Since the two matrices have the same characteristic polynomial, they must have the same eigenvalues. ■

EXAMPLE 7 Given

$$T = \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix} \quad \text{and} \quad S = \begin{pmatrix} 5 & 3 \\ 3 & 2 \end{pmatrix}$$

It is easily seen that the eigenvalues of T are $\lambda_1 = 2$ and $\lambda_2 = 3$. If we set $A = S^{-1}TS$, then the eigenvalues of A should be the same as those of T.

$$A = \begin{pmatrix} 2 & -3 \\ -3 & 5 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 5 & 3 \\ 3 & 2 \end{pmatrix} = \begin{pmatrix} -1 & -2 \\ 6 & 6 \end{pmatrix}$$

We leave it to the reader to verify that the eigenvalues of this matrix are $\lambda_1 = 2$ and $\lambda_2 = 3$. ■

SECTION 6.1 EXERCISES

1. Find the eigenvalues and the corresponding eigenspaces for each of the following matrices:

(a) $\begin{pmatrix} 3 & 2 \\ 4 & 1 \end{pmatrix}$ **(b)** $\begin{pmatrix} 6 & -4 \\ 3 & -1 \end{pmatrix}$

(c) $\begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix}$ **(d)** $\begin{pmatrix} 3 & -8 \\ 2 & 3 \end{pmatrix}$

(e) $\begin{pmatrix} 1 & 1 \\ -2 & 3 \end{pmatrix}$ **(f)** $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

3. Let A be an $n \times n$ matrix. Prove that A is singular if and only if $\lambda = 0$ is an eigenvalue of A.

4. Let A be a nonsingular matrix and let λ be an eigenvalue of A. Show that $1/\lambda$ is an eigenvalue of A^{-1}.

4. If A is a nonsingular matrix and λ is an eigenvalue of A, then there exists a nonzero vector $\mathbf{x}$ such that

$$\begin{aligned} A\mathbf{x} &= \lambda\mathbf{x} \\ A^{-1}A\mathbf{x} &= \lambda A^{-1}\mathbf{x} \end{aligned}$$

It follows from Exercise 3 that $\lambda \neq 0$. Therefore

$$A^{-1}\mathbf{x} = \frac{1}{\lambda}\mathbf{x}$$

and hence $1/\lambda$ is an eigenvalue of A^{-1}.

6. Let λ be an eigenvalue of A and let $\mathbf{x}$ be an eigenvector belonging to λ. Use mathematical induction to show that, for $m \geq 1$, λ^m is an eigenvalue of A^m and $\mathbf{x}$ is an eigenvector of A^m belonging to λ^m.

8. An $n \times n$ matrix A is said to be *idempotent* if $A^2 = A$. Show that if λ is an eigenvalue of an idempotent matrix, then λ must be either 0 or 1.

9. An $n \times n$ matrix is said to be *nilpotent* if $A^k = O$ for some positive integer k. Show that all eigenvalues of a nilpotent matrix are 0.

12. Show that A and A^T have the same eigenvalues. Do they necessarily have the same eigenvectors? Explain.

14. Let A be a 2×2 matrix. If $\operatorname{tr}(A) = 8$ and $\det(A) = 12$, what are the eigenvalues of A?

8. If A is idempotent and λ is an eigenvalue of A with eigenvector $\mathbf{x}$, then

$$\begin{aligned} A\mathbf{x} &= \lambda\mathbf{x} \\ A^2\mathbf{x} &= \lambda A\mathbf{x} = \lambda^2\mathbf{x} \end{aligned}$$

and

$$A^2\mathbf{x} = A\mathbf{x} = \lambda\mathbf{x}$$

Therefore

$$(\lambda^2 - \lambda)\mathbf{x} = \mathbf{0}$$

Since $\mathbf{x} \neq \mathbf{0}$ it follows that

$$\lambda^2 - \lambda = 0$$
$$\lambda = 0 \quad \text{or} \quad \lambda = 1$$

9. If λ is an eigenvalue of A, then λ^k is an eigenvalue of A^k (Exercise 6). If $A^k = O$, then all of its eigenvalues are 0. Thus $\lambda^k = 0$ and hence $\lambda = 0$.

12. $\det(A-\lambda I) = \det((A-\lambda I)^T) = \det(A^T - \lambda I)$. Thus A and A^T have the same characteristic polynomials and consequently must have the same eigenvalues. The eigenspaces however will not be the same. For example if

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

then the eigenvalues of A and A^T are both given by

$$\lambda_1 = \lambda_2 = 1$$

The eigenspace of A corresponding to $\lambda = 1$ is spanned by $(1,\ 0)^T$ while the eigenspace of A^T is spanned by $(0,\ 1)^T$.