Homework 2 (chapter 3) Reem Ali

Abistract 1 .

Q1: For each Group in the following list, find the order of the group and the order of each element in the group. and notice the Relation between it.

- $\mathbb{Z}_{12} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$, $|\mathbb{Z}_{12}| = 12$.

$|\overset{e}{0}| = 1$ $|1| = 12$ $|2| = 6$ $|3| = 4$ $|4| = 3$ $|5| = 12$

$|6| = 2$ $|7| = 12$ $|8| = 3$ $|9| = 4$ $|10| = 6$ $|11| = 12$

- $U(10) = \{1, 3, 7, 9\}$, $|U(10)| = 4$

$|1| = 1$ $|3| = 4$ $|7| = 4$ $|9| = 2$.

- $U(12) = \{1, 5, 7, 11\}$, $|U(12)| = 4$.

$|1| = 1$ $|5| = 2$ $|7| = 2$ $|11| = 2$

- $U(20) = \{1, 3, 7, 9, 11, 13, 17, 19\}$, $|U(20)| = 8$

$|1| = 1$ $|3| = 4$ $|7| = 4$ $|9| = 2$ $|11| = 2$ $|13| = 4$ $|17| = 4$ $|19| = 2$

- D_4

Q2: let Q be the group of Rational numbers under addition and let Q^* be the group of nonzero rational numbers under multiplication. In Q list the elements in $\langle \frac{1}{2} \rangle$. In Q^* list the elements in $\langle \frac{1}{2} \rangle$.

$\frac{1}{2}^{-3}, \frac{1}{2}^{-2}, \frac{1}{2}^{-1}, \frac{1}{2}^{0}, \frac{1}{2}^{1}, \frac{1}{2}^{1}, \frac{1}{2}^{1}, \frac{1}{2}^{3}, \frac{1}{2}^{4}, \frac{1}{2}^{5}, \frac{1}{2}^{6}$

① In Q: $\langle \frac{1}{2} \rangle = \{\ldots, -\frac{3}{2}, -1, -\frac{1}{2}, 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, 3, \ldots\}$

② In Q^*: $\langle \frac{1}{2} \rangle = \{\ldots, \overset{-4}{16}, \overset{-3}{8}, \overset{-2}{4}, \overset{-1}{2}, \overset{1}{\frac{1}{2}}, \overset{2}{\frac{1}{4}}, \overset{3}{\frac{1}{8}}, \overset{4}{\frac{1}{16}}, \ldots\}$

$= \{\frac{1}{16}, \frac{1}{8}, \frac{1}{4}, \frac{1}{2}, 2, 4, 8, 16\}$

Q4: prove that in any group, an element and its inverse have the same order.

let $|a| = n$, $|a^{-1}| = m$, $m, n > 0$:

$\Rightarrow a^n = e$, $(a^{-1})^m = (a^m)^{-1} = e^{-1} = e$

$\Rightarrow e = e.e = (a^n)(an)^{-1} = a^{n-m} = |a| = n-m$ X

By Conradiction.

Q5: without actually computing the orders, explain why the two elements in each of the following pairs of elements from Z_{30} must have the same order: $\{2, 28\}$, $\{8, 22\}$, Do the same for the following pairs of elements from $U(15)$: $\{2, 8\}$, $\{7, 13\}$.

Z_{30}: $\{2, 28\} \rightarrow 2 + 28 = 0$ so $2^{-1} = 28$, $28^{-1} = 2$ (2 and 28 inverse to each other)

$\{8, 22\} \rightarrow 8 + 22 = 0$ so $8^{-1} = 22$, $22^{-1} = 8$ (8 and 22 inverse to each other).

$\overset{15\times 1}{}$

$U(15)$: $\{2, 8\} \rightarrow 2.8 = 1$ so $2^{-1} = 8$, $8^{-1} = 2$

$\{7, 13\} \rightarrow 7.13 = 1$ so $7^{-1} = 13$, $13^{-1} = 7$

$15\times 6 \rightarrow$

$\Rightarrow$ Any element and its inverse have the same order.

Q6: suppose that a is a group element and $a^6 = e$. What are the possibilities for $|a|$. provide reasons for your answer? $|a| = n$ st $a^n = e$.

$|a|$ = divides 6 $\rightsquigarrow$ 1, 2, 3, 6.

$a^6 = e \rightarrow 6$ ✓

$a^6 = a^5 a = e \Rightarrow a = e$ X

$a^6 = a^4 a^2 = e \Rightarrow a^2 = e$ X

$a^6 = a^3 a^3 = e \Rightarrow e = e$ ✓

$a^6 = a^2 a^2 a^2 = e \Rightarrow e = e$ ✓

$a^6 = aaaaaa = e \Rightarrow e = e$ ✓

$\Rightarrow$ Possibilities = $\{1, 2, 3, 6\}$

Not possible = $\{4, 5\}$.

Q7: If a is a group element and a has infinite order, prove that $a^m \neq a^n$ when $m \neq n$.

By contradiction:

suppose $m \neq n$ and $a^m = a^n$

$a^m = a^n$ (multiply by a^{-n} from right).

$a^m a^{-n} = a^n a^{-n}$

$a^{m-n} = a^0$

خواص الأسس

$\Rightarrow m - n = 0$

$\Rightarrow m = n$ X.

Q9: Show that if a is an element of a group G, then $|a| \leq |G|$.

case 1: if a has infinite order: $e, a, a^n, \dots$ distinct and belong to G.

number of G is a finite $\Rightarrow |a| = |G|$.

case 2: if $|a| = n \rightarrow a^i = a^j$, $0 < i < j < n$

$a^{i-j} = e \rightarrow$ X $(i-j = 0 \rightarrow i = j)$ X

thus, $e, a, a^2, \dots, a^{n-1}$ are all distinct and belong to G.

So G has aleast n element.

$\Rightarrow |a| \leq |G|$ #

Q16: Show that $U(14) = \langle 3 \rangle = \langle 5 \rangle$, Is $(14) = \langle 11 \rangle$? ⊗14

$U(14) = \{1, 3, 5, 9, 11, 13\}$

$\langle 3 \rangle = \{3^1, 3^2, 3^3, 3^4, 3^5, 3^6\} = \{3, 9, 13, 11, 5, 1\} = U(14)$.

$\langle 5 \rangle = \{5^1, 5^2, 5^3, 5^4, 5^5, 5^6\} = \{5, 11, 13, 9, 3, 1\} = U(14)$

$\langle 11 \rangle = \{11^1, 11^2, 11^3, 11^4\} = \{11, 9, 1\} \neq U(14)$.

Q12: prove that an Abelian group with two elements of order 2 must have a subgroup of order 4.

suppose a, b are 2 elements of order 2

then $\{a, b, e, ab\}$ closed and subgroup of order 4.

- $e^2 = e$ ✓
- $a, b \in$ group $\rightarrow a^2 = e$, $b^2 = e$ since $|a| = 2$, $|b| = 2$.
- $(ab)^2 = ab\,ab = aabb = a^2 b^2 = e \rightarrow e.e = e$.

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Q14: suppose that H is a proper subgroup of $\mathbb{Z}$ under addition and H contains 18, 30 and 40. Determine H.

كيف؟؟ $H = \langle 2 \rangle$

Q18: If H and K are subgroups of G, show that $H \cap K$ is a subgroup of G.

$H \cap K \neq \phi$, since $e \in H \cap K$.

طريقة 1

① let $x, y \in H \cap K$, then since H and K are subgroups

$\Rightarrow xy^{-1} \in H$ and $xy^{-1} \in K$

$\Rightarrow xy^{-1} \in H \cap K$ By-one step subgroup test.

طريقة 2

② ① let $x \in H \cap K \Rightarrow x \in H$ and $x \in K$

$\Rightarrow x^{-1} \in H$ and $x^{-1} \in K$

$\Rightarrow x^{-1} \in H \cap K$

② let $x, y \in H \cap K \rightarrow x, y \in H$ and $x, y \in K$

$\rightarrow xy \in H$ and $xy \in K$ since H and K subgroups

$\rightarrow xy \in H \cap K$

By two step subgroup test, $H \cap K$ is a subgroup of G.

Q19: Let G be a group. Show that $Z(G) = \bigcap_{a \in G} C(a)$.

If $x \in Z(G)$ the $x \in C(a)$ for all a, so $x \in \bigcap_{a \in G} C(a)$

If $x \in \bigcap_{a \in G} C(a)$ then $xa = ax$ for all a in G so $x \in Z(G)$.

Q20. Let G be a group, and let $a \in G$. prove that $C(a) = C(a^{-1})$.

suppose $x \in C(a) \Rightarrow ax = xa$

$a^{-1}(ax) = a^{-1}(xa)$

$x = a^{-1}(xa)$,

Thus, $(a^{-1}x)a = x$ and therefor $a^{-1}x = xa^{-1}$

$\Rightarrow x \in C(a^{-1})$ #

Q26: If H is a subgroup of G, then by the centralizer $C(H)$ of H we meen the set $\{xh = hx$ for all $h \in H\}$. prove that $C(H)$ is a subgroup of G.

By 2-step test:

$C(H) \neq \emptyset$ since $eh = he = e \rightarrow e \in C(H)$.

① let $x, y \in C(H) \Rightarrow xh = hx$

$yh = hy$

$(xy)h = x(yh) = x(hy) = (xh)y = h(xy)$

$\Rightarrow xy \in C(H)$.

② let $x \in C(H) \Rightarrow xh = hx$

$x^{-1}(xh = hx)x^{-1}$ from right and left

$hx^{-1} = x^{-1}h$

$\Rightarrow x^{-1} \in C(H)$

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