10.2 Infinite Series:

Def: An infinite series is the sum of an infinite sequence of numbers:

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots = \sum_{n=1}^{\infty} a_n$$

where a_n is the nth term of the series.

Sequence of Partial sums:

The sequence $\{S_n\}$ defined by:

$S_1 = a_1$, (1st partial sum).

$S_2 = a_1 + a_2$, (2nd partial sum).

$\vdots$

$S_n = a_1 + a_2 + \cdots + a_n = \sum_{k=1}^{n} a_k$. (nth Partial sum)

is the sequence of partial sums of the series

$a_1 + a_2 + a_3 + \cdots + a_n + \cdots$

Remark :

1) If $\lim_{n\to\infty} S_n = L$ (converges), We say that the series converges and its sum is L, that is, $a_1 + a_2 + a_3 + \dots + a_n + \dots = L$

2) If the sequence of partial sums $\{S_n\}$ does not converge, we say that the series diverges.

Notation : $\sum_{n=1}^{\infty} a_n$, $\sum_{k=1}^{\infty} a_k$ or simply $\sum a_n$.

Examples : (Using n-th partial Sum).

① $\sum_{n=1}^{\infty} n = 1 + 2 + 3 + 4 + \dots + n + \dots$

$S_1 = 1$

$S_2 = 1 + 2 = 3$

$S_3 = 1 + 2 + 3 = 6$

$\vdots$

$S_n = 1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2}$.

So, the sequence of partial sum is

$$\{S_n\} = \{1, 3, 6, \dots, \frac{n(n+1)}{2}, \dots\}$$

Since, $\lim\limits_{n\to\infty} \frac{n(n+1)}{2} = \infty$, then the series

$\sum\limits_{n=1}^{\infty} n$ is diverge.

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(2) $\sum\limits_{n=1}^{\infty} \frac{1}{2^{n-1}} = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots$

$S_1 = 1$, $(2-1)$

$S_2 = 1 + \frac{1}{2} = \frac{3}{2}$, $(2 - \frac{1}{2})$

$S_3 = 1 + \frac{1}{2} + \frac{1}{4} = \frac{7}{4}$, $(2 - \frac{1}{4})$

$\vdots$

$$S_n = 1 + \frac{1}{2} + \frac{1}{4} + \dots + \frac{1}{2^{n-1}} = \frac{2^n - 1}{2^{n-1}}$$

$$= 2 - \frac{1}{2^{n-1}}.$$

Now, $\lim\limits_{n\to\infty} S_n = \lim\limits_{n\to\infty}\left(2 - \frac{1}{2^{n-1}}\right) = 2 - 0 = \boxed{2}$

$\therefore \sum\limits_{n=1}^{\infty} \frac{1}{2^{n-1}}$ Converges & its sum $= 2$

$$\left(\sum_{n=1}^{\infty} \frac{1}{2^{n-1}} = 2\right)$$

③ $$\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1} \right)$$

$$= \left(\frac{1}{1} - \frac{1}{2} \right) + \left(\frac{1}{2} - \frac{1}{3} \right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1} \right) + \cdots$$

$$S_1 = 1 - \frac{1}{2}$$

$$S_2 = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) = 1 - \frac{1}{3}$$

$$S_3 = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) = 1 - \frac{1}{4}$$

$$\vdots$$

$$S_n = 1 - \frac{1}{n+1}$$

$$\Rightarrow \lim_{n\to\infty} S_n = \lim_{n\to\infty} \left(1 - \frac{1}{n+1} \right) = 1 - 0 = 1$$

$$\therefore \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1} \right) = 1 \quad \text{(converges).}$$

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Remark : $$\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1} \right) = \sum_{n=1}^{\infty} \frac{1}{n(n+1)} ,$$

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this series is called Telescoping Series.

④ $$\sum_{n=1}^{\infty} \left(\tan^{-1}(n) - \tan^{-1}(n+1) \right)$$

$$= \left(\tan^{-1}(1) - \tan^{-1}(2)\right) + \left(\tan^{-1}(2) - \tan^{-1}(3)\right) + \cdots$$

$$S_1 = \tan^{-1}(1) - \tan^{-1}(2) = \frac{\pi}{4} - \tan^{-1} 2.$$

$$S_2 = \left(\frac{\pi}{4} - \tan^{-1} 2\right) + \left(\tan^{-1} 2 - \tan^{-1} 3\right) = \frac{\pi}{4} - \tan^{-1} 3$$

$\vdots$

$$S_n = \frac{\pi}{4} - \tan^{-1}(n+1).$$

$$\Rightarrow \lim_{n\to\infty} S_n = \lim_{n\to\infty} \left(\frac{\pi}{4} - \tan^{-1}(n+1)\right) = \frac{\pi}{4} - \frac{\pi}{2} = -\frac{\pi}{4}$$

$$\Rightarrow \sum_{n=1}^{\infty} \left(\tan^{-1}(n) - \tan^{-1}(n+1)\right) = -\frac{\pi}{4} \quad \text{(Converges).}$$

Remark : The series in ④ and ⑤ and ⑥ are called Telescoping series.

⑤ $$\sum_{n=1}^{\infty} \left(\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}} \right) = 1 \quad \text{(Converges).}$$

$S_1 = 1 - \frac{1}{\sqrt{2}}$, $S_2 = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{3}}$, ...

$$S_n = \left(1 - \cancel{\frac{1}{\sqrt{2}}}\right) + \left(\cancel{\frac{1}{\sqrt{2}}} - \cancel{\frac{1}{\sqrt{3}}}\right) + \left(\cancel{\frac{1}{\sqrt{3}}} - \cancel{\frac{1}{\sqrt{4}}}\right) + \cdots + \left(\cancel{\frac{1}{\sqrt{n}}} - \frac{1}{\sqrt{n+1}}\right)$$

$$\Rightarrow \lim_{n\to\infty} S_n = \lim_{n\to\infty} \left(1 - \frac{1}{\sqrt{n+1}}\right) = 1$$

⑥ $\sum_{n=1}^{\infty} \ln\left(\frac{n}{n+1}\right) = \sum_{n=1}^{\infty} \ln n - \ln(n+1)$

$S_1 = (\ln 1 - \ln 2) = -\ln 2$

$S_2 = (\ln 1 - \ln 2) + (\ln 2 - \ln 3) = -\ln 3$

$\vdots$

$S_n = -\ln(n+1)$

$\lim_{n\to\infty} S_n = -\lim_{n\to\infty} \ln(n+1) = -\infty$

$\therefore \sum_{n=1}^{\infty} \ln\left(\frac{n}{n+1}\right)$ diverges.

Geometric series: (G.S)

Def: Geometric series are series of the form

$$a + ar + ar^2 + \dots + ar^{n-1} + \dots = \sum_{n=1}^{\infty} ar^{n-1}$$

in which a and r are fixed real numbers

with $a \neq 0$, (1st term).

Note: r is called the ratio and it can be positive or negative. $\left(\sum_{n=1}^{\infty} ar^{n-1} = \sum_{n=0}^{\infty} ar^n\right)$

Example: ① $1 + \frac{1}{2} + \frac{1}{4} + \cdots + \left(\frac{1}{2}\right)^{n-1} + \cdots$

is a geometric series with $a = 1$, $r = \frac{1}{2}$

② $1 - \frac{1}{3} + \frac{1}{9} + \cdots + \left(-\frac{1}{3}\right)^{n-1} + \cdots$

is a geometric series with $a = 1$ and $r = -\frac{1}{3}$.

To determine the Convergence or divergence of a geometric series, we have the following Cases:

1) If $\boxed{r = 1}$, then the G.S has the form:

$$\sum_{n=1}^{\infty} a r^{n-1} = a + a + a + \cdots + a + \cdots$$

So, the nth partial Sum of the G.S:

$S_1 = a$.

$S_2 = a + a = 2a$

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$S_3 = a + a + a = 3a$

$\vdots$

$S_n = na \implies \lim_{n\to\infty} S_n = \pm\infty$ (Diverges)

$\implies$ Geometric series diverges if $r = 1$

2) If $\boxed{r = -1}$, then:

$$\sum_{n=1}^{\infty} a(-1)^{n-1} = a\sum_{n=1}^{\infty}(-1)^{n-1} = a(1-1+1-\cdots)$$

Now: $S_1 = 1$

$S_2 = 1 - 1 = 0$

$S_3 = 1 - 1 + 1 = 1$

$S_4 = 1 - 1 + 1 - 1 = 0$

$\therefore \{S_n\} = \{1, 0, 1, 0, 1, 0, \cdots\}$

Clearly, $\lim_{n\to\infty} S_n$ D.N.E $\Rightarrow \sum_{n=1}^{\infty}(-1)^{n-1}$ diverges

$\therefore$ $a\sum_{n=1}^{\infty}(-1)^{n-1}$ diverges.

Corollary: If $r = \pm 1$, the Geometric series diverges.

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3) If $|r| \neq 1$, then the G.S:

$$\sum_{n=1}^{\infty} a r^{n-1} = a + ar + ar^2 + \cdots + ar^{n-1} + \cdots$$

Using the nth partial sum of the G. S:

$S_1 = a$

$S_2 = a + ar$

$S_3 = a + ar + ar^2$

$\vdots$

$$\left(S_n = a + ar + ar^2 + \cdots + ar^{n-1} \right)$$

Multiply by r:

$$\left(r S_n = ar + ar^2 + ar^3 + \cdots + ar^n \right)$$

Now, $S_n - r S_n = a - ar^n = a(1 - r^n)$

$$(1-r) S_n = a(1 - r^n).$$

$$\Rightarrow \quad S_n = \frac{a(1 - r^n)}{1-r}, \quad |r| \neq 1$$

If $|r| < 1$, then $r^n \to 0$ as $n \to \infty$

and $\lim_{n\to\infty} S_n = \dfrac{a}{1-r}$.

If $|r| > 1$, then $|r^n| \to \infty$ as $n \to \infty$

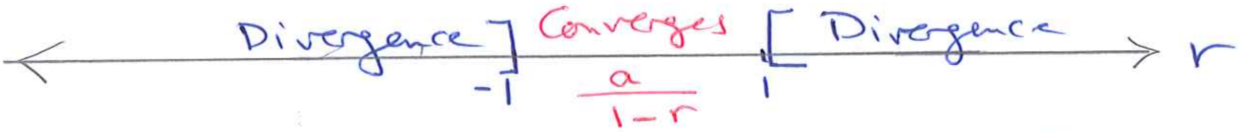
and $\lim_{n\to\infty} S_n$ diverges.

Conclusion: Consider the Geometric series

$$a + ar + ar^2 + \cdots + ar^{n-1} + \cdots$$

(1) If $|r| < 1$, then $\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}$

(2) If $|r| \geq 1$, then $\sum_{n=1}^{\infty} ar^{n-1}$ diverges.



Examples:

(a) $\sum_{n=1}^{\infty} \frac{1}{2^{n-1}} = 1 + \frac{1}{2} + \frac{1}{4} + \cdots$ is a G.S.

with $a = 1$, $r = \frac{1}{2}$.

Since $|r| < 1$, then $\sum_{n=1}^{\infty} \frac{1}{2^{n-1}} = \frac{a}{1-r} = \frac{1}{1-\frac{1}{2}} = \boxed{2}$

(b) $\sum_{n=0}^{\infty} \frac{(-1)^n 5}{4^n} = 5 + -\frac{5}{4} + \frac{5}{16} - \frac{5}{64} + \cdots$

is a G.S with $a = 5$, and $r = -\frac{1}{4}$

Since $\left|-\frac{1}{4}\right| < 1$, then

$$\sum_{n=0}^{\infty} \frac{(-1)^n 5}{4^n} = \frac{a}{1-r} = \frac{5}{1+\frac{1}{4}} = \boxed{4} \text{ (Converges)}.$$

(c) $\sum_{n=1}^{\infty} \frac{1}{9}\left(\frac{1}{3}\right)^{n-1} = \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \cdots$

with $a = \frac{1}{9}$ and $r = \frac{1}{3}$.

since $\left|\frac{1}{3}\right| < 1$, then

$$\sum_{n=1}^{\infty} \frac{1}{9}\left(\frac{1}{3}\right)^{n-1} = \frac{a}{1-r} = \frac{\frac{1}{9}}{1-\frac{1}{3}} = \boxed{\frac{1}{6}}$$

(d) $\sum_{n=1}^{\infty} 4 = 4 + 4 + 4 + 4 + \cdots$ Geometric series

with $r = 1 \Rightarrow \sum_{n=1}^{\infty} 4$ diverges.

Example: Express the repeating decimal $5.\overline{23}$ as the ratio of two integers.

$$5.\overline{23} = 5.23232323\ldots$$

$$= 5 + \frac{23}{100} + \frac{23}{(100)^2} + \frac{23}{(100)^3} + \cdots$$

$$= 5 + \frac{23}{100}\left[\underbrace{1 + \frac{1}{100} + \frac{1}{(100)^2} + \cdots}_{\text{G.S : } a=1,\ r = 1/100}\right]$$

$$= 5 + \frac{23}{100}\left(\frac{1}{1-\frac{1}{100}}\right) = 5 + \frac{23}{99} = \boxed{\frac{518}{99}}$$

Example: Express the repeating decimal $0.0\overline{6}$ as a ratio of two integers.

$$0.0\overline{6} = 0.06666\ldots = \frac{6}{100} + \frac{6}{1000} + \frac{6}{10000} + \cdots$$

$a = \frac{6}{100}$, $r = \frac{1}{10}$, then

$$0.0\overline{6} = \frac{\frac{6}{100}}{1 - \frac{1}{10}} = \frac{\frac{6}{100}}{\frac{9}{10}} = \frac{6}{90} = \frac{1}{15}.$$

Example: Find $\sum_{n=0}^{\infty} \left(\frac{5}{2^n} + \frac{1}{3^n} \right)$

$$= [5+1] + \left[\frac{5}{2} + \frac{1}{3}\right] + \left[\frac{5}{2^2} + \frac{1}{3^2}\right] + \cdots$$

$$= \left[5 + \frac{5}{2} + \frac{5}{2^2} + \cdots\right] + \left[1 + \frac{1}{3} + \frac{1}{3^2} + \cdots\right]$$

$$= \left[\frac{5}{1 - \frac{1}{2}}\right] + \left[\frac{1}{1 - \frac{1}{3}}\right].$$

$$= \frac{5}{\frac{1}{2}} + \frac{1}{\frac{2}{3}}$$

$$= 10 + \frac{3}{2} = \boxed{\frac{23}{2}}$$

The nth term test for a divergent series.

Theorem: If $\sum_{n=1}^{\infty} a_n$ Converges, then $a_n \xrightarrow[n\to\infty]{} 0$

Remark (*): The Converse of the theorem (↖) is not true, i-e, If $a_n \xrightarrow[n\to\infty]{} 0 \not\Rightarrow \sum_{n=1}^{\infty} a_n$ Converges.

Example: $\sum_{n=1}^{\infty} \frac{1}{n}$ (Harmonic series)

Notice that $\lim_{n\to\infty} \frac{1}{n} = 0$, but the series diverges. (We will see Later). section (10-3)

The nth term test for divergent series **Theorem:**

If $\lim_{n\to\infty} a_n \neq 0$ or fails to exist, then

 $\sum_{n=1}^{\infty} a_n$ diverges.

Example on Remark (*):

$$1 + \left(\frac{1}{2} + \frac{1}{2}\right) + \left(\frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4}\right) + \cdots + \underbrace{\left(\frac{1}{2^n} + \cdots + \frac{1}{2^n}\right)}_{n\text{-times}} + \cdots$$

$= 1 + 1 + 1 + \cdots$ which is diverges, Although $a_n \xrightarrow[]{n\to\infty} 0$

Examples: (Using nth term test for divergent).

1. $\sum_{n=1}^{\infty} n^2$ diverges, since $\lim_{n\to\infty} a_n = \lim_{n\to\infty} n^2 = \infty$.

2. $\sum_{n=1}^{\infty} \frac{n+1}{n}$ diverges, since $\lim_{n\to\infty} \frac{n+1}{n} = 1 \neq 0$

3. $\sum_{n=1}^{\infty} \left(1 - \frac{2023}{n}\right)^n$ diverges, since

$$\lim_{n\to\infty} \left(1 - \frac{2023}{n}\right)^n = e^{-2023} \neq 0. \quad \begin{pmatrix} \text{Thm } (*) \\ \text{in section} \\ 10.1 \end{pmatrix}$$

4. $\sum_{n=1}^{\infty} (-1)^{n+1}$ diverges, since $\lim_{n\to\infty} (-1)^{n+1}$ DNE.

5. $\sum_{n=1}^{\infty} \frac{-n}{2n+10}$ diverges, since $\lim_{n\to\infty} \frac{-n}{2n+10} = \frac{-1}{2} \neq 0$

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6. $\sum_{n=1}^{\infty} \frac{1}{2^{n-1}}$. Here the nth term test fails

since $\lim_{n\to\infty} a_n = 0$, but we showed that

its Converge by Geometric series test
or by nth partial Sum.

Combining Series :

Theorem : If $\sum_{n=1}^{\infty} a_n = A$ and $\sum_{n=1}^{\infty} b_n = B$

are convergent series, then : (k constant).

1) $\sum_{n=1}^{\infty} (a_n \pm b_n) = \sum_{n=1}^{\infty} a_n \pm \sum_{n=1}^{\infty} b_n = A \pm B$

2) $\sum_{n=1}^{\infty} k\, a_n = k \sum_{n=1}^{\infty} a_n = k A$

Collories :

1. Every nonzero constant multiple of a divergent series is divergent.

2. If $\sum_{n=1}^{\infty} a_n$ converges and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} (a_n + b_n)$ and $\sum_{n=1}^{\infty} (a_n - b_n)$ both diverge

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Caution : If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both diverge then $\sum_{n=1}^{\infty} (a_n + b_n)$ Can be Converge or diverge.

Example: $\sum_{n=1}^{\infty} a_n = 1+1+1+\cdots$, and

$\sum_{n=1}^{\infty} b_n = (-1)+(-1)+(-1)+\cdots$, both series

are divergent series, but:

$$\sum_{n=1}^{\infty}(a_n+b_n) = 0+0+0+\cdots = 0. \text{ (Converges)}$$

Example: Find the sums:

(a) $$\sum_{n=1}^{\infty} \frac{3^{n-1}-1}{6^{n-1}} = \sum_{n=1}^{\infty}\left(\frac{1}{2^{n-1}} - \frac{1}{6^{n-1}}\right)$$

$$= \underbrace{\sum_{n=1}^{\infty}\left(\frac{1}{2^{n-1}}\right)}_{\text{G.S}} - \underbrace{\sum_{n=1}^{\infty}\left(\frac{1}{6^{n-1}}\right)}_{\text{G.S}}$$

$$= \frac{1}{1-\frac{1}{2}} - \frac{1}{1-\frac{1}{6}} = 2 - \frac{6}{5} = \boxed{\frac{4}{5}}.$$

(b) $$\sum_{n=0}^{\infty} \frac{4}{2^n} = 4\sum_{n=0}^{\infty}\frac{1}{2^n}$$

$$= 4\left(1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots\right)$$

$$= 4\left(\frac{1}{1-\frac{1}{2}}\right) = 4.2 = \boxed{8}$$

Adding or Deleting terms:

(a) $\sum_{n=1}^{\infty} a_n$ Converges iff $\sum_{n=k}^{\infty} a_n$ Converges

for any $k > 1$

(b) $\sum_{n=1}^{\infty} a_n$ diverges iff $\sum_{n=k}^{\infty} a_n$ diverges;

for any $k > 1$

Example: $\sum_{n=1}^{\infty} \frac{1}{5^n}$ is a Geometric series which is Convergent.

$$\sum_{n=1}^{\infty} \frac{1}{5^n} = \frac{1}{5} + \frac{1}{25} + \frac{1}{125} + \sum_{n=4}^{\infty} \frac{1}{5^n} \rightarrow \text{converges}$$

$$\Leftrightarrow \sum_{n=4}^{\infty} \frac{1}{5^n} = \sum_{n=1}^{\infty} \frac{1}{5^n} - \left(\frac{1}{5} + \frac{1}{25} + \frac{1}{125}\right).$$

Reindexing:

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1+h}^{\infty} a_{n-h} = a_1 + a_2 + a_3 + \cdots$$

" Let $m = n - h$, when $n = 1+h \Rightarrow m = 1$ "

Example: For the series $\sum_{n=2}^{\infty} n\, 2^{n-1}$.

(a) start the index at $n=0$.

Let $m = n-2$, then when $n=2 \Rightarrow m=0$

$$\therefore \sum_{n=2}^{\infty} n\,2^{n-1} = \sum_{m=0}^{\infty} (m+2)\, 2^{((m+2)-1)} = \sum_{n=0}^{\infty} (n+2)\, 2^{n+1}.$$

(b) Write the power of 2 in the form $n+6$

Let $m+6 = n-1 \Rightarrow n = m+7$

and when $n=2 \Rightarrow m=-5$

$$\therefore \sum_{n=2}^{\infty} n\,2^{n-1} = \sum_{m=-5}^{\infty} (m+7)\, 2^{m+6} = \sum_{n=-5}^{\infty} (n+7)\, 2^{n+6}$$

Example: For which values of a does the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1} \cdot a \cdot 2^{n}}{a^{n-1}} \quad \text{Converges?}$$

sol: $\sum_{n=1}^{\infty} \frac{\cancel{(-1)^2}^{\,1}\,(-1)^{n-1} \cdot a \cdot 2 \cdot 2^{n-1}}{a^{n-1}}$ (توحيد القوى)

$$= \sum_{n=1}^{\infty} 2a \left(\frac{-2}{a}\right)^{n-1} \text{, which is Geometric series}$$

with $r = -\frac{2}{a}$.

The Geometric series Converges if $|r| < 1$

$\therefore \left| -\frac{2}{a} \right| < 1 \iff \left| \frac{a}{2} \right| > 1$

$\iff \frac{a}{2} > 1$ or $\frac{a}{2} < -1$

$\iff a > 2$ or $a < -2$

$\therefore a \in (-\infty, -2) \cup (2, \infty)$.

Example: Find $\sum_{n=1}^{\infty} \frac{6}{(2n-1)(2n+1)}$

Using partial fraction:

$$\sum_{n=1}^{\infty} \frac{6}{(2n-1)(2n+1)} = \sum_{n=1}^{\infty} \left(\frac{3}{(2n-1)} - \frac{3}{(2n+1)} \right) \quad \text{(Telescoping)}$$

$$S_1 = 3 - \frac{3}{3}$$

$$S_2 = (3-1) + \left(1 - \frac{3}{5}\right) = 3 - \frac{3}{5}$$

$$S_3 = (3-1) + \left(1 - \frac{3}{5}\right) + \left(\frac{3}{5} - \frac{3}{7}\right) = 3 - \frac{3}{7}$$

$\vdots$

$$S_n = 3 - \frac{3}{2n+1}$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \left(3 - \frac{3}{2n+1}\right) = 3 - 0 = \boxed{3}$$

$$\therefore \sum_{n=1}^{\infty} \frac{6}{(2n-1)(2n+1)} = \boxed{3}. \text{ (Converges)}.$$

Lecture Problems :

(37) Use nth partial sum to determine if the series $\sum_{n=1}^{\infty} \left(\ln\sqrt{n+1} - \ln\sqrt{n} \right)$ Converges or div.?

$$S_1 = \ln\sqrt{2} - \ln 1 = \ln\sqrt{2}$$

$$S_2 = (\ln\sqrt{2} - \ln 1) + (\ln\sqrt{3} - \ln\sqrt{2}) = \ln\sqrt{3}$$

$$\vdots$$

$$S_n = \ln\left(\sqrt{n+1}\right)$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \ln\left(\sqrt{n+1}\right) = +\infty \quad \text{(diverges)}.$$

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(51) Find $\sum_{n=1}^{\infty} (-1)^{n+1} \cdot \frac{3}{2^n} = \frac{3}{2} - \frac{3}{4} + \frac{3}{8} + \cdots$

The Series is Geometric series, with

$a = \frac{3}{2}$ and $r = -\frac{1}{2}$

$$\Rightarrow \sum_{n=1}^{\infty} (-1)^{n+1} \cdot \frac{3}{2^n} = \frac{a}{1-r} = \frac{3/2}{1 - -\frac{1}{2}} =$$

$$= \frac{3/2}{3/2} = \boxed{1} \quad \text{(Converges)}$$

(64) $\displaystyle\sum_{n=1}^{\infty} \frac{2^n+4^n}{3^n+4^n}$

$$\lim_{n\to\infty} \frac{2^n+4^n}{3^n+4^n} = \lim_{n\to\infty} \frac{4^n\left[\frac{2^n}{4^n}+1\right]}{4^n\left[\frac{3^n}{4^n}+1\right]}$$

$$= \lim_{n\to\infty} \frac{\left(\frac{1}{2}\right)^n+1}{\left(\frac{3}{4}\right)^n+1} = \frac{0+1}{0+1} = 1 \neq 0.$$

So, by the nth term test, the series diverges.

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(75) Find the values of x for which the geometric series converges for $\displaystyle\sum_{n=0}^{\infty}(-1)^n(x+1)^n$

$$\sum_{n=0}^{\infty}(-1)^n(x+1)^n = 1-(x+1)+(x+1)^2-(x+1)^3+\cdots$$

$a=1$ and $r=-(x+1)$.

The Geometric series converges if $|r|<1$

$\Rightarrow \quad |-(x+1)| = |x+1| < 1$

$\Leftrightarrow \quad -1 < x+1 < 1 \quad \Leftrightarrow \quad -2 < x < 0$

10.3 The Integral Test.

Recall : Theorem (6) (section 10.1) :

The monotonic sequence Theorem :

If a sequence $\{a_n\}$ is both bounded and monotonic, then the sequence Converges.

Corollary of Theorem (6) :

A series $\sum_{n=1}^{\infty} a_n$ of nonnegative terms Converges if and only if its partial sums are bounded from above.

Example : $\sum_{n=1}^{\infty} \frac{1}{n}$ (Harmonic series).

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots$$

The harmonic series is divergent. (Although $\underbrace{\lim \frac{1}{n} = 0}_{\text{Fails}}$) (n th term test)

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \underbrace{\left(\frac{1}{3} + \frac{1}{4}\right)}_{> \frac{2}{4} = \frac{1}{2}} + \underbrace{\left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right)}_{> \frac{4}{8} = \frac{1}{2}} + \cdots$$

The sequence of the partial sums is not bounded from above $\Rightarrow$ The harmonic series diverges.
The divergence is very slow.

Theorem: The integral Test:

Let $\{a_n\}$ be a sequence of positive terms.

Suppose that $a_n = f(n)$, where f is a ① Continuous, ② positive, ③ decreasing function of x, $\forall x \geq N$ (where N is a positive integer). Then the series $\sum_{n=N}^{\infty} a_n$ and the integral $\int_N^{\infty} f(x)\,dx$ both converge or both diverge.

Example: Do the following series Converge? diverge?

① $\sum_{n=1}^{\infty} \frac{1}{n}$

Let $f(x) = \frac{1}{x}$, $x \geq 1$. f is Continuous, positive and decreasing for $x \geq 1$. $\left[f'(x) = \frac{-1}{x^2} < 0 \right]$

Now, $\int_1^{\infty} \frac{1}{x}\,dx = \lim_{A\to\infty} \int_1^{A} \frac{1}{x}\,dx = \lim_{A\to\infty} \ln|x| \Big]_1^A$

$= \lim_{A\to\infty} (\ln A - \ln 1) = \infty$ (Diverges)

Therefore, Using the Integral Test $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

② $\sum_{n=1}^{\infty} \frac{1}{n^2}$

Let $f(x) = \frac{1}{x^2}$, $x \geq 1$. f is Continuous, positive and decreasing on $x \geq 1$. $\left[f'(x) = \frac{-2}{x^3} < 0, \ \forall x \geq 1 \right]$

Now, $\int_1^{\infty} \frac{1}{x^2}\, dx$ Converges to 1 (Using p-test section 8.7)

There fore, $\sum_{n=1}^{\infty} \frac{1}{x^2}$ also converges by Integral Test.

Note : $\sum_{n=1}^{\infty} \frac{1}{x^2}$ need not equal 1 (we don't know).

③ $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$

$f(x) = \frac{1}{x^2+1}$, $x \geq 1$. f is Continuous, positive and decreasing on $x \geq 1$. $\left[f'(x) = \frac{-2x}{(x^2+1)^2} < 0, \ \forall x \geq 1 \right]$

Now, $\int_1^{\infty} \frac{1}{x^2+1}\, dx = \lim_{A \to \infty} \int_1^{A} \frac{1}{x^2+1}\, dx = \lim_{A \to \infty} \tan^{-1} x \Big|_1^{A}$

$= \lim_{A \to \infty} \left(\tan^{-1} A - \tan^{-1} 1 \right) = \lim_{A \to \infty} \tan^{-1} A - \frac{\pi}{4}$

$= \frac{\pi}{2} - \frac{\pi}{4} = \boxed{\frac{\pi}{4}}$ Converge

(sum need not be $\frac{\pi}{4}$ for the series.)

$\therefore$ By Integral Test $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ Converges.

④ $\sum_{n=1}^{\infty} \frac{1}{n^P}$ (P– series) $= \begin{cases} \text{Converges}, & P>1 \\ \text{Diverges}, & P \leq 1 \end{cases}$

• If $P>1$, Let $f(x) = \frac{1}{x^P}$, $x \geq 1$.

f is Continuous, positive, decreasing on $x \geq 1$.

Now, $\int_1^{\infty} \frac{1}{x^P}\,dx = \lim_{A\to\infty} \int_1^{A} \frac{1}{x^P}\,dx = \lim_{A\to\infty} \int_1^{A} x^{-P}\,dx$

$= \lim_{A\to\infty} \left(\frac{x^{-P+1}}{-P+1} \Big|_1^A \right) = \lim_{A\to\infty} \left(\frac{A^{1-P}}{1-P} - \frac{1}{1-P} \right)$

$= \left(0 - \frac{1}{1-P} \right) = \frac{1}{P-1}$ (Converge)

∴ By Integral Test $\sum_{n=1}^{\infty} \frac{1}{n^P}$ Converges for $P>1$.

Note: The sum of the series is Not $\frac{1}{P-1}$.

• If $P<1$, then $1-P>0$, then

$\int_1^{\infty} \frac{1}{x^P}\,dx = \lim_{A\to\infty} \left(\frac{A^{1-P}}{1-P} - \frac{1}{1-P} \right) = \infty$ (diverge)

• If $P=1$, then the series becomes Harmonic series which is divergent.

(5) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ Converges

(p-series with $p = \frac{3}{2} > 1$).

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(6) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{\pi - e}}$ diverges. (p-series, $p = \pi - e < 1$)

---

(7) $\displaystyle\sum_{n=1}^{\infty} n \sin\left(\frac{1}{n}\right)$, diverges by the nth

term test for divergence.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}} = \lim_{x\to 0} \frac{\sin x}{x} = 1 \neq 0.$$

---

(8) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n-1}$, ($a_n \to 0$ as $n \to \infty$) (It may Converge & may not)

Let $f(x) = \frac{1}{2x-1}$, $x \geq 1$. f is Continuous, positive

and decreasing on $x \geq 1$.

Now, $$\int_1^{\infty} \frac{1}{2x-1}\,dx = \lim_{A\to\infty} \frac{1}{2} \ln|2x-1| \Big|_1^A = \lim_{A\to\infty} \frac{1}{2}\ln(2A-1) = \infty$$

Therefore, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n-1}$ diverges by the Integral Test.

⑨ $\sum_{n=1}^{\infty} n^2 \, 2^{-n^3}$ $\left(a_n \xrightarrow[n\to\infty]{} 0 \right.$, nth test fails).

Let $f(x) = \dfrac{x^2}{2^{x^3}}$, $x \geq 1$.

f is continuous and positive and decreasing $\forall x \geq 1$

([Decreasing: $f'(x) = \dfrac{2x\, 2^{x^3} - (x^2)(\ln 2)\, 2^{x^3} \cdot (3x^2)}{\left(2^{x^3}\right)^2}$

$f'(x) = \dfrac{x\, \cancel{2^{x^3}} \left(2 - 3\ln 2\, x^3\right)}{\left(2^{x^3}\right)^{\cancel{2}}} < 0, \ \forall x \geq 1$].)

Now, $\int_1^{\infty} \dfrac{x^2}{2^{x^3}} \, dx$, $\left\{ \begin{array}{l} \text{Let } u = -x^3 \Rightarrow du = -3x^2 dx \\ \text{when } x = 1 \Rightarrow u = -1 \\ \quad\quad x = A \Rightarrow u = -A^3. \end{array} \right.$

$= \lim_{A\to\infty} \int_1^{A} x^2 \, 2^{-x^3} \, dx$

$= \lim_{A\to\infty} \int_{-1}^{-A^3} -\frac{1}{3} 2^u \, du = -\frac{1}{3} \lim_{A\to\infty} \left. \frac{2^u}{\ln 2} \right|_{-1}^{-A^3}$

$= -\frac{1}{3} \lim_{A\to\infty} \left[\frac{2^{-A^3}}{\ln 2} - \frac{2^{-1}}{\ln 2} \right] = -\frac{1}{3}\left(\frac{-2^{-1}}{\ln 2} \right) = \frac{1}{6\ln 2}$

which is Convergent, so by Integral test:

$\sum_{n=1}^{\infty} n^2 \, 2^{-n^3}$ is Convergent.

Lecture Problems:

(Q8) $\displaystyle\sum_{n=2}^{\infty} \frac{\ln(n^2)}{n}$

Let $f(x) = \dfrac{\ln(x^2)}{x}$, $x \geq 2$

f is Continuous, positive and decreasing on $x \geq 2$

(Decreasing : $f'(x) = \dfrac{\frac{x}{x^2}(2x) - \ln(x^2)}{x^2} = \dfrac{2-\ln(x^2)}{x^2} \overset{???}{<} 0$)

$2 - \ln(x^2) < 0$ iff $2 < \ln(x^2)$

iff $e^2 < x^2$

iff $e < |x|$,

Since $x \geq 2$, thus f is decreasing for $x \geq 3$

Now, $\displaystyle\int_3^{\infty} \frac{\ln(x^2)}{x}\, dx = \lim_{A\to\infty} \int_3^{A} \frac{\ln(x^2)}{x}\, dx$

substitution

$\uparrow$

$= \displaystyle\lim_{A\to\infty} (\ln x)^2 \Big|_3^A = \lim_{A\to\infty} \left((\ln A)^2 - (\ln 3)^2\right) = \infty$ (Diverge)

$\Rightarrow \displaystyle\sum_{n=3}^{\infty} \frac{\ln(n^2)}{n}$ diverges by Integral Test.

Therefore, $\displaystyle\sum_{n=2}^{\infty} \frac{\ln(n^2)}{n} = \frac{\ln 4}{2} + \sum_{n=3}^{\infty} \frac{\ln(n^2)}{n}$ diverges

Q26 $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}\,(\sqrt{n}+1)}$

Let $f(x) = \dfrac{1}{\sqrt{x}\,(\sqrt{x}+1)}$, $x \geq 1$

f is positive, continuous and decreasing on $x \geq 1$.

check !!!

Now, $\displaystyle\int_1^{\infty} \frac{1}{\sqrt{x}\,(\sqrt{x}+1)}\,dx = \lim_{A\to\infty} \int_1^{A} \frac{1}{\sqrt{x}\,(\sqrt{x}+1)}\,dx$

$\displaystyle = \lim_{A\to\infty} 2\ln(\sqrt{x}+1)\Big|_1^A = \lim_{A\to\infty} 2\ln(\sqrt{A}+1) - 2\ln 2$

$= \infty$, diverges.

$\Rightarrow$ The series diverges by the Integral Test.

Q34 $\displaystyle\sum_{n=1}^{\infty} n \tan\left(\frac{1}{n}\right)$ diverges

By nth term test for divergence, since

$\displaystyle\lim_{n\to\infty} n\tan\left(\frac{1}{n}\right) = \lim_{n\to\infty} \frac{\tan\left(\frac{1}{n}\right)}{\frac{1}{n}}$

$\displaystyle = \lim_{n\to\infty} \frac{\sec^2\left(\frac{1}{n}\right)\cdot\left(\frac{-1}{n^2}\right)}{\left(\frac{-1}{n^2}\right)} = 1 \neq 0 \Rightarrow$ Series diverges.

10.4 Comparison Test.

Theorem: The Comparison Test. (D.C.T)

Let $\sum_{n=1}^{\infty} a_n$, $\sum_{n=1}^{\infty} c_n$ and $\sum_{n=1}^{\infty} d_n$ be series with nonnegative terms. Suppose that for some integer N

$$d_n \leq a_n \leq c_n \quad , \quad \forall n > N$$

(a) If $\sum_{n=1}^{\infty} c_n$ Converges, then $\sum_{n=1}^{\infty} a_n$ also Converges.

(b) If $\sum_{n=1}^{\infty} d_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ also diverges.

Example: Determine if each series Converges or diverges.

① $\sum_{n=1}^{\infty} \frac{5}{5n-1}$

Notice that $5n > 5n-1 \iff \frac{1}{5n} < \frac{1}{5n-1}$

$$\iff \frac{5}{5n} < \frac{5}{5n-1} \iff \underbrace{\frac{1}{n}}_{d_n} < \underbrace{\frac{5}{5n-1}}_{a_n} \quad , \forall n \geq 1$$

Since $\underbrace{\sum_{n=1}^{\infty} \frac{1}{n}}_{\text{Harmonic series}}$ diverges $\Rightarrow \sum_{n=1}^{\infty} \frac{5}{5n-1}$ diverges by D.C.T

② $\sum_{n=1}^{\infty} \frac{1+\cos n}{n^2} \leq \sum_{n=1}^{\infty} \frac{2}{n^2}$

But : $2\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a Convergent p-series $(p=2)$

$\Rightarrow \sum_{n=1}^{\infty} \frac{1+\cos n}{n^2}$ Converges.

③ $\sum_{n=1}^{\infty} \frac{2n}{3n-1}$ diverges by nth term test

since $\lim_{n\to\infty} \frac{2n}{3n-1} = \frac{2}{3} \neq 0$

④ $\sum_{n=1}^{\infty} \frac{\sqrt{n}+1}{n+3}$

Notice that $\sqrt{n}+1 > \sqrt{n}$ and $n+3 < n+n, \forall n>3$

$\sqrt{n}+1 > \sqrt{n}$ and $\frac{1}{n+3} > \frac{1}{2n}$ $, \forall n>3$

$\Leftrightarrow \frac{\sqrt{n}+1}{n+3} > \frac{\sqrt{n}}{2n} = \frac{1}{2\sqrt{n}}$ $, \forall n>3$

But $\sum_{n=1}^{\infty} \frac{1}{2\sqrt{n}}$ diverges (p-series with $p=\frac{1}{2}$).

then By D.C.T : $\sum_{n=4}^{\infty} \frac{\sqrt{n}+1}{n+3}$ diverges

$\Rightarrow \sum_{n=1}^{\infty} \frac{\sqrt{n}+1}{n+3}$ diverges (adding 3 terms)

⑤ $\displaystyle\sum_{n=1}^{\infty} \frac{3}{n+\sqrt{n}}$

Notice that $n+n+n > 0+n+\sqrt{n}$

$\Leftrightarrow \quad 3n > n+\sqrt{n} \quad \Leftrightarrow \quad n > \dfrac{n+\sqrt{n}}{3}$

$\Leftrightarrow \quad \dfrac{1}{n} < \dfrac{3}{n+\sqrt{n}}$.

Notice that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges. , then by D.C.T

$\displaystyle\sum_{n=1}^{\infty} \frac{3}{n+\sqrt{n}}$ diverges.

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⑥ $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n}{3n+1}\right)^n < \sum_{n=1}^{\infty} \left(\frac{n}{3n}\right)^n = \sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n$

The last series is Geometric series, with $r = \frac{1}{3}$

$\therefore \displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n = \frac{1}{1-\frac{1}{3}} = \frac{3}{2}$ Converges.

$\therefore$ By D.C.T : $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n}{3n+1}\right)^n$ Converges.

$= \dfrac{1}{1-\frac{1}{2}} = \boxed{2}$

⑦ $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n!} = 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots \leq 1 + \sum_{n=0}^{\infty} \frac{1}{2^n}$

For $n \geq 2$, $n! = n(n-1)\cdots 3.2 \geq 2.2.2.2\ldots.2 = 2^{n-1}$

$\therefore \displaystyle\sum_{n=0}^{\infty} \frac{1}{n!}$ Converges by D.C.T.

$\displaystyle\sum_{n=1}^{\infty}$ ← multiply by ②

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Theorem: Limit Comparison Test. (L.C.T)

Suppose that $a_n > 0$ and $b_n > 0$, $\forall n \geq N$ where N is an integer.

1) If $\lim_{n\to\infty} \frac{a_n}{b_n} = c > 0$, then $\sum a_n$ and $\sum b_n$ both Converge or both diverge.

2) If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\sum b_n$ Converges, then $\sum a_n$ Converges.

3) If $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ and $\sum b_n$ diverges, then $\sum a_n$ diverges.

Examples: Discuss the Convergence of the following series.

(1) $$\sum_{n=1}^{\infty} \frac{\sqrt{n}+1}{n+3}$$

We showed that the series diverges Using D.C.T

Now, Using L.C.T :

Let $a_n = \frac{\sqrt{n}+1}{n+3}$ and $b_n = \frac{\sqrt{n}}{n} = \frac{1}{\sqrt{n}}$

Notice that $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges (p-series , $p = \frac{1}{2}$)

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{\sqrt{n}+1}{n+3}}{\frac{1}{\sqrt{n}}} = \lim_{n\to\infty} \frac{n+\sqrt{n}}{n+3} = 1$$

so, by L.C.T , $\sum_{n=1}^{\infty} \frac{\sqrt{n}+1}{n+3}$ diverges.

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② $\sum_{n=1}^{\infty} \frac{n-2}{n^3-n^2+3}$ (We expect to behave like $\frac{n}{n^3}$)

Let $a_n = \frac{n-2}{n^3-n^2+3}$ and $b_n = \frac{1}{n^2}$.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty}\left(\frac{n-2}{n^3-n^2+3}\right)\cdot\left(\frac{n^2}{1}\right) = \lim_{n\to\infty} \frac{n^3-2n^2}{n^3-n^2+3} = \boxed{1}$$

We know that $\sum_{n=1}^{\infty} \frac{1}{n^2}$ Converges (p-series , $p = 2$)

$\therefore \sum_{n=1}^{\infty} \frac{n-2}{n^3-n^2+3}$ Converges. by L.C.T.

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(3) $\sum_{n=1}^{\infty} \frac{1}{1+\ln n}$. Take $b_n = \frac{1}{n}$.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1}{(1+\ln n)} \cdot \left(\frac{n}{1}\right)$$

$$\overset{L.H}{=} \lim_{n\to\infty} \frac{1}{\frac{1}{n}} = \lim_{n\to\infty} n = \infty.$$

Since $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, then $\sum_{n=1}^{\infty} \frac{1}{1+\ln n}$ diverges. By L.C.T

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(4) $\sum_{n=1}^{\infty} \sin\frac{1}{n}$. Take $b_n = \frac{1}{n}$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\sin \frac{1}{n}}{\frac{1}{n}} = \lim_{x\to 0} \frac{\sin x}{x} = 1 > 0$$

$= x$

Since $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, then $\sum_{n=1}^{\infty} \sin\frac{1}{n}$ diverges.

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(5) $\sum_{n=1}^{\infty} \frac{1-n}{n2^n} = \sum_{n=1}^{\infty} \frac{1}{n2^n} - \sum_{n=1}^{\infty} \frac{1}{2^n}$ → G.S $r = \frac{1}{2}$

Notice that $\frac{1}{n2^n} < \frac{1}{2^n} \Rightarrow \sum \frac{1}{2^n}$ Converges

By D.C.T: $\sum_{n=1}^{\infty} \frac{1}{n2^n}$ Converges. $\Rightarrow \sum_{n=1}^{\infty} \frac{1-n}{n2^n}$ Converges.

⑥ $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n^{3/2}}$

Recall: $\ln x$ grows slower than x^r, $r>0$, $x\to\infty$

(i.e) $\ln n < n^r$. (Calculus I (7.8)).

Let $a_n = \dfrac{\ln n}{n^{3/2}}$ and $b_n = \dfrac{1}{n^{5/4}}$.

Notice that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{5/4}}$ converges (p-series, $p = \frac{5}{4} > 1$)

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \left(\frac{\ln n}{n^{3/2}}\right)\cdot\left(\frac{n^{5/4}}{1}\right) = \lim_{n\to\infty} \frac{\ln n}{n^{1/4}} \overset{L.H}{=} 0$$

$\therefore \displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n^{3/2}}$ converges by L.C.T.

⑦ $\displaystyle\sum_{n=2}^{\infty} \frac{1+n\ln n}{n^2+5}$. Take $b_n = \dfrac{1}{n}$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \left(\frac{1+n\ln n}{n^2+5}\right)\cdot\left(\frac{n}{1}\right) = \lim_{n\to\infty} \frac{n+n^2\ln n}{n^2+5}$$

$$\overset{L.H}{=} \lim_{n\to\infty} \frac{1+\frac{n^2}{n}+2n\ln n}{2n} \overset{L.H}{=} \lim_{n\to\infty} \frac{1+2n\left(\frac{1}{n}\right)+2\ln n}{2} \boxed{=\infty}$$

Notice that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges

$\Rightarrow \displaystyle\sum_{n=1}^{\infty} \frac{1+n\ln n}{n^2+5}$ diverges by L.C.T.

Lecture Problems :

Q10 Use L.C.T to determine whether the following series converges or diverges.

$$\sum_{n=1}^{\infty} \sqrt{\frac{n+1}{n^2+2}} .$$

$\sqrt{\frac{n}{n^2}}$

Let $a_n = \sqrt{\frac{n+1}{n^2+2}}$ and $b_n = \frac{1}{\sqrt{n}}$

Notice that $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ is divergent p-series $(P = \frac{1}{2} < 1)$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \left(\sqrt{\frac{n+1}{n^2+2}}\right) \cdot \left(\frac{\sqrt{n}}{1}\right)$$

$$= \lim_{n\to\infty} \sqrt{\frac{n^2+n}{n^2+2}}$$

$$= \sqrt{\lim_{n\to\infty} \frac{n^2+n}{n^2+2}} = 1 > 0$$

$\Rightarrow$ By L.C.T : $\sum_{n=1}^{\infty} \sqrt{\frac{n+1}{n^2+2}}$ diverges.

Q22 $\sum_{n=1}^{\infty} \frac{n+1}{n^2\sqrt{n}}$

Let $a_n = \frac{n+1}{n^2\sqrt{n}}$ and $b_n = \frac{1}{n^{3/2}}$.

Notice that $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ is Convergent p-series $(p = \frac{3}{2} > 1)$

$\Rightarrow \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \left(\frac{n+1}{n^2\sqrt{n}}\right) \cdot \left(\frac{n^{3/2}}{1}\right)$

$= \lim_{n\to\infty} \frac{n+1}{n} = 1 > 0$

$\Rightarrow$ By L.C.T : $\sum_{n=1}^{\infty} \frac{n+1}{n^2\sqrt{n}}$ is Convergent series.

Q36 $\sum_{n=1}^{\infty} \frac{n+2^n}{n^2 2^n} = \sum_{n=1}^{\infty} \frac{1}{n2^n} + \sum_{n=1}^{\infty} \frac{1}{n^2}$

Notice that $\frac{1}{n2^n} < \frac{1}{2^n}$.

$\sum_{n=1}^{\infty} \frac{1}{2^n}$ is Geometric series with $p = \frac{1}{2}$

So By D.C.T : $\sum_{n=1}^{\infty} \frac{1}{n2^n}$ is Convergent

and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent p-series $(p = 2 > 1)$

$\Longrightarrow \sum_{n=1}^{\infty} \frac{n+2^n}{n^2 2^n}$ is Convergent series.

Q46) $\sum_{n=1}^{\infty} \tan\left(\frac{1}{n}\right)$

Let $a_n = \tan\left(\frac{1}{n}\right)$ and $b_n = \frac{1}{n}$.

Notice that $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent Harmonic series.

$$\lim_{n\to\infty} \frac{\tan\left(\frac{1}{n}\right)}{\left(\frac{1}{n}\right)} = \lim_{n\to\infty} \left(\frac{1}{\cos\frac{1}{n}}\right)\left(\frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}}\right)$$

$$= \lim_{x\to 0} \left(\frac{1}{\cos x}\right)\left(\frac{\sin x}{x}\right) = 1.1 = 1$$

$\therefore$ By L.C.T : $\sum_{n=1}^{\infty} \tan\left(\frac{1}{n}\right)$ diverges.

Extra Problems: (10.3 + 10.4)

① $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n + \sqrt{n}}$

Notice that $2^n + \sqrt{n} > 2^n \quad , \forall n$

$\Rightarrow \quad \dfrac{1}{2^n + \sqrt{n}} < \dfrac{1}{2^n}$

$\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n}$ is convergent Geometric series $(r = \frac{1}{2})$

$\therefore$ By D.C.T : $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n + \sqrt{n}}$ converges.

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② $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n}}{\cos(2n-1) + n^2}$

Notice that $\cos(2n-1) > -1$ (add n^2)

$\Rightarrow \cos(2n-1) + n^2 > n^2 - 1 > n^2 - \frac{n^2}{2} = \frac{n^2}{2} \quad , \forall n \geq 2$

$\Rightarrow \dfrac{1}{\cos(2n-1) + n^2} < \dfrac{2}{n^2}$ (Multiply by $\sqrt{n}$)

$\Rightarrow \displaystyle\sum_{\boxed{n=2}}^{\infty} \frac{\sqrt{n}}{\cos(2n-1) + n^2} < \sum_{\boxed{n=2}}^{\infty} \frac{2\sqrt{n}}{n^{3/2}} = \underbrace{\sum_{n=2}^{\infty} \frac{2}{n^{3/2}}}_{\Downarrow}$

Converges p-series $p = \frac{3}{2} > 1$

$\therefore$ By D.C.T : $\displaystyle\sum_{\boxed{n=1}}^{\infty} \frac{\sqrt{n}}{\cos(2n-1) + n^2}$ Converges.

③ $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$.

$\boxed{i)}$. If we take $a_n = \frac{1}{n \ln n}$ and $b_n = \frac{1}{n}$

then $\lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n \to \infty} \frac{\left(\frac{1}{n \ln n}\right)}{\left(\frac{1}{n}\right)} = \lim_{n \to \infty} \frac{1}{\ln n} = 0$.

since $\sum_{n=2}^{\infty} \frac{1}{n}$ diverges, then L.C.T fails.

$\boxed{ii)}$ If we take $a_n = \frac{1}{n \ln n}$ and $b_n = \frac{1}{n^{5/4}}$

then $\lim_{n \to \infty} \frac{a_n}{b_n} = \infty$ (check !!!).

Since $\sum_{n=2}^{\infty} \frac{1}{n^{5/4}}$ converges, then L.C.T fails.

$\boxed{iii)}$ Using Integral Test: Let $f(x) = \frac{1}{x \ln x}$, $x \geq 2$

check !!!

f is continuous, positive & decreasing $\forall n \geq 2$, then

$\int_2^{\infty} \frac{dx}{x \ln x} = \lim_{A \to \infty} \int_2^{A} \frac{dx}{x \ln x} \overset{\text{substitution}}{=} \lim_{A \to \infty} \int_{\ln 2}^{\ln A} \frac{du}{u} =$

$\lim_{A \to \infty} \Big[\ln|u|\Big]_{\ln 2}^{\ln A} = \lim_{A \to \infty} \Big[\ln(\ln A) - \ln(\ln 2)\Big] = \infty$

$\therefore \sum_{n=1}^{\infty} \frac{1}{n \ln n}$ diverges,

10.5 The ratio and Root tests.

Theorem: The Ratio Test:

Let $\sum a_n$ be a series with positive terms and suppose that $\lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n} = \rho$. Then:

(a) The series Converges if $\rho < 1$.

(b) The series diverges if $\rho > 1$. or ρ is infinite

(c) The test is inconclusive if $\rho = 1$.

Example: Investigate the Convergence of the following:

(1) $\sum\limits_{n=1}^{\infty} \frac{n^2}{e^n}$.

Let $a_n = \frac{n^2}{e^n} \Rightarrow a_{n+1} = \frac{(n+1)^2}{e^{n+1}}$

$$\rho = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)^2}{e^{n+1}} \frac{e^n}{n^2} = \lim_{n\to\infty} \left(\frac{1}{e}\right)\left(\frac{n+1}{n}\right)^2$$

$$\rho = \frac{1}{e} < 1$$

$\therefore$ By ratio test, the series Converges.

(2) $\sum \frac{n!}{e^n}$

$a_{n+1} \cdot \frac{1}{a_n}$

$$\rho = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \left(\frac{(n+1)!}{e^{n+1}}\right) \cdot \left(\frac{e^n}{n!}\right) = \frac{1}{e} \lim_{n\to\infty} (n+1)$$

$\Rightarrow \rho = \infty$. Hence the series diverges by Ratio Test

Remark: When $\rho = 1$, then the series could be converge or diverge.

(3) $\sum_{n=1}^{\infty} \frac{1}{n}$ (Harmonic series).

$$\rho = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \left(\frac{1}{n+1}\right)\left(\frac{n}{1}\right) = 1. \text{ (Inconclusive)}$$

But we know that the series diverges.

(4) $\sum_{n=1}^{\infty} \frac{1}{n^2}$ (p- series).

$$\rho = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \left(\frac{1}{(n+1)^2}\right)\left(\frac{n^2}{1}\right) = 1. \text{ (Inconclusive)}.$$

But the series converges (p- series, $p=2$).

(5) $\sum_{n=0}^{\infty} \frac{2^n+5}{3^n}$

$$\rho = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\left(\frac{2^{n+1}+5}{3^{n+1}}\right)\cdot\left(\frac{3^n}{2^n+5}\right) = \lim_{n\to\infty}\frac{2^{n+1}+5}{3(2^n+5)}$$

$$\overset{L.H}{=} \frac{1}{3}\lim_{n\to\infty}\frac{2^{n+1}\ln 2}{2^n \ln 2} = \frac{1}{3}\lim_{n\to\infty} 2 = \frac{2}{3} < 1.$$

$\therefore$ By Ratio Test, the series Converges.

(Be aware that $\frac{2}{3}$ is not the sum of the series).

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(6) $\sum_{n=1}^{\infty} \frac{(n+3)!}{3!\, n!\, 3^n}$

$$\rho = \lim_{n\to\infty}\left(\frac{(n+4)!}{3!\,(n+1)!\,3^{n+1}}\right)\cdot\left(\frac{3!\,n!\,3^n}{(n+3)!}\right)$$

$$= \lim_{n\to\infty}\frac{(n+4)}{3(n+1)} = \frac{1}{3} < 1.$$

Then the series Converges by Ratio Test.

Remark: Usually, when we have factorial, we use Ratio Test provided $\rho \neq 1$.

(7) $\displaystyle\sum_{n=1}^{\infty} \frac{(2n)!}{n!\,n!}$

$$\rho = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \left(\frac{(2n+2)!}{(n+1)!\,(n+1)!}\right) \cdot \left(\frac{n!\,n!}{(2n)!}\right)$$

$$= \lim_{n\to\infty} \frac{(2n+2)(2n+1)(2n)!\,n!\,n!}{(n+1)n!\,(n+1)n!\,(2n)!}$$

$$= \lim_{n\to\infty} \frac{(2n+2)(2n+1)}{(n+1)(n+1)} = \lim_{n\to\infty} \frac{4n+2}{n+1} = 4 > 1$$

By Ratio Test, the series diverges.

(8) $\displaystyle\sum_{n=1}^{\infty} \frac{4^n\,n!\,n!}{(2n)!}$

$$\rho = \lim_{n\to\infty} \frac{4^{n+1}(n+1)!\,(n+1)!\,(2n)!}{(2n+2)!\,4^n\,n!\,n!} \qquad \frac{a_{n+1}}{a_n}$$

$$= \lim_{n\to\infty} \frac{\cancel{4^n}\cdot 4\cdot (n+1)\cancel{n!}\,(n+1)\cancel{n!}\,\cancel{(2n)!}}{(2n+2)(2n+1)\cancel{(2n)!}\,\cancel{4^n}\,\cancel{n!}\,\cancel{n!}}$$

$$= \lim_{n\to\infty} \frac{4\cancel{(n+1)}(n+1)}{2\cancel{(n+1)}(2n+1)} = \lim_{n\to\infty} \frac{2n+2}{2n+1} = 1$$

Since $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1 \Rightarrow$ Ratio test is inconclusive.

But we notice that

$$\frac{a_{n+1}}{a_n} = \frac{2n+2}{2n+1} > 1 \qquad \text{(why?)}$$

$$\Rightarrow \quad a_{n+1} > a_n \quad , \forall n$$

(i.e) $\quad a_2 > a_1$ and $a_3 > a_2 > a_1 \cdots$

$$\Rightarrow \quad a_n > a_1 = 2 \quad , \forall n.$$

$$\Rightarrow \quad \lim_{n\to\infty} a_n \neq 0$$

So by the nth term test, the series diverges.

(a) $\quad a_1 = 1 \quad , \quad a_{n+1} = \left(\frac{1+\tan^{-1} n}{n}\right) a_n$

$$\rho = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\left(\frac{1+\tan^{-1} n}{n}\right) \cancel{a_n}}{\cancel{a_n}}$$

$$= \lim_{n\to\infty} \frac{1+\tan^{-1} n}{n} = 0 < 1$$

∴ By Ratio test, the series Converges.

Theorem: The Root Test.

Let $\sum a_n$ be a series with $a_n \geq 0$, $\forall n \geq N$

and suppose that $\lim_{n\to\infty} \sqrt[n]{a_n} = \rho$, then

(a) The series Converges if $\rho < 1$.

(b) The series diverges if $\rho > 1$. or ρ is infinite

(3) The test is inconclusive if $\rho = 1$.

Example: which of the following series Converges and which diverges?

$\boxed{1}$ $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$.

By Root Test, the series Converges, since:

$$\rho = \lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \frac{\sqrt[n]{n^2}}{\sqrt[n]{2^n}} = \lim_{n\to\infty} \frac{(\sqrt[n]{n})^2}{2} = \frac{1}{2} < 1$$

By Ratio Test, the series Converges, since:

$$\rho = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \left(\frac{(n+1)^2}{2^{n+1}}\right) \cdot \left(\frac{2^n}{n^2}\right) = \frac{1}{2} < 1.$$

Remark: Notice that if we can solve Using Root test & Ratio test, then ρ's are equal.

$\boxed{2}$ $\sum_{n=1}^{\infty} \left(\frac{1}{1+n}\right)^n$

$\rho = \lim_{n\to\infty} \sqrt[n]{\left(\frac{1}{1+n}\right)^n} = \lim_{k\to\infty} \frac{1}{1+n} = 0 < 1.$

Therefore, The series Converges Using Root Test.

$\boxed{3}$ $\sum_{n=1}^{\infty} \frac{n^{2n}}{(1+2n^2)^n}$

$\rho = \lim_{k\to\infty} \sqrt[n]{\frac{n^{2n}}{(1+2n^2)^n}} = \lim_{n\to\infty} \frac{n^2}{1+2n^2} = \frac{1}{2} < 1.$

$\therefore$ By Root Test, the series Converges.

$\boxed{4}$ $\sum_{n=1}^{\infty} \frac{2^n}{n^2}$

(i) $\lim_{n\to\infty} a_n = \infty \neq 0$, then by nth term test, it diverges.

(ii) $\rho = \lim_{n\to\infty} \left(\frac{2^{n+1}}{(n+1)^2}\right)\left(\frac{n^2}{2^n}\right) = 2 > 1$, so by Ratio test, it diverges.

(iii) $\rho = \lim_{n\to\infty} \sqrt[n]{\frac{2^n}{n^2}} = \lim_{n\to\infty} \frac{2}{(\sqrt[n]{n})^2} = \frac{2}{1} = 2 > 1$

$\therefore$ By Root test, the series diverges.

Example: Consider the following Recurssive Sequence

$$a_1 = 2 \quad , \quad a_{n+1} = \frac{2}{n} a_n .$$

Does $\sum_{n=1}^{\infty} a_n$ Converge ?

Sol: $\boxed{a_1 = 2} = \frac{2^1}{0!}$, $a_2 = \frac{2}{1} a_1 = \left(\frac{2}{1}\right)(2) = \frac{2^2}{1!}$

$$a_3 = \frac{2}{2} a_2 = \left(\frac{2}{2}\right)(2)(2) = \frac{2^3}{2!}$$

$$a_4 = \frac{2}{3} a_3 = \frac{2}{3}\left(\frac{2}{2}\right)(2)(2) = \frac{2^4}{3!}$$

$$a_5 = \frac{2}{4} a_4 = \frac{2}{4} \cdot \frac{2^4}{3!} = \frac{2^5}{4!}$$

$$\vdots$$

$$a_n = \frac{2^n}{(n-1)!} \quad , n \geq 1$$

So the series is $\sum_{n=1}^{\infty} \frac{2^n}{(n-1)!}$

$$\rho = \lim_{n\to\infty} \left(\frac{2^{n+1}}{n!}\right)\left(\frac{(n-1)!}{2^n}\right) = \lim_{n\to\infty} \frac{2}{n} = 0 < 1$$

$\therefore$ By Ratio Test , the series Converges .

Example : Consider the series $\sum_{n=1}^{\infty} a_n$, where

$$a_n = \begin{cases} \frac{n}{2^n} & , n \text{ odd} \\ \frac{1}{2^n} & , n \text{ even} \end{cases}$$

$$\Rightarrow \sum_{n=1}^{\infty} a_n = \frac{1}{2} + \frac{1}{2^2} + \frac{3}{2^3} + \frac{1}{2^4} + \frac{5}{2^5} + \cdots$$

Notice that : $\sqrt[n]{a_n} = \begin{cases} \frac{\sqrt[n]{n}}{2} & , n \text{ odd} \\ \frac{1}{2} & , n \text{ even} \end{cases}$

$$\Rightarrow \lim_{n\to\infty} \sqrt[n]{a_n} = \frac{1}{2} < 1$$

$\therefore$ By Root test, the series Converges.

Example: $\sum_{n=2}^{\infty} \frac{\log_n (n!)}{n^3}$. (Its not easy to use Ratio Test)

Log is Increasing

We know that $n! < n^n \Rightarrow \log_n (n!) < n \log_n (n)$

$$\Rightarrow \log_n (n!) < n \cdot \frac{\ln n}{\ln n} = n$$

$$\Rightarrow \frac{\log_n (n!)}{n^3} < \frac{1}{n^2} .$$ But $\sum_{n=2}^{\infty} \frac{1}{n^2}$ Converges

$\therefore$ By D.C.T, the series Converges.

Lecture Problems:

Q6 $\displaystyle\sum_{n=2}^{\infty} \frac{3^{n+2}}{\ln n}$

$$\rho = \lim_{n\to\infty} \left(\frac{3^{n+3}}{\ln(n+1)}\right)\left(\frac{\ln n}{3^{n}}\right) = \lim_{n\to\infty} \frac{3\ln n}{\ln(n+1)}$$

$$\overset{L.H}{=} \lim_{n\to\infty} \frac{\frac{3}{n}}{\frac{1}{n+1}} = 3 > 1$$

By Ratio Test, the series diverges.

Q15 $\displaystyle\sum_{n=1}^{\infty} \left(1-\frac{1}{n}\right)^{n^2}$

$$\rho = \lim_{n\to\infty} \sqrt[n]{\left(1-\frac{1}{n}\right)^{n^2}} = \lim_{n\to\infty} \left(1-\frac{1}{n}\right)^{n} = e^{-1} = \frac{1}{e} < 1.$$

By Root Test, the series Converges.

Q20 $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{10^n}$

$$\rho = \lim_{n\to\infty} \left(\frac{(n+1)!}{10^{n+1}}\right)\cdot\left(\frac{10^n}{n!}\right) = \lim_{n\to\infty} \frac{n+1}{10} = \infty$$

By Ratio Test, the series diverges.

Q30 $\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n^2}\right)^n$

$$\rho = \lim_{n\to\infty} \sqrt[n]{\left(\frac{1}{n}-\frac{1}{n^2}\right)^n} = \lim_{n\to\infty}\left(\frac{1}{n}-\frac{1}{n^2}\right)$$

$$= 0 < 1$$

By Root Test, the series Converges.