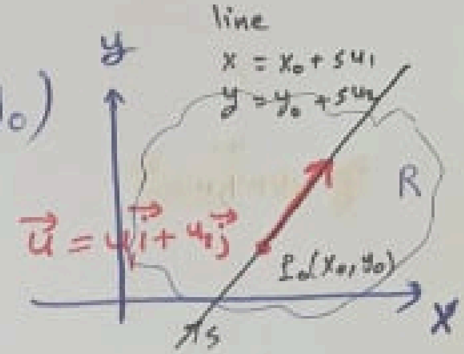
14.5 Directional Derivatives and Gradient Vectors

Def The derivative of $f(x,y)$ at $P_0(x_0,y_0)$ in the direction of the unit vector $\vec{u} = u_1\vec{i} + u_2\vec{j}$ is the number



$$\left(D_{\vec{u}} f\right)(x_0,y_0) = \left(\frac{df}{ds}\right)_{\vec{u}}(x_0,y_0) = \lim_{s\to 0} \frac{f(x_0+su_1, y_0+su_2) - f(x_0,y_0)}{s}$$

provided the limit exists.

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* $f(x,y)$ is defined on the region R in the xy-plane.
* The parametrization of the line through $P_0(x_0,y_0)$ and // to $\vec{u} = u_1\vec{i} + u_2\vec{j}$ is
  $x = x_0 + su_1$, $y = y_0 + su_2$ where the parameter s measures the arc length from P_0 in the direction of $\vec{u}$. (see 12.5)
* $\left(D_{\vec{u}} f\right)(x_0,y_0) = \left(\frac{df}{ds}\right)_{\vec{u}}(x_0,y_0)$ is also called the rate of change of f at P_0 in the direction of $\vec{u}$.
* $f_x(x_0,y_0)$ is the directional derivative of f at P_0 in the $\vec{i}$ direction.
* $f_y(x_0,y_0)$ " " " " " " " " " $\vec{j}$ " .

Exp Use the definition to find the derivative of $f(x,y) = x^2 + y$ at $P_0(1,2)$ in the direction of $\vec{w} = 3\vec{i} + 4\vec{j}$ $\vec{u} = \frac{3}{5}\vec{i} + \frac{4}{5}\vec{j}$ since $|\vec{w}| = \sqrt{9+16} = 5$

$$\left(D_{\vec{u}} f\right)(1,2) = \lim_{s\to 0} \frac{f(1+\frac{3}{5}s, 2+\frac{4}{5}s) - f(1,2)}{s} = \lim_{s\to 0} \frac{(1+\frac{3}{5}s)^2 + (2+\frac{4}{5}s) - [1+2]}{s}$$

$$= \lim_{s\to 0} \frac{\cancel{1} + \frac{6}{5}s + \frac{9}{25}s^2 + \cancel{2} + \frac{4}{5}s - \cancel{3}}{s} = \lim_{s\to 0} \frac{s(2+\frac{9}{25}s)}{s} = 2$$

Hence, the rate of change of $f(x,y) = x^2+y$ at $P_0(1,2)$ in the direction of $\vec{u}$ is 2.

Gradient Vector (Gradient):

* Recall the line parameterized by

$$x = x_0 + s u_1 \quad , \quad y = y_0 + s u_2$$

through $P_0(x_0, y_0)$ with parameter s increasing in the direction of $\vec{u} = u_1 \vec{i} + u_2 \vec{j}$.

* Using Chain Rule:

$$\left(\frac{df}{ds}\right)_{\vec{u}} (x_0, y_0) = \frac{\partial f}{\partial x}(x_0, y_0) \frac{dx}{ds} + \frac{\partial f}{\partial y}(x_0, y_0) \frac{dy}{ds}$$

$$= f_x(x_0, y_0)\, u_1 + f_y(x_0, y_0)\, u_2$$

$$= \left[f_x(x_0, y_0)\vec{i} + f_y(x_0, y_0)\vec{j}\right] \cdot \left[u_1 \vec{i} + u_2 \vec{j}\right]$$

$$= \nabla f(x_0, y_0) \cdot \vec{u}$$

Def: The gradient vector (gradient) of $f(x,y)$ at a point $P_0(x_0, y_0)$ is the vector:

$$\nabla f(x_0, y_0) = f_x(x_0, y_0)\vec{i} + f_y(x_0, y_0)\vec{j}$$

Th: (The directional derivative is a dot product):

If $f(x,y)$ is differentiable on an open region containing $P_0(x_0,y_0)$ then $(D_{\vec{u}} f)(x_0, y_0) = \left(\frac{df}{ds}\right)_{\vec{u}}(x_0, y_0) = \nabla f(x_0, y_0) \cdot \vec{u} = |\nabla f(x_0, y_0)| \cos\theta$

unit vector

Exp Find the gradient of $f(x,y) = \ln(x^2 + y^2)$ at $(1,1)$. Sketch the gradient together with the level curve passes through the point.

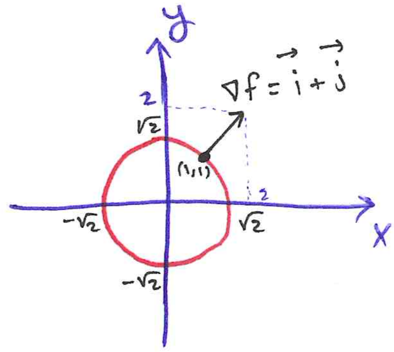
- $\nabla f(1,1) = f_x(1,1)\vec{i} + f_y(1,1)\vec{j}$

$$\nabla f = \left.\frac{2x}{x^2+y^2}\right|_{(1,1)} \vec{i} + \left.\frac{2y}{x^2+y^2}\right|_{(1,1)} \vec{j} = \vec{i} + \vec{j}$$

- The level curve is $f(1,1) = \ln(x^2+y^2)$

$$\ln 2 = \ln(x^2+y^2)$$

$$x^2+y^2 = 2$$



Exp Find the derivative of $f(x,y) = 2xy - 3y^2$ at $P_0(5,5)$ in the direction of $\vec{u} = 4\vec{i} + 3\vec{j}$

$f_x = 2y$, $f_y = 2x - 6y$

$$(D_{\vec{u}} f)(5,5) = \nabla f(5,5) \cdot \frac{4\vec{i} + 3\vec{j}}{|4\vec{i} + 3\vec{j}|}$$

$$= \left[f_x(5,5)\vec{i} + f_y(5,5)\vec{j}\right] \cdot \left[\frac{4}{5}\vec{i} + \frac{3}{5}\vec{j}\right]$$

$$= \left[10\vec{i} - 20\vec{j}\right] \cdot \left[\frac{4}{5}\vec{i} + \frac{3}{5}\vec{j}\right]$$

$$= 8 - 12 = -4$$

Exp Find the derivative of $f(x,y,z) = x^2 + 2y^2 - 3z^2$ at $P_0(1,1,1)$ in the direction of $\vec{u} = \vec{i} + \vec{j} + \vec{k}$

$f_x = 2x$

$f_y = 4y$

$f_z = -6z$

$$(D_{\vec{u}} f)(1,1,1) = \nabla f(1,1,1) \cdot \frac{\vec{i} + \vec{j} + \vec{k}}{|\vec{i} + \vec{j} + \vec{k}|}$$

$$= \left[f_x(1,1,1)\vec{i} + f_y(1,1,1)\vec{j} + f_z(1,1,1)\vec{k}\right] \cdot \left[\frac{1}{\sqrt{3}}\vec{i} + \frac{1}{\sqrt{3}}\vec{j} + \frac{1}{\sqrt{3}}\vec{k}\right]$$

$$= \left[2\vec{i} + 4\vec{j} - 6\vec{k}\right] \cdot \left[\frac{1}{\sqrt{3}}\vec{i} + \frac{1}{\sqrt{3}}\vec{j} + \frac{1}{\sqrt{3}}\vec{k}\right]$$

$$= \frac{2}{\sqrt{3}} + \frac{4}{\sqrt{3}} - \frac{6}{\sqrt{3}}$$

$$= 0$$

Properities of the Directional Derivative:

$$(D_u f)(x_0, y_0) = \nabla f(x_0, y_0) \cdot \vec{u} = |\nabla f(x_0, y_0)| \cos \theta$$

$\boxed{1}$ • The function f increases most rapidly in the direction of the gradient vector ∇f at $P(x_0, y_0)$. That is, when $\theta = 0$ and $\vec{u} = \dfrac{\nabla f(x_0, y_0)}{|\nabla f(x_0, y_0)|}$.

• The derivative in this direction is

$$(D_{\vec{u}} f)(x_0, y_0) = |\nabla f(x_0, y_0)| \cos(0) = |\nabla f(x_0, y_0)|$$

$\boxed{2}$ • The function f decreases most rapidly in the direction of $-\nabla f$ at $P(x_0, y_0)$. That is, when $\theta = \pi$ and $-\vec{u} = -\dfrac{\nabla f(x_0, y_0)}{|\nabla f(x_0, y_0)|}$.

• The derivative in this direction is

$$(D_{\vec{u}} f)(x_0, y_0) = |\nabla f(x_0, y_0)| \cos \pi = -|\nabla f(x_0, y_0)|$$

$\boxed{3}$ • The function f has no change in the direction of any vector $\vec{u}$ orthogonal to $\nabla f \neq 0$. That is, when $\theta = \frac{\pi}{2}$ and $\vec{u}$ is the unit normal $\vec{n}$ or $-\vec{n}$.

• The derivative in these direction is

$$(D_{\vec{u}} f)(x_0, y_0) = |\nabla f(x_0, y_0)| \cos \frac{\pi}{2} = 0$$

<u>Exp</u> Find the directions in which $f(x,y) = x^2 + xy + y^2$

[1] increases most rapidly at the point $P_0(-1,1)$. Find the derivative in this direction also.

- The direction is $\vec{u} = \dfrac{\nabla f(x_0,y_0)}{|\nabla f(x_0,y_0)|} = \dfrac{\nabla f(-1,1)}{|\nabla f(-1,1)|}$

- $\nabla f(-1,1) = f_x(-1,1)\,\vec{i} + f_y(-1,1)\,\vec{j}$

$= -\vec{i} + \vec{j}$

$f_x = 2x + y$

$f_y = x + 2y$

Hence $\vec{u} = \dfrac{-\vec{i}+\vec{j}}{|-\vec{i}+\vec{j}|} = \dfrac{-1}{\sqrt{2}}\vec{i} + \dfrac{1}{\sqrt{2}}\vec{j}$

- The derivative in this direction is

$(D_{\vec{u}}f)(-1,1) = |\nabla f(-1,1)| = |-\vec{i}+\vec{j}| = \sqrt{2}$

This is the rate of change in the direction of $\vec{u}$

[2] decreases most rapidly at $P_0(-1,1)$. Find the derivative in this direction.

- The direction is $-\vec{u} = \dfrac{1}{\sqrt{2}}\vec{i} - \dfrac{1}{\sqrt{2}}\vec{j}$

- The derivative in this direction is

$(D_{\vec{u}}f)(-1,1) = -|\nabla f(-1,1)| = -\sqrt{2}$

this is the rate of change in the direction of $-\vec{u}$

[3] has zero change at $P_0(-1,1)$. Find the derivative in this direction.

- The directions of zero change at $(-1,1)$ are the directions orthogonal to $\nabla f(-1,1)$: $\vec{i}+\vec{j}$, $-\vec{i}-\vec{j}$

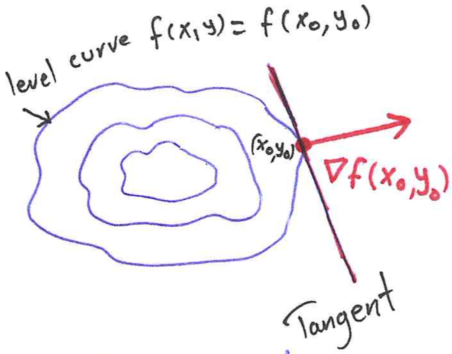
$\vec{n} = \dfrac{1}{\sqrt{2}}\vec{i} + \dfrac{1}{\sqrt{2}}\vec{j}$ and $-\vec{n} = \dfrac{-1}{\sqrt{2}} - \dfrac{1}{\sqrt{2}}\vec{j}$

"note that if $\vec{u}$ was given by $\vec{u} = \frac{3}{5}\vec{i} - \frac{4}{5}\vec{j}$, then $\vec{n} = \frac{4}{5}\vec{i} + \frac{3}{5}\vec{j}$"

- $(D_{\vec{u}}f)(-1,1) = 0$

Gradients and Tangents to Level Curves

- At every point (x_0, y_0) in the domain of a differentiable function $f(x,y)$, the gradient of f is normal to the level curve through (x_0, y_0).



- This is because for a given diff. $f(x,y)$ with constant value c alonge a smooth curve $\vec{r} = g(t)\,\vec{i} + h(t)\,\vec{j}$

$$\frac{d}{dt} f(g(t), h(t)) = \frac{d}{dt} c$$

$$\frac{\partial f}{\partial x}\frac{dg}{dt} + \frac{\partial f}{\partial h}\frac{dh}{dt} = 0 \quad \Leftrightarrow \quad \underbrace{\left(f_x \vec{i} + f_y \vec{j}\right)}_{\nabla f} \cdot \underbrace{\left(g'\vec{i} + h'\vec{j}\right)}_{\frac{d\vec{r}}{dt}} = 0$$

Hence, $\nabla f \perp$ tangent vector $\frac{d\vec{r}}{dt}$

$\Rightarrow \nabla f \perp$ level curve through (x_0, y_0)

* Equation for the tangent line :

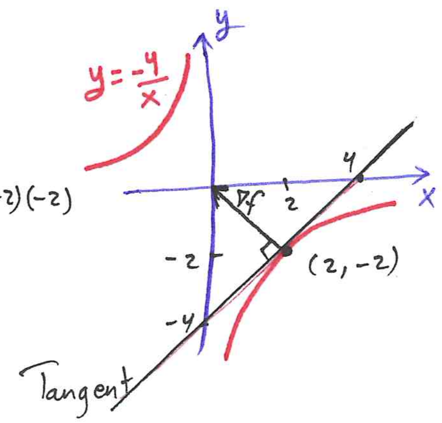
- The normal is $\nabla f(x_0,y_0) = f_x(x_0,y_0)\,\vec{i} + f_y(x_0,y_0)\,\vec{j}$
- The equation for the line through (x_0, y_0) normal to the vector $\nabla f(x_0,y_0)$ is

$$f_x(x_0,y_0)\,x + f_y(x_0,y_0)\,y = f_x(x_0,y_0)\,x_0 + f_y(x_0,y_0)\,y_0$$

Exp Find the equation of the tangent of $xy = -4$ at the point $(2,-2)$. sketh the level curve together with the tangent and ∇f.

$f(x,y) = xy \quad \Rightarrow f_x = y \ , \ f_y = x$

- $\nabla f(2,-2) = f_x(2,-2)\,\vec{i} + f_y(2,-2)\,\vec{j}$
  $= -2\vec{i} + 2\vec{j}$
- Tangent line : $f_x(2,-2)\,x + f_y(2,-2)\,y = f_x(2,-2)(2) + f_y(2,-2)(-2)$
  $-2x + 2y = -4 - 4$
  $\boxed{y = x - 4}$



Algebra Rules for Gradients

1. Sum Rule : $\nabla(f+g) = \nabla f + \nabla g$
2. Difference Rule: $\nabla(f-g) = \nabla f - \nabla g$
3. Constant Multiple Rule: $\nabla(cf) = c\,\nabla f$ "any number c"
4. Product Rule : $\nabla(fg) = f\,\nabla g + g\,\nabla f$
5. Quotient Rule: $\nabla\left(\frac{f}{g}\right) = \frac{g\,\nabla f - f\,\nabla g}{g^2}$

Exp $f(x,y) = x - y \Rightarrow \nabla f = \vec{i} - \vec{j}$

$g(x,y) = 3y \Rightarrow \nabla g = 3\vec{j}$

- $\nabla(f-g) = \nabla(x-4y) = \vec{i} - 4\vec{j} = \nabla f - \nabla g$
- $\nabla(fg) = \nabla(3xy - 3y^2) = 3y\,\vec{i} + (3x-6y)\,\vec{j}$

$$= 3y(\vec{i}-\vec{j}) + 3y\,\vec{j} + (3x-6y)\,\vec{j}$$
$$= 3y(\vec{i}-\vec{j}) + (3x-3y)\,\vec{j}$$
$$= 3y(\vec{i}-\vec{j}) + (x-y)\,3\vec{j}$$
$$= g\,\nabla f + f\,\nabla g$$

Def Plane tangent to the surface $z = f(x,y)$ of a diff function f at point $P_0(x_0, y_0, z_0) = (x_0, y_0, f(x_0,y_0))$ is

$f(x,y) - z = 0$

$F \Rightarrow F_z = -1$

$$f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) - (z-z_0) = 0$$

Exp Find eq. of tangent plane to the surface $f(x,y) = x^2y - xy + y + 1$ at $(1,1)$

- $f_x = 2xy - y \Rightarrow f_x(1,1) = 1$

$f_y = x^2 - x + 1 \Rightarrow f_y(1,1) = 1$

$z_0 = f(1,1) = 2$

$z = 2 + (x-1) + (y-1)$

$= x + y$