3.1 2^{nd} order linear and homogenous DE with constant coefficients

Second order and non-Linear => 2.9

- Any 2^{nd} order DE has the form

$$\boxed{y'' = f(t, y, y')} \quad (1)$$

general form of 2^{nd} O.D.E

- The DE (1) is linear if f is linear in y and y'. Otherwise, (1) is nonlinear.
- The solution of (1) is $y(t)$ where

  t is the indep. variable
  y is the depen. variable

- We will focus on the linear case of (1) which has the form

general form of Linear for the 2^{nd} O.D.E

$$\boxed{y'' + p(t)\, y' + q(t)\, y = g(t)} \quad (2)$$

- If $g(t) = 0$, then (2) is called homogenous DE
- If $g(t) \neq 0$, then (2) is called nonhomogenous DE
- More precisely, we will learn how to solve (2) when $g(t) = 0$ and $p(t)$ and $q(t)$ are constants ⇒

$$\boxed{ay'' + by' + cy = 0} \quad (3)$$

$y(t_0) = y_0$, $y'(t_0) = y'_0$

General form for homog. Linear equation

- The DE (3) is 2^{nd} order linear homogenous with constant coefficients (2^{nd} OLHCC)

Question: How to solve the DE (3) ?

Answer: To find the solution of (3) $\Rightarrow$
we assume exponensial solution of the form:

$$\boxed{y(t) = e^{rt}}$$

Assume it always

where r is constant

To find r we substitute y, y', y'' in (3) $\Rightarrow$

$$y'(t) = r e^{rt}$$
$$y''(t) = r^2 e^{rt}$$

$$a r^2 e^{rt} + b r e^{rt} + c e^{rt} = 0$$

$$\boxed{a r^2 + b r + c = 0} \quad (4) \rightarrow$$ Characterstic Equation (Ch. Eq.)

Compare (4) with (3)

- To solve the Ch. Eq. (4) for the roots r_1, r_2

$$r_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

So we have three cases for the roots:

Three Cases for the root :-

1) If $r_1 \neq r_2 \in \mathbb{R}$ "Real Different", then

Case 1

the first solution is $y_1(t) = e^{r_1 t}$

and the second solution is $y_2(t) = e^{r_2 t}$

the general solution is $y(t) = c_1 y_1(t) + c_2 y_2(t)$

$$\boxed{y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}}$$

2) If $r_1 = r_2 = r \in \mathbb{R}$ "Real Repeated", then

Case 2

the first solution is $y_1(t) = e^{rt}$

and the second solution is $y_2(t) = t\, e^{rt}$

the general solution is $y(t) = c_1 y_1(t) + c_2 y_2(t)$

$$\boxed{y(t) = c_1 e^{rt} + c_2 t\, e^{rt}}$$

3) If $r_{1,2} = \lambda \pm \mu i$ "Complex Roots", then

Case 3

the first solution is $y_1(t) = e^{\lambda t} \cos(\mu t)$

and the second solution is $y_2(t) = e^{\lambda t} \sin(\mu t)$

the general solution is

$\mu > 0$

$$y(t) = c_1 y_1(t) + c_2 y_2(t)$$

$$\boxed{y(t) = c_1 e^{\lambda t} \cos(\mu t) + c_2 e^{\lambda t} \sin(\mu t)}$$

$y'' + 5y' + 6y = 0 \quad , \; y(0) = 2 \quad , \; y'(0) = 3$

$r^2 + 5r + 6 = 0$

$(r+5)(r+1) = 0 \Rightarrow y_1(t) = e^{r_1 t} \Rightarrow y_1(t) = e^{-5t}$

$r = -5 \quad , \; r = -1 \qquad y_2(t) = e^{r_2 t} \Rightarrow y_2(t) = e^{-t}$

$y = C_1 y_1 + C_2 y_2 \Rightarrow C_1 e^{-5t} + C_2 e^{-t} = y$

$y'(t) = -5C_1 e^{-5t} - C_2 e^{-t}$

$y(0) = C_1 + C_2 = 2$

$y'(0) = -5C_1 - C_2 = 3 \Rightarrow -4C_1 = 5 \Rightarrow C_1 = -\frac{5}{4}$

$y'' - 4y' + 4y = 0$

$r^2 - 4r + 4 = 0$

$(r-2)(r-2)$

$r = 2 \Rightarrow$ Real Repeated $\Rightarrow y_1 = e^{rt} \quad , \; y_2 = te^{rt}$

$y = C_1 e^{2t} + C_2 t e^{2t}$

$y'' + 2y' + 2y = 0$

$r^2 + 2r + 2 = 0$

$r = \frac{-b \pm \sqrt{b^2 - 4AC}}{2A} \Rightarrow$

$r = \frac{-2 \pm \sqrt{4-8}}{2} \Rightarrow r = \frac{-2 \pm \sqrt{-4}}{2} \Rightarrow r = \frac{-2 \pm \sqrt{4} * \sqrt{i}}{2}$

$r = \frac{-2 \pm 2i}{2} \Rightarrow r = -1 + i \qquad \lambda = -1 \quad , \; \omega = 1$

$y_1 = e^{\lambda t} \cos(\omega i) \Rightarrow y_1 = e^{-t} \cos(1)$

$y_2 = e^{\lambda t} \sin(\omega i) \qquad y_2 = e^{-t} \sin(1)$

$y'' + 9y = 0$

$r^2 + 9 = 0 \Rightarrow r = \sqrt{-9} \Rightarrow r = \sqrt{9} * \sqrt{i} \Rightarrow r = 3\sqrt{-i}$

$r = 3i \Rightarrow \lambda = 0 \quad , \; \omega = 3$

$y_1 = e^{\lambda t} \cos(\omega t) \Rightarrow y_1 = \cos(3t)$

$y_2 = e^{\lambda t} \sin(\omega t) \qquad y_2 = \sin(3t)$

Exp Find the general solution of the following

(1) IVP: $y'' + 5y' + 6y = 0$, $y(0) = 2$, $y'(0) = 3$

2^{nd} OLHCC

Ch. Eg $r^2 + 5r + 6 = 0$

$(r+2)(r+3) = 0$

$r_1 = -2$, $r_2 = -3$ "Real Different"

missing t

$y_1(t) = e^{-2t}$, $y_2(t) = e^{-3t}$

gen. sol. $y(t) = c_1 y_1(t) + c_2 y_2(t)$

$y(t) = c_1 e^{-2t} + c_2 e^{-3t}$ To find $c_1, c_2 \Rightarrow$

$y(0) = \boxed{c_1 + c_2 = 2}$

$y'(t) = -2c_1 e^{-2t} - 3c_2 e^{-3t}$

$y'(0) = \boxed{-2c_1 - 3c_2 = 3}$

$c_1 = 9$
$c_2 = -7$

$y(t) = 9e^{-2t} - 7e^{-3t}$ unique solution

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(2) DE: $y'' - 4y' + 4y = 0$ 2^{nd} OLHCC

missing t

Ch. Eg. $r^2 - 4r + 4 = 0$

$(r-2)(r-2) = 0$

$r_1 = r_2 = 2$ "Real Repeated"

$y_1(t) = e^{2t}$, $y_2(t) = t e^{2t}$

gen. sol. $y(t) = c_1 y_1(t) + c_2 y_2(t)$

$y(t) = c_1 e^{2t} + c_2 t e^{2t}$

in Complex => فقط استخدم القانون العام

3) DE: $y'' + 2y' + 2y = 0$

2nd OLHCC

missing t

ch. Eq $r^2 + 2r + 2 = 0$

$$r_{1,2} = \frac{-2 \pm \sqrt{4 - 4(2)(1)}}{2} = \frac{-2 \pm \sqrt{-4}}{2}$$

$$= \frac{-2 \pm \sqrt{4}\sqrt{-1}}{2} = \frac{-2 \pm 2i}{2} = -1 \pm i$$

$\lambda = -1$, $\mu = 1$

$$y_1(t) = e^{\lambda t}\cos(\mu t) = e^{-t}\cos t$$

$$y_2(t) = e^{\lambda t}\sin(\mu t) = e^{-t}\sin t$$

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

gen. sol. $y(t) = c_1 y_1(t) + c_2 y_2(t)$

$$= c_1 e^{-t}\cos t + c_2 e^{-t}\sin t$$

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4) IVP: $2y'' + 3y' = 0$, $y(0) = 1$, $y'(0) = 3$

2nd OLHCC

missing t
missing y

ch. Eq. $2r^2 + 3r = 0$

$$r(2r+3) = 0$$

$r_1 = 0$, $r_2 = -\frac{3}{2}$ "Real Different"

$y_1(t) = 1$, $y_2(t) = e^{-\frac{3}{2}t}$

gen. sol. $y(t) = c_1 y_1(t) + c_2 y_2(t)$

$$y(t) = c_1 + c_2 e^{-\frac{3}{2}t}$$

$$y'(t) = -\frac{3}{2} c_2 e^{-\frac{3}{2}t}$$

To find c_1 and c_2:

$y(0) = \boxed{c_1 + c_2 = 1}$

$y'(0) = -\frac{3}{2} c_2 = 3$

$\boxed{c_2 = -2}$

$\Rightarrow \boxed{c_1 = 3}$

$$y(t) = 3 - 2e^{-\frac{3}{2}t}$$

$\boxed{5}$ DE: $y'' + 9y = 0$ 2nd OLHCC missing t

Ch. Eq $r^2 + 9 = 0$

$r^2 = -9 \Rightarrow \sqrt{r^2} = \sqrt{-9}$

$|r| = \sqrt{9}\,\sqrt{-1}$

$|r| = 3i$

$r_{1,2} = \pm 3i$ $\lambda = 0$

$\mu = 3$

$y_1(x) = e^{\lambda x}\cos\mu x = \cos 3x$

$y_2(x) = e^{\lambda x}\sin\mu x = \sin 3x$

gen. sol. $y(x) = c_1 y_1(x) + c_2 y_2(x)$

$= c_1\cos 3x + c_2\sin 3x$

$\boxed{6}$ IVP: $R'' + R = 0$, $R(0) = 3$, $R'(0) = 2$

Ch. Eq. $r^2 + 1 = 0 \Rightarrow r_{1,2} = \pm i$ $\lambda = 0$

$\mu = 1$

2nd OLHCC

$R_1(x) = \cos x$

$R_2(x) = \sin x$

gen. sol. $R(x) = c_1 R_1(x) + c_2 R_2(x)$

$= c_1\cos x + c_2\sin x$

To find c_1 and c_2 $\Rightarrow$

$R'(x) = -c_1\sin x + c_2\cos x$

$R(0) = c_1 + 0 = 3 \Rightarrow \boxed{c_1 = 3}$

$R'(0) = 0 + c_2 = 2 \Rightarrow \boxed{c_2 = 2}$

$R(x) = 3\cos x + 2\sin x$

نص يلي فوقه بس :-

Euler DE - يلي فوقه معادلة ثوابت

Euler DE has the form $\boxed{x^2 y'' + \alpha x y' + \beta y = 0}$ (E)

where α and β are constant and $x \neq 0$

Question: How to solve the Euler DE (E) ?

Answer: Assume power solution of the form:

$y = x^r$, $r \in \mathbb{R}$ and consider $x > 0$

To find r we substitute y, y', y'' in (E) $\Rightarrow$

$y' = r x^{r-1}$

$y'' = r(r-1) x^{r-2}$

$t = \ln x$
$e^t = e^{\ln x}$
$e^t = x$
$e^{rt} = x^r$
$e^{rt} = y$

$$x^2 \, r(r-1) x^{r-2} + \alpha x \, r x^{r-1} + \beta x^r = 0$$

$$(r^2 - r) x^r + \alpha r x^r + \beta x^r = 0$$

$$x^r \left[r^2 - r + \alpha r + \beta \right] = 0$$ note that $x^r \neq 0$

$$\boxed{r^2 + (\alpha - 1) r + \beta = 0}$$ (E*) How to solve it

Now we solve the quadratic equation (E*) for the roots r_1 and r_2

So we have three cases for these roots:

$$r_{1,2} = \frac{(1-\alpha) \pm \sqrt{(\alpha-1)^2 - 4\beta}}{2}$$

$\Downarrow$

1E If $r_1 \neq r_2 \in \mathbb{R}$ "Real Different", then

$$y_1(x) = x^{r_1} \quad \text{and} \quad y_2(x) = x^{r_2}$$

The gen. sol. is $y(x) = c_1 y_1(x) + c_2 y_2(x)$

$$y(x) = c_1 x^{r_1} + c_2 x^{r_2}$$

2E If $r_1 = r_2 = r \in \mathbb{R}$ "Real Repeated", then

$$y_1(x) = x^r \quad \text{and} \quad y_2(x) = (\ln x)\, x^r$$

The gen. sol. is $y(x) = c_1 y_1(x) + c_2 y_2(x)$

$$y(x) = c_1 x^r + c_2 (\ln x)\, x^r$$

3E If $r_{1,2} = \lambda \pm \mu i$ "Complex Roots", then

$$y_1(x) = x^{\lambda} \cos(\mu \ln x) \quad \text{and} \quad y_2(x) = x^{\lambda} \sin(\mu \ln x)$$

The gen. sol. is $y(x) = c_1 y_1(x) + c_2 y_2(x)$

$$y(x) = c_1 x^{\lambda} \cos(\mu \ln x) + c_2 x^{\lambda} \sin(\mu \ln x)$$

$\Downarrow$

$$y_1(x) = e^{\lambda t} \cos \mu t = e^{\lambda \ln x} \cos(\mu \ln x)$$

$$= e^{\ln x^{\lambda}} \cos(\mu \ln x)$$

$$= x^{\lambda} \cos(\mu \ln x)$$

same for $y_2(x)$

Exp Solve the DE

① $2x^2 y'' + 3xy' - y = 0$, $x > 0$ 2^{nd} OLH $\rightarrow$ Ⓔ

- This DE is Euler with $\alpha = \frac{3}{2}$ and $\beta = -\frac{1}{2}$
- Solve E^* $\Rightarrow$ $r^2 + (\alpha - 1) r + \beta = 0$

$$r^2 + \left(\frac{3}{2} - 1\right) r - \frac{1}{2} = 0$$

$$r^2 + \frac{1}{2} r - \frac{1}{2} = 0$$

$$(r+1)\left(r - \frac{1}{2}\right) = 0$$

$r_1 = -1$, $r_2 = \frac{1}{2}$ " Real Different "

$y_1(x) = x^{-1} = \frac{1}{x}$ and $y_2(x) = x^{\frac{1}{2}} = \sqrt{x}$

gen. sol. $y(x) = c_1 y_1(x) + c_2 y_2(x)$

$$y(x) = \frac{c_1}{x} + c_2 \sqrt{x}$$

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② $x^2 y'' + 5xy' + 4y = 0$, $x > 0$ 2^{nd} OLH $\rightarrow$ Ⓔ

- This DE is Euler with $\alpha = 5$ and $\beta = 4$
- Solve E^* $\Rightarrow$ $r^2 + (\alpha - 1) r + \beta = 0$ $\Rightarrow$ $r^2 + 4r + 4 = 0$

$\Rightarrow (r+2)(r+2) = 0$

$\Rightarrow r_1 = r_2 = r = -2$

$y_1(x) = x^{-2}$

$y_2(x) = (\ln x)\, x^{-2}$

gen. sol. $y(x) = c_1 y_1(x) + c_2 y_2(x)$

$$y(x) = \frac{c_1}{x^2} + \frac{c_2 \ln x}{x^2}$$

$x^2 y'' + 5xy' + 4y = 0 \qquad x > 0$

$\alpha = 5 \qquad \beta = 4$

$r^2 + (\alpha - 1)r + \beta = 0 \implies r^2 + 4r + 4 = 0$

$(r+2)(r+2)$

$r = -2$

Real repeated

$y_1 = x^r$

$y_2 = \ln x * x^r$

$y = C_1 y_1 + C_2 y_2$

$y = C_1 x^r + C_2 \ln x \, x^r$

$\frac{2x^2y''}{2} + \frac{3xy'}{2} \frac{-y}{2} = 0$

$x^2 y'' + \frac{3}{2} x y' - \frac{1}{2} y = 0 \qquad \alpha = \frac{3}{2} \qquad \beta = \frac{-1}{2}$

$r^2 + (\alpha - 1)r + \beta = 0$

$r^2 + \frac{1}{2} r - \frac{1}{2} = 0 \implies (r+1)(r - \frac{1}{2})$

$x^2 y'' + xy' + y = 0 \qquad \alpha = 1, \ \beta = 1$

$r^2 + 1 = 0 \implies r = \sqrt{-1} \implies r = i \qquad \lambda = 0 \qquad \omega = 1$

$y_1 = x^\lambda \cos(\omega \ln x) \implies y_1 = \cos(\ln x)$

$y_2 = x^\lambda \sin(\omega \ln x) \qquad y_2 = \sin(\ln x)$

$\boxed{y = C_1 \cos(\ln x) + C_2 \sin(\ln x)}$

(3) $x^2y'' + xy' + y = 0$, $x > 0$ (2nd OLH) → (E)

- This DE is Euler with $\alpha = \beta = 1$
- solve (E^*) $\Rightarrow$ $r^2 + (\alpha-1)r + \beta = 0$

$$r^2 + 1 = 0$$

$$r_{1,2} = \pm i \qquad \lambda = 0 \; , \; \mu = 1$$

$$y_1(x) = x^\lambda \cos(\mu \ln x) = \cos(\ln x)$$

$$y_2(x) = x^\lambda \sin(\mu \ln x) = \sin(\ln x)$$

gen. sol. is $y(x) = c_1 y_1(x) + c_2 y_2(x)$

$$= c_1 \cos(\mu \ln x) + c_2 \sin(\mu \ln x)$$

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(4) $x y'' = \frac{1}{x} y$, $x > 0$

$x^2 y'' - y = 0$ $\Rightarrow$ This is Euler DE with $\alpha = 0$ and $\beta = -1$

$\Rightarrow$ solve (E^*): $r^2 + (\alpha - 1) r + \beta = 0$

$$r^2 - r - 1 = 0$$

$r_{1,2} = \dfrac{1 \pm \sqrt{1+4}}{2} = \dfrac{1 \pm \sqrt{5}}{2}$ "Real Different"

$y_1(x) = x^{r_1}$ and $y_2 = x^{r_2}$

gen. sol. $y(x) = c_1 y_1(x) + c_2 y_2(x)$

$$y(x) = c_1 x^{\frac{1+\sqrt{5}}{2}} + c_2 x^{\frac{1-\sqrt{5}}{2}}$$

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(5) $x y'' = y'$, $x > 0$ $\qquad x^2 y'' - x y' = 0 \Rightarrow$ Euler

with $\alpha = -1$ and $\beta = 0$ $\Rightarrow r^2 + (\alpha - 1) r + \beta = 0 \Rightarrow r^2 - 2r = 0$

$r(r-2) = 0 \Rightarrow r_1 = 0 , r_2 = 2 \Rightarrow y_1(x) = 1$ and $y_2(x) = x^2$

gen. sol. $y(x) = c_1 y_1(x) + c_2 y_2(x) = c_1 + c_2 x^2$

3.2 Solutions for linear DE's of order 2
(Wronskian, Fundemental Solutions, Abel's Theorem)

Th 3.2.1 (Existance and Uniquness)

① نحطها مثل المعادلة
② نوجد أصفار المقام
③ نحدد الفترة التي يتواجد فيها t المعطاة

Consider the IVP:

$$y'' + p(t)\,y' + q(t)\,y = g(t) \quad , \quad y(t_0) = y_0 \ , \ y'(t_0) = y'_0 \qquad ①$$

If $p(t), q(t), g(t)$ are cont. on an open interval I containing t_0, then $\exists$ a unique solution $y = \phi(t)$ satisfying the IVP ① on I.

Exp Find the largest interval in which the solution of the following IVP's is valid (defined):

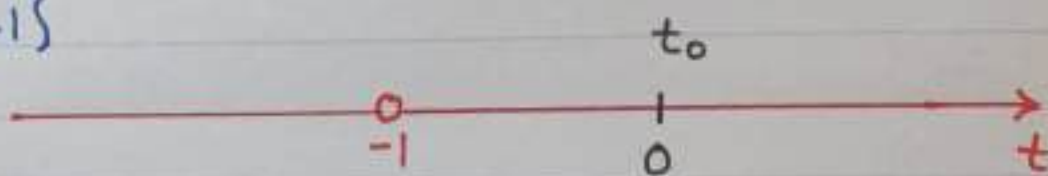
A $(t+1)\,y'' - (\cos t)\,y' = 1 - 3y \ , \ y(0) = 1 \ , \ y'(0) = 0$

Compare this IVP with ① $\Rightarrow$

$$y'' - \left(\frac{\cos t}{t+1}\right) y' + \left(\frac{3}{t+1}\right) y = \frac{1}{t+1}$$

$$p(t) = -\frac{\cos t}{t+1} \quad , \quad q(t) = \frac{3}{t+1} \quad , \quad g(t) = \frac{1}{t+1}$$

All cont. on $\mathbb{R} \setminus \{-1\}$



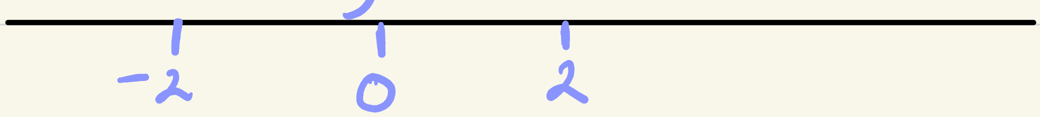
$I = (-1, \infty)$

$$(t^2-4)y'' + (\sin t)y' + \ln|t|\, y = t$$

$$y(-1)=0 \quad , \quad y'(-1)=0$$

$$y'' + \left(\frac{\sin t}{t^2-4}\right)y' + \left(\frac{\ln|t|}{t^2-4}\right)y = \left(\frac{t}{t^2-4}\right)$$

$(-2, 0)$



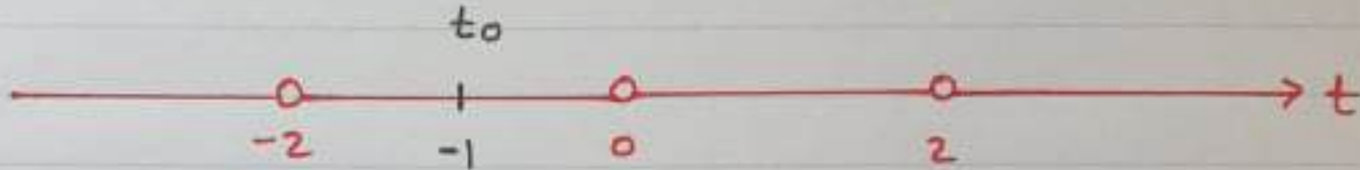
B $(t^2-4)y'' + (\sin t)y' + \ln|t|\, y = t$, $y(-1)=0$, $y'(-1)=0$

Compare this IVP with ① $\Rightarrow$

$$y'' + \underbrace{\left(\frac{\sin t}{t^2-4}\right)}_{p(t)} y' + \underbrace{\left(\frac{\ln|t|}{t^2-4}\right)}_{q(t)} y = \underbrace{\frac{t}{t^2-4}}_{g(t)}$$

All cont. on $\mathbb{R}\setminus\{-2,0,2\}$

$I = (-2, 0)$



C $\sqrt{2t+6}\, y'' + y' = \ln(2-t)$, $y(1)=4$, $y'(1)=2$

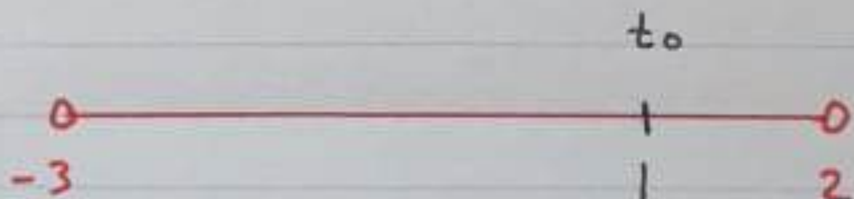
Compare this IVP with ① $\Rightarrow$

$$y'' + \frac{1}{\sqrt{2t+6}}\, y' = \frac{\ln(2-t)}{\sqrt{2t+6}}$$

$2t+6>0$

$2t>-6$

$t>-3$



$2-t>0$

$2>t$

$I = (-3, 2)$

Exp Consider the IVP: $y'' + p(t)y' + q(t)y = 0$, $y(t_0) = 0$, $y'(t_0) = 0$

where $p(t)$, $q(t)$ are cont. on an open interval I contains t_0.

Find the solution of this IVP. Is it unique?

- $g(t) = 0$ which is cont. on $\mathbb{R}$ $\Rightarrow$ it's also cont. on I
- Conditions of Th3.2.1 hold $\Rightarrow \exists$ unique sol.
- The unique sol. must satisfy the DE and the IC's:

$y(t) = 0$ is the unique solution

Th 3.2.2 (Principle of Superposition)

Suppose y_1 and y_2 are solutions for the DE:

$y'' + p(t)y' + q(t)y = 0$. Then the linear combination $c_1 y_1 + c_2 y_2$ is also solution for any constants c_1 and c_2.

Proof

$$(c_1 y_1 + c_2 y_2)'' + p(t)(c_1 y_1 + c_2 y_2)' + q(t)(c_1 y_1 + c_2 y_2) =$$

$$c_1 y_1'' + c_2 y_2'' + p(t)(c_1 y_1' + c_2 y_2') + g(t)(c_1 y_1 + c_2 y_2) =$$

$$c_1 \underbrace{(y_1'' + p(t)y_1' + g(t)y_1)}_{\text{zero since } y_1 \text{ sol.}} + c_2 \underbrace{(y_2'' + p(t)y_2' + g(t)y_2)}_{\text{zero since } y_2 \text{ sol.}} =$$

$$c_1(0) + c_2(0) = 0 \quad \Rightarrow c_1 y_1 + c_2 y_2 \text{ is sol.}$$

Remark The linear combination $c_1 y_1 + c_2 y_2$ is called the general solution and we write $y(t) = c_1 y_1(t) + c_2 y_2(t)$

• If the DE is supported by two IC's :
$y(t_0) = y_0$ and $y'(t_0) = y'_0$ then we can find the constants c_1 and c_2 in the general solution:

$$y(t) = c_1 y_1(t) + c_2 y_2(t)$$
$$y'(t) = c_1 y'_1(t) + c_2 y'_2(t)$$

$$y(t_0) = \boxed{c_1 y_1(t_0) + c_2 y_2(t_0) = y_0} \quad \text{---} ①$$
$$y'(t_0) = \boxed{c_1 y'_1(t_0) + c_2 y'_2(t_0) = y'_0} \quad \text{---} ②$$

We use Cramer's Rule to solve ① and ② for c_1 and c_2:

البسط: تبديل عامود K بعامود النواتج

$$c_1 = \frac{\begin{vmatrix} y_0 & y_2(t_0) \\ y'_0 & y'_2(t_0) \end{vmatrix}}{\begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y'_1(t_0) & y'_2(t_0) \end{vmatrix}} = \frac{y_0\, y'_2(t_0) - y_2(t_0)\, y'_0}{y_1(t_0)\, y'_2(t_0) - y_2(t_0)\, y'_1(t_0)} \quad \text{---} *^1$$

المقام: الصف الأول هو صف y_k
الصف الثاني هو صف y'_k

$$c_2 = \frac{\begin{vmatrix} y_1(t_0) & y_0 \\ y'_1(t_0) & y'_0 \end{vmatrix}}{\begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y'_1(t_0) & y'_2(t_0) \end{vmatrix}} = \frac{y_1(t_0)\, y'_0 - y_0\, y'_1(t_0)}{y_1(t_0)\, y'_2(t_0) - y_2(t_0)\, y'_1(t_0)} \quad \text{---} *^2$$

For c_1 and c_2 to be well defined, we must have
$$W(y_1, y_2)(t_0) = y_1(t_0)\, y'_2(t_0) - y_2(t_0)\, y'_1(t_0) \neq 0$$

Def If y_1 and y_2 are solutions to the DE

$$y'' + p(t)y' + q(t)y = 0 \qquad (2)$$

then the Wronskian of y_1 and y_2 is defined by

$$W(y_1, y_2)(t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix} = y_1(t)\,y_2'(t) - y_2(t)\,y_1'(t)$$

$t \in I$.

Th 3.2.3 Assume y_1 and y_2 are solutions of (2).
Then $\exists$ c_1 and c_2 s.t $c_1 y_1 + c_2 y_2$ satisfies (2)
iff $W(y_1, y_2)(t_0) \neq 0$, $t_0 \in I$.

Proof $\Rightarrow$ If $\exists$ c_1 and c_2 s.t $c_1 y_1 + c_2 y_2$ satisfies (2)
then c_1 and c_2 defined by $*^1$ and $*^2$
are well-defined
$\Rightarrow W(y_1, y_2)(t_0) \neq 0$

$\Leftarrow$ Assume y_1 and y_2 are solution of (2) $\Rightarrow$
by Th 3.2.2 $c_1 y_1 + c_2 y_2$ is also solution.

Since $W(y_1, y_2)(t_0) \neq 0$, $t_0 \in I$ $\Rightarrow$ we can
find c_1 and c_2 using Cramer's Rule $*^1$ and $*^2$

Def • $y_1, y_2, \cdots, y_n$ are Linearly Dependent if $\exists$

$c_1, c_2, \cdots, c_n$ not all zeros s.t

$$c_1 y_1 + c_2 y_2 + \cdots + c_n y_n = 0$$

• $y_1, y_2, \cdots, y_n$ are Linearly Independent if

whenever $c_1 y_1 + c_2 y_2 + \cdots + c_n y_n = 0$ implies that

$$c_1 = c_2 = \cdots = c_n = 0$$

Remark • If $W(y_1, y_2)(t) \neq 0$, then y_1 and y_2 are linearly independent

That is:

• If y_1 and y_2 are linearly dependent, then $W(y_1, y_2)(t) = 0$

Exp $y_1 = e^t$ and $y_2 = e^{-t}$ are Linearly Independent

since $W(e^t, e^{-t})(t) = \begin{vmatrix} e^t & e^{-t} \\ e^t & -e^{-t} \end{vmatrix} = -1 - 1 = -2 \neq 0$

---

or $c_1 y_1 + c_2 y_2 = 0 \Rightarrow c_1 e^t + c_2 e^{-t} = 0$

$$\left.\begin{aligned} t = 0 &\Rightarrow c_1 + c_2 = 0 \\ t = \ln 2 &\Rightarrow 2c_1 + \tfrac{1}{2} c_2 = 0 \end{aligned}\right\} \Rightarrow c_1 = c_2 = 0$$

Exp $y_1 = \sin 2x$ and $y_2 = \sin x \cos x$ are L. dependent

$c_1 y_1 + c_2 y_2 = 0 \Rightarrow c_1 \sin 2x + c_2 \sin x \cos x = 0 \Rightarrow c_1 = 1$, $c_2 = -2$

Th 3.2.4 (Fundemental Set of Solution)

Assume y_1 and y_2 are solutions for the DE:

$$y'' + p(t)\, y' + q(t)\, y = 0 \quad \text{—}(2)$$

on I. Then the family of all solutions $c_1 y_1 + c_2 y_2$ satisfies (2) iff $\exists\, t_0 \in I$ s.t $W(y_1, y_2)(t_0) \neq 0$.

Proof: similar to proof of Th 3.2.3

Remark: If y_1 and y_2 satisfy Th 3.2.4, then

(1) y_1 and y_2 are solutions for (2) and

(2) y_1 and y_2 are L. indep. since $W(y_1, y_2)(t_0) \neq 0$

So $\{y_1, y_2\}$ is called Fundemental set of solutions.

Exp Find the fundemental set of solutions for $y'' - y = 0$

ch. Eq $\quad r^2 - 1 = 0 \Rightarrow (r-1)(r+1) = 0 \Rightarrow r_1 = 1 \Rightarrow y_1 = e^t$

$r_2 = -1 \Rightarrow y_2 = e^{-t}$

(1) e^t, e^{-t} are solutions

(2) $W(e^t, e^{-t})(t) = \begin{vmatrix} e^t & e^{-t} \\ e^t & -e^{-t} \end{vmatrix} = -1 - 1 = -2 \neq 0$

$\Rightarrow e^t, e^{-t}$ are L. indep.

Hence, $\{e^t, e^{-t}\}$ is fundemental set of solutions

Exp Show that $y_1 = \sqrt{t}$ and $y_2 = \frac{1}{t}$ form fendemental set of solutions for the DE :

$$2t^2 y'' + 3t y' - y = 0 \quad , \quad t > 0$$

① we need to show y_1 and y_2 are solutions

$$y_1 = \sqrt{t} \Rightarrow y_1' = \frac{1}{2\sqrt{t}} = \frac{1}{2} t^{-\frac{1}{2}} \Rightarrow y_1'' = -\frac{1}{4} t^{-\frac{3}{2}}$$

$$2t^2 y_1'' + 3t y_1' - y_1 = 2t^2\left(-\frac{1}{4} t^{-\frac{3}{2}}\right) + 3t\left(\frac{1}{2} t^{-\frac{1}{2}}\right) - t^{\frac{1}{2}}$$

$$= -\frac{1}{2} t^{\frac{1}{2}} + \frac{3}{2} t^{\frac{1}{2}} - t^{\frac{1}{2}}$$

$$= 0 \quad \text{so } y_1 \text{ is solution}$$

$$y_2 = \frac{1}{t} = t^{-1} \Rightarrow y_2' = -t^{-2} \Rightarrow y_2'' = 2t^{-3}$$

$$2t^2 y_2'' + 3t y_2' - y_2 = 2t^2(2t^{-3}) + 3t(-t^{-2}) - t^{-1}$$

$$= 4t^{-1} - 3t^{-1} - t^{-1}$$

$$= 0 \quad \text{so } y_2 \text{ is solution}$$

---

② $$W\left(\sqrt{t}, \frac{1}{t}\right)(t) = \begin{vmatrix} \sqrt{t} & \frac{1}{t} \\ \frac{1}{2\sqrt{t}} & -\frac{1}{t^2} \end{vmatrix} = -t^{-\frac{3}{2}} - \frac{1}{2} t^{-\frac{3}{2}} = \frac{-3}{2\sqrt{t^3}} \neq 0$$

since $t > 0$

Hence, y_1 and y_2 are L. independent.

Thus they form fendemental set of solutions.

Exp Find the fundemental set of solutions for the DE:

$$3y'' + y' - 2y = 0$$

Ch. Eq.

$$3r^2 + r - 2 = 0$$
$$(3r-2)(r+1) = 0$$

$r_1 = \frac{2}{3} \Rightarrow y_1 = e^{\frac{2}{3}t}$

$r_2 = -1 \Rightarrow y_2 = e^{-t}$

$\Rightarrow y_1$ and y_2 are solutions

$$W(e^{\frac{2}{3}t}, e^{-t})(t) = \begin{vmatrix} e^{2/3\,t} & e^{-t} \\ \frac{2}{3} e^{2t/3} & -e^{-t} \end{vmatrix}$$

$$= -e^{-t/3} - \frac{2}{3} e^{-t/3}$$

$$= \frac{-5}{3} e^{-t/3}$$

$\neq 0$ since $e^{-t/3} \neq 0$

Hence, y_1 and y_2 are L. indep.

Thus, $\{e^{\frac{2}{3}t}, e^{-t}\}$ form fundemental set of solutions

Th (Abel's Theorem)

Assume y_1 and y_2 are solutions for the DE :

$$y'' + p(t)y' + q(t)y = 0 \quad (2)$$

where $p(t)$ and $q(t)$ are cont. on interval I. Then the Wronskian of y_1 and y_2 is given by :

$$W(y_1, y_2)(t) = c\, e^{-\int p(t)dt}$$

where c is constant that depends on the form of y_1 and y_2.

Furthermore, $W(y_1, y_2)(t) = 0 \quad \forall\, t \in I$
or $W(y_1, y_2)(t) \neq 0 \quad \forall\, t \in I$

Proof since y_1 and y_2 sol. for the DE (2) $\Rightarrow$

$$y_1'' + p(t)y_1' + q(t)y_1 = 0 \quad \dots A$$
$$y_2'' + p(t)y_2' + q(t)y_2 = 0 \quad \dots B$$

multiply A by $-y_2$
mutiply B by y_1
Thed add the results

$$\underbrace{(y_1 y_2'' - y_2 y_1'')}_{W'} + p(t)\underbrace{(y_1 y_2' - y_2 y_1')}_{W} = 0$$

$$W' = y_1 y_2'' + \cancel{y_2' y_1'} - y_2 y_1'' - \cancel{y_1' y_2'}$$
$$= y_1 y_2'' - y_2 y_1''$$

$$W' + p(t)W = 0$$

$$\int \frac{W'}{W} = \int - p(t) \Rightarrow \ln|W| = -\int p(t) + d$$

$$|W| = e^{-\int p(t)dt}\, e^{d} \Rightarrow W = \pm e^{d}\, e^{-\int p(t)dt}$$

$$W(y_1, y_2)(t) = c\, e^{-\int p(t)dt}$$

$W(y_1, y_2)(t) = 0$ iff $c = 0$ since $e^{-\int p(t)dt} \neq 0 \quad \forall\, t \in I$

Exp Find the wronskian for the solutions of the DE

1) $y'' - y' - 2y = 0$ $\quad$ $p(t) = -1$

$W(y_1, y_2)(t) = ce^{-\int p(t)\,dt} = ce^{-\int -dt} = ce^{t}$ $\quad$ "we can find c" $\quad$ $c = -3$

or ch.Eq. $r^2 - r - 2 = 0 \Rightarrow (r-2)(r+1) = 0$

$r_1 = 2 \Rightarrow y_1(t) = e^{2t}$

$r_2 = -1 \Rightarrow y_2(t) = e^{-t}$

$W(e^{2t}, e^{-t})(t) = \begin{vmatrix} e^{2t} & e^{-t} \\ 2e^{2t} & -e^{-t} \end{vmatrix} = -e^{t} - 2e^{t} = -3e^{t}$

---

2) $(t-1)y'' - ty' + y = 0$ $\quad , t > 1$

Compare this DE with 2 $\Rightarrow y'' - \left(\frac{t}{t-1}\right)y' + \left(\frac{1}{t-1}\right)y = 0$

$p(t) = \frac{-t}{t-1}$ $\Rightarrow W(y_1, y_2)(t) = ce^{-\int p(t)\,dt}$

$= ce^{-\int \frac{-t}{t-1}\,dt}$

$W(y_1, y_2)(t) = ce^{\int \frac{t}{t-1}\,dt}$

$= ce^{\int \frac{t-1+1}{t-1}\,dt} = ce^{\int \left(1 + \frac{1}{t-1}\right)dt}$

$= ce^{t + \ln|t-1|} = ce^{t}e^{\ln(t-1)}$ $\quad t > 1$

$= c(t-1)e^{t}$

To find $W(y_1, y_2)(t)$:

- $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$ if we know y_1 and y_2
- $W = ce^{-\int p(t)\,dt}$ if we don't know y_1, y_2

Exp Assume y_1 and y_2 are solutions for the DE:

$$y'' - 2t\,y' + e^t y = 0 \,, \quad t > 0 \quad \text{with}$$

$W(y_1, y_2)(2) = 8$. Find $W(y_1, y_2)(3)$.

$p(t) = -2t$

$$W(y_1, y_2)(t) = c\,e^{-\int p(t)\,dt} = c\,e^{-\int(-2t)\,dt}$$

$$= c\,e^{t^2}$$

$$W(y_1, y_2)(2) = c\,e^{2^2} = c\,e^4$$

$$8 = c\,e^4$$

$$c = \frac{8}{e^4} \Rightarrow W(y_1, y_2)(t) = \frac{8}{e^4}\,e^{t^2}$$

$$W(y_1, y_2)(3) = \frac{8}{e^4}\,e^{3^2}$$

$$= \frac{8}{e^4}\,e^9$$

$$= 8\,e^5$$

3.3 Complex Roots

2nd OLHCC

Exp Solve the DE: $y'' + 4y' + 5y = 0$

Ch. Eq. $r^2 + 4r + 5 = 0 \Rightarrow r_{1,2} = \frac{-4 \pm \sqrt{16-20}}{2}$

$$r_{1,2} = \frac{-4 \pm \sqrt{-4}}{2} = \frac{-4 \pm 2i}{2} = -2 \pm i$$

$\lambda = -2 \ , \ \mu = 1$

$$y_1(x) = e^{\lambda x} \cos \mu x = e^{-2x} \cos x$$
$$y_2(x) = e^{\lambda x} \sin \mu x = e^{-2x} \sin x$$

gen. sol. $y(x) = c_1 y_1 + c_2 y_2 = c_1 e^{-2x} \cos x + c_2 e^{-2x} \sin x$

Taylor Series

- If $f(x)$ is infinitly many differentiable, then Taylor expansion of $f(x)$ about $x=a$ is

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots$$

- Special case when $a=0$ "Maclurin Series"

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(a) + f'(a)x + \frac{f''(a)}{2!}x^2 + \cdots$$

Exp Derive Euler Formula $e^{ix} = \cos x + i \sin x$

Recall Maclurine Series of

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$i = \sqrt{-1}$
$i^2 = -1$
$i^3 = -i$
$i^4 = 1$
$i^5 = i$
$\vdots$

$$e^{ix} = 1 + (ix) + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \frac{(ix)^5}{5!} + \frac{(ix)^6}{6!} + \cdots$$

$$= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right) + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots\right)$$

$$= \cos x + i \sin x$$

Exp Rewrite $e^{2+\frac{\pi}{2}i}$ as $a + bi$

$$e^{2+\frac{\pi}{2}i} = e^2 \, e^{\frac{\pi}{2}i} = e^2\left[\cos\frac{\pi}{2} + i \sin\frac{\pi}{2}\right] = e^2[0 + i]$$

$$= e^2 i \qquad a = 0 \, , \; b = e^2$$

Exp $e^{-i\theta} = \cos\theta - i\sin\theta$ show this form of Euler Formula

$$e^{-i\theta} = e^{i(-\theta)} = \cos(-\theta) + i\sin(-\theta) = \cos\theta - i\sin\theta$$

Exp Use Euler Formula to write $e^{1-\frac{\pi}{3}i}$ in the form of $a+bi$

$$e^{1-\frac{\pi}{3}i} = e \, e^{-\frac{\pi}{3}i} = e\left(\cos\frac{\pi}{3} - i\sin\frac{\pi}{3}\right) = e\left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) = \frac{e}{2} - \frac{e\sqrt{3}}{2}i$$

$\boxed{3.4}$ Repeated Roots; Reduction of Order Method

Exp Solve the IVP: $y'' + 2y' + y = 0$, $y(0) = y'(0) = 1$

2nd OLHCC

Ch. Eq $r^2 + 2r + 1 = 0$

$$(r+1)(r+1) = 0$$

$$r_1 = r_2 = r = -1$$

$$y_1(t) = e^{rt} = e^{-t}$$

$$y_2(t) = t e^{rt} = t e^{-t}$$

gen. sol. $y(t) = c_1 y_1 + c_2 y_2 = c_1 e^{-t} + c_2 t e^{-t}$

$$y'(t) = -c_1 e^{-t} + c_2 e^{-t} - c_2 t e^{-t}$$

$y(0) = c_1 + 0 = 1 \Rightarrow c_1 = 1$

$y'(0) = -c_1 + c_2 - 0 = 1 \Rightarrow c_2 = 2$

$$y(t) = e^{-t} + 2t e^{-t}$$

$\lim_{t\to\infty} y(t) = 0$

Exp Find Fundemental solutions of Exp above.

$$w(y_1, y_2)(t) = \begin{vmatrix} e^{-t} & te^{-t} \\ -e^{-t} & -te^{-t} + e^{-t} \end{vmatrix} = \cancel{-te^{-2t}} + e^{-2t} - \cancel{-te^{-2t}} = e^{-2t} \neq 0$$

Hence, $y_1 = e^{-t}$ and $y_2 = te^{-t}$ are L. Indep.

$\Rightarrow \{e^{-t}, te^{-t}\}$ form fundemental Set of solutions

Reduction of Order Method (ROM)

Given $y_1(t)$ solution for the 2^{nd} order linear homogeneous DE:

$$y'' + p(t)y' + q(t)y = 0 \quad (2)$$

How to find 2^{nd} independent solution $y_2(t)$?

We use ROM to reduce the order of (2) as follow:

- Assume $\boxed{y_2(t) = v(t)\,y_1(t)}$ (C) is solution for (2)

$\Rightarrow y_2' = v y_1' + y_1 v' \qquad \Rightarrow y_2'' = v y_1'' + y_1' v' + y_1' v' + v' y_1'$

$\qquad\qquad\qquad\qquad\qquad\qquad\quad = v y_1'' + 2y_1' v' + y_1 v''$

- Substitute y_2, y_2', y_2'' in (2) $\Rightarrow$

$$v y_1'' + 2y_1' v' + y_1 v'' + p(t)(v y_1' + y_1 v') + q(t)v(t)y_1 = 0$$

$$y_1 v'' + (2y_1' + p(t)y_1)v' + v\underbrace{(y_1'' + p(t)y_1' + q(t)y_1)}_{\text{zero since } y_1 \text{ solves } (2)} = 0$$

$$y_1 v'' + (2y_1' + p(t)y_1)v' = 0$$

Let $\boxed{F = v'}$ (B) $\Rightarrow F' = v''$

$$\boxed{y_1 F' + (2y_1' + p(t)y_1)F = 0} \quad (A)$$

First solve (A) for F
then solve (B) for v
then solve (C) for y_2

"Note that can be solved using B^* since it is 1^{st} order linear"

<u>Exp</u> Given $y_1(x) = \frac{1}{x}$ is a solution for the DE

Euler DE

$$x^2 y'' + 3xy' + y = 0 \quad , \quad x > 0$$

Use ROM to find a second independent solution.

- We write the DE in the form of (2) $\Rightarrow$

$$y'' + \frac{3}{x} y' + \frac{1}{x^2} y = 0 \qquad \Rightarrow p(x) = \frac{3}{x}$$

- $y_1(x) = \frac{1}{x} \quad \Rightarrow \quad y_1' = -\frac{1}{x^2}$

- First solve (A) $\Rightarrow \quad y_1 F' + (2y_1' + p(x)\, y_1) F = 0$

$$\frac{1}{x} F' + \left(\frac{-2}{x^2} + \frac{3}{x} \frac{1}{x}\right) F = 0$$

$$\frac{1}{x} F' + \frac{1}{x^2} F = 0 \qquad \Rightarrow \quad F' + \frac{1}{x} F = 0$$

$$\mu(x) = e^{\int \frac{1}{x} dx} = e^{\ln|x|} = e^{\ln x} = x$$

$$F(x) = \frac{1}{\mu}\left[\int \mu g \, dx + c\right] = \frac{1}{x}\left[\int x(0) dx + c\right] = \frac{c}{x}$$

- Then solve (B) $\Rightarrow \quad F = v'$

$$\int \frac{c}{x} = \int v' \qquad \Rightarrow \quad v = c \ln x + d$$

- Then solve (A) $\Rightarrow \quad y_2(x) = v(x)\, y_1(x)$

gen. sol.

$$= (c \ln x + d) \frac{1}{x}$$

$$= c \underbrace{\frac{\ln x}{x}}_{y_2} + d \underbrace{\frac{1}{x}}_{y_1}$$

$$y_2(x) = \frac{\ln x}{x}$$

Exp Use ROM to show that if $y_1(t)$ is solution to the DE ②:

$$y'' + p(t)y' + q(t)y = 0$$

then the 2^{nd} independent solution is given by

$$y_2(t) = y_1(t) \int \frac{W(y_1, y_2)(t)}{y_1^2(t)}\, dt$$

- First solve Ⓐ for F: $y_1 F' + (2y_1' + p(t)y_1)F = 0$

$$\frac{F'}{F} + \left(2\frac{y_1'}{y_1} + p(t)\right) = 0 \Rightarrow \int \frac{F'}{F} = \int -\left(2\frac{y_1'}{y_1} + p(t)\right)$$

$$\ln|F| = -2\ln|y_1| - \int p(t)\,dt + d$$

$$|F| = e^{\ln|y_1|^{-2}}\, e^{-\int p(t)dt}\, e^{d}$$

$$F = \pm e^{d}\, \frac{1}{y_1^2}\, e^{-\int p(t)dt} = \frac{c\, e^{-\int p(t)dt}}{y_1^2(t)}$$

$c = \pm e^{d}$

- Then solve Ⓑ for $V \Rightarrow V' = F$

$$V = \int F\, dt = \int \frac{W(y_1, y_2)(t)}{y_1^2(t)}\, dt$$

- Then solve Ⓒ for $y_2 \Rightarrow y_2(t) = y_1(t)\, V(t)$

$$y_2(t) = y_1(t) \int \frac{W(y_1, y_2)(t)}{y_1^2(t)}\, dt$$

Remark: $\left(\frac{y_2}{y_1}\right)' = \frac{y_1 y_2' - y_2 y_1'}{y_1^2} = \frac{W}{y_1^2} \Rightarrow \frac{y_2}{y_1} = \int \frac{W}{y_1^2}\, dt \Rightarrow y_2 = y_1 \int \frac{W}{y_1^2}\, dt$

Exp Find second independent solution for the DE

$$2t^2y'' + 3ty' - y = 0 \ , \ t > 0$$ Euler DE

if $y_1(t) = \frac{1}{t}$ is solution.

First we write the DE in the form of ②

$$y'' + \frac{3}{2t}y' - \frac{1}{2t^2}y = 0 \quad \Rightarrow p(t) = \frac{3}{2t}$$

$$W(y_1, y_2)(t) = c\,e^{-\int p(t)\,dt} = c\,e^{-\int \frac{3}{2}\frac{1}{t}\,dt} = c\,e^{-\frac{3}{2}\ln|t|}$$

$$= c\,e^{-\frac{3}{2}\ln t} = c\,t^{-\frac{3}{2}}$$

$$y_2(t) = y_1(t)\int \frac{W(y_1,y_2)(t)}{y_1^2(t)}\,dt$$

$$= \frac{1}{t}\int \frac{c\,t^{-\frac{3}{2}}}{\frac{1}{t^2}}\,dt = \frac{1}{t}\int c\,t^{2-\frac{3}{2}}\,dt$$

$$= c\,\frac{1}{t}\int t^{\frac{1}{2}}\,dt = \frac{c}{t}\left[\frac{2}{3}t^{\frac{3}{2}} + d\right]$$

$$= c_1\sqrt{t} + c_2\frac{1}{t} \qquad c_1 = \frac{2}{3}c$$

$y_2(t)$ $\quad$ $y_1(t)$ $\qquad c_2 = cd$

Exp Find y_2 if $y_1 = t^3$ solves $t^2y'' - 6y = 0$ Euler DE

Solving Linear Nonhomogenous DE's of order 2

We will learn two methods to solve 2^{nd} order linear nonhomogenous DE's :

A (section 3.5): The Method of Undetermined Coefficients

B (section 3.6): The Variation of Parameter Method
"More General" $y'' + p(t)y' + q(t)y = g(t)$

A 3.5 The Method of Undetermined Coefficients

We use this method solve 2^{nd} order linear nonhomogenous DE's of the form:

$$\boxed{a y'' + b y' + c y = g(t)} \quad (1)$$

where $g(t)$ is one of the following functions:

1. exponential
2. polynomial
3. Sin or Cos
4. multiple or addition of 1, 2, 3
5. Constant ثوابت

and "a, b, c constant".

Remark : In section 3.6 we use The Variation of Parameter Method to solve 2^{nd} order linear nonhomogenous DE's of the form

$$y'' + p(t)y' + q(t)y = g(t)$$

where $g(t)$ is other than 1, 2, 3, 4 and $p(t)$, $q(t)$ are functions.

$y_1 = t^3$, $t^2 y'' - 6y = 0$ $\quad P(t) = 0$

$y'' - \frac{6}{t^2} y = 0$

$y_2 = y_1 \int \frac{w}{y_1^2}$

$w(y_1, y_2)(t) = c\, e^{-\int pt}$

$w = c\, e^{\int 0} \Rightarrow w = ce^{d}$

$y_2 = t^3 \int \frac{ce^{d}}{t^6} \Rightarrow t^3 ce^{d} \int \frac{1}{t^6}$

$\int t^{-6} \quad = -\frac{t^{-\cancel{5}^{2}}}{5} \times \cancel{t^5}^{3}\, ce^{d}$

$y_2 = -\frac{5ce^{d}}{t^2}$

Question: How do we use The Method of Undetermined Coefficients to solve the DE ①:

$$\boxed{a y'' + b y' + c y = g(t)}$$

Answer: The gen. sol. of ① is

$$\boxed{y(t) = y_h(t) + y_p(t)}$$

where

$y_h(t)$: is the homogenous solution obtained by solving the corresponding homogenous DE of ①

$a y'' + b y' + c y = 0$ using Ch. Eq

steps to Solve

① y_h "homog. Solution"

② y_p "Particular Solution"

③ $y_g = y_h + y_p$

$a r^2 + b r + c = 0$

Find r_1 and r_2

Find y_1 and y_2

$$\boxed{y_h(t) = c_1 y_1(t) + c_2 y_2(t)}$$

$y_p(t)$: is the particular solution which depends on the form of $g(t)$:

[1] If $g(t) = c e^{rt}$ then we let $y_p(t) = A e^{rt}$

Then substitute y_p, y_p', y_p'' in ① to find A

[2] If $g(t) = c_n t^n + c_{n-1} t^{n-1} + \cdots + c_1 t + c_0$ then we let $y_p(t) = A_n t^n + A_{n-1} t^{n-1} + \cdots + A_1 t + A_0$

Then substitute y_p, y_p', y_p'' in ① to find the constants $A_n, A_{n-1}, \cdots, A_1, A_0$

3) If $g(t) = c_1 \sin rt$ or
$g(t) = c_2 \cos rt$ or
$g(t) = c_1 \sin rt + c_2 \cos rt$ then we let

$$y_p(t) = A_1 \sin rt + A_2 \cos rt$$

Then substitute y_p, y_p', y_p'' in ① to find A_1, A_2

Remark • The form of the particular solution $y_p(t)$ must be independent of the form of the homogenous solution $y_h(t) = c_1 y_1(t) + c_2 y_2(t)$

• If y_p is part of y_h then we multiply y_p by t or t^2 or t^3 ... depending on the case.

Exp Solve the following DE's:

① $y'' - 5y' + 6y = 3e^{4t}$ non hom. $\Rightarrow$ we can apply 3.5

gen. sol. $y(t) = y_h(t) + y_p(t)$

• To find $y_h(t)$ $\Rightarrow$ we solve $y'' - 5y' + 6y = 0$

$$\begin{cases} r^2 - 5r + 6 = 0 \\ (r-2)(r-3) = 0 \\ r_1 = 2 \ , \ r_2 = 3 \\ y_1(t) = e^{2t} \ , \ y_2(t) = e^{3t} \end{cases}$$

$$\Rightarrow y_h(t) = c_1 y_1(t) + c_2 y_2(t) = c_1 e^{2t} + c_2 e^{3t}$$

- To find $y_p(t) \Rightarrow$ Let $y_p(t) = A e^{4t}$ → Indep. from y_1 and y_2 (R*) ✓

$$y_p' = 4A e^{4t}$$
$$y_p'' = 16A e^{4t}$$

substitute y_p, y_p', y_p'' in the nonhomogenous DE to find A:

$$y_p'' - 5y_p' + 6y_p = 3e^{4t}$$

$$16A e^{4t} - 5(4A e^{4t}) + 6(A e^{4t}) = 3e^{4t}$$

$$16A - 20A + 6A = 3$$

$$2A = 3 \Rightarrow A = \frac{3}{2}$$

$$\Rightarrow \boxed{y_p(t) = \frac{3}{2} e^{4t}}$$

gen. sol. $\Rightarrow y(t) = y_h(t) + y_p(t)$

$$= c_1 y_1 + c_2 y_2 + y_p$$

$$= c_1 e^{2t} + c_2 e^{3t} + \frac{3}{2} e^{4t}$$

---

(2) $y'' - 5y' + 6y = 10 e^{3x}$

nonhomo. $\Rightarrow$ we can apply 3.5

gen. sol. $y(x) = y_h(x) + y_p(x)$

$$= c_1 y_1(x) + c_2 y_2(x) + y_p(x)$$

$$= c_1 e^{2x} + c_2 e^{3x} + y_p(x)$$

$y_p(x) = xA e^{3x}$ (R*) ✓

$$y_p' = 3Ax e^{3x} + A e^{3x}$$
$$y_p'' = 3A e^{3x} + 9Ax e^{3x} + 3A e^{3x}$$
$$= 6A e^{3x} + 9Ax e^{3x}$$

$\Rightarrow$ To find A we substitute y_p, y_p', y_p'' in the nonhomogenuous DE

$$y_p'' - 5y_p' + 6y_p = 10e^{3x}$$

$$6Ae^{3x} + 9Axe^{3x} - 5(3Axe^{3x} + Ae^{3x}) + 6(xAe^{3x}) = 10e^{3x}$$

$$6A - 5A = 10$$ $$9A - 15A + 6A = 0$$

$$A = 10$$ $$0 = 0$$

Hence, $y_p(x) = 10xe^{3x}$ and the gen. sol. becomes

$$y(x) = c_1e^{2x} + c_2e^{3x} + 10xe^{3x}$$

---

③ $y'' - 5y' + 6y = 18x^2$ nonhomo. $\Rightarrow$ we can apply 3.5

gen. sol. $y(x) = y_h(x) + y_p(x)$

$$= c_1y_1(x) + c_2y_2(x) + y_p(x)$$

$$= c_1e^{2x} + c_2e^{3x} + y_p(x)$$

$y_p(x) = Ax^2 + Bx + C$ R* ✓

$y_p' = 2Ax + B$

$y_p'' = 2A$

substitute y_p, y_p', y_p'' in the nonhomo. DE

$$y_p'' - 5y_p' + 6y_p = 18x^2$$

$$2A - 5(2Ax + B) + 6(Ax^2 + Bx + C) = 18x^2$$

$6A = 18 \Rightarrow A = 3$

$-10A + 6B = 0 \Rightarrow 6B = 30 \Rightarrow B = 5$

$2A - 5B + 6C = 0 \Rightarrow 6 - 25 + 6C = 0 \Rightarrow C = \frac{19}{6}$

Hence, $y_p(x) = Ax^2 + Bx + C$
$= 3x^2 + 5x + \frac{19}{6}$

and the gen. sol. $y(x) = c_1 e^{2x} + c_2 e^{3x} + 3x^2 + 5x + \frac{19}{6}$

Exp Find the particular solution of the following DE's:

① $y'' + y' = 10t^2$

non homo. $\Rightarrow$ we can apply 3.5

First we find $y_h(t) \Rightarrow y'' + y' = 0$

$r^2 + r = 0$
$r(r+1) = 0$
$r_1 = 0 \ , \ r_2 = -1$
$y_1(t) = 1 \ , \ y_2(t) = e^{-t}$

$y_h(t) = c_1 y_1(t) + c_2 y_2(t)$
$= c_1 + c_2 e^{-t}$

$y_p(t) = (At^2 + Bt + C)t$ $\boxed{R*}$ ✓
$= At^3 + Bt^2 + Ct$

$y_p'(t) = 3At^2 + 2Bt + C$
$y_p''(t) = 6At + 2B$

$\Rightarrow$ substitute $y_p, y_p', y_p'' \Rightarrow$

$y_p'' + y_p' = 10t^2$

$6At + 2B + 3At^2 + 2Bt + C = 10t^2$

$3A = 10 \Rightarrow A = \frac{10}{3}$

$6A + 2B = 0 \Rightarrow 20 + 2B = 0 \Rightarrow B = -10$

$2B + C = 0 \Rightarrow -20 + C = 0 \Rightarrow C = 20$

$y_p(t) = At^3 + Bt^2 + ct$
$= \frac{10}{3}t^3 - 10t^2 + 20t$

(2) $y'' + y' - 6y = 10\cos 3x$

nonhomo. $\Rightarrow$ we can apply 3.5

First we find $y_h(x) \Rightarrow$ $y'' + y' - 6y = 0$

$$r^2 + r - 6 = 0$$

$$(r+3)(r-2) = 0$$

$$r_1 = -3 \quad , \quad r_2 = 2$$

$$y_1(x) = e^{-3x} \quad , \quad y_2(x) = e^{2x}$$

$$y_h(x) = c_1 y_1(x) + c_2 y_2(x) \Leftarrow$$

$$= c_1 e^{-3x} + c_2 e^{2x}$$

$y_p(x) = A\cos 3x + B\sin 3x$ (R*) ✓

$$y_p' = -3A\sin 3x + 3B\cos 3x$$

$$y_p'' = -9A\cos 3x - 9B\sin 3x$$

$\Rightarrow$ substitute $\Rightarrow$

$$y_p'' + y_p' - 6y_p = 10\cos 3x$$

$$-9\underline{A\cos 3x} - 9\underline{\underline{B\sin 3x}} - 3\underline{\underline{A\sin 3x}} + 3\underline{B\cos 3x}$$
$$- 6\underline{A\cos 3x} - 6\underline{\underline{B\sin 3x}} = 10\underline{\cos 3x}$$

$$-9A + 3B - 6A = 10 \Rightarrow 3B - 15A = 10$$

$$-9B - 3A - 6B = 0 \Rightarrow 15B + 3A = 0$$

$$A = -\frac{50}{78}$$

$$B = \frac{10}{78}$$

$$y_p(x) = A\cos 3x + B\sin 3x \Leftarrow$$

$$= -\frac{50}{78}\cos 3x + \frac{10}{78}\sin 3x$$

Exp Find y_p "Don't Evaluate Coefficients"

① $y'' + y = 5 \sin x$

$y_h(x)$: $\Rightarrow r^2 + 1 = 0 \Rightarrow r_{1,2} = \pm i \Rightarrow \lambda = 0$, $\mu = 1$

$y_1(x) = e^{\lambda x} \cos \mu x = \cos x$

$y_2(x) = e^{\lambda x} \sin \mu x = \sin x$

$y_h(x) = c_1 y_1(x) + c_2 y_2(x)$

$= c_1 \cos x + c_2 \sin x$

$y_p(x) = (A \sin x + B \cos x) x$ R*

$= Ax \sin x + Bx \cos x$

② $y'' + y = 5x \sin x$

$y_h(x) = c_1 \cos x + c_2 \sin x$

$y_p(x) = (Ax + B)(C \cos x + D \sin x) x$ R*

$= (Ax^2 + Bx)(C \cos x + D \sin x)$

③ $y'' + y = \sin 5x$

$y_h(x) = c_1 \cos x + c_2 \sin x$

$y_p(x) = A \sin 5x + B \cos 5x$ R*

④ $y'' - 2y' + y = 2e^{t} + 3$

$y_h(t) \Rightarrow \quad y'' - 2y' + y = 0$

$r^2 - 2r + 1 = 0$

$(r-1)(r-1) = 0$

$r_1 = r_2 = r = 1 \quad \Rightarrow y_1(t) = e^{t}, \quad y_2(t) = te^{t}$

$y_h(t) = c_1 y_1(t) + c_2 y_2(t)$

$= c_1 e^{t} + c_2 t e^{t}$

$y_p(t) = y_{p_1}(t) + y_{p_2}(t)$

$= A e^{t} t^{2} + B$ R* ✓

---

⑤ $y'' - y' = 2e^{2t} + 3t$

$y_h(t) \Rightarrow \quad y'' - y' = 0$

$r^2 - r = 0$

$r(r-1) = 0$

$r_1 = 0, \quad r_2 = 1$

$y_1(t) = 1, \quad y_2(t) = e^{t}$

$y_h(t) = c_1 y_1(t) + c_2 y_2(t)$

$= c_1 + c_2 e^{t}$

$y_p(t) = y_{p_1}(t) + y_{p_2}(t)$

$= A e^{2t} + (Bt + C)\, t$

(6) $y'' - y = x^2 e^x$

$y_h(x) \Rightarrow y'' - y = 0$

$r^2 - 1 = 0$

$(r-1)(r+1) = 0$

$r_1 = 1 \ , \ r_2 = -1$

$y_1(x) = e^x \ , \ y_2(x) = e^{-x}$

$\Rightarrow y_h(x) = c_1 y_1(x) + c_2 y_2(x)$

$= c_1 e^x + c_2 e^{-x}$

$y_p(x) = (Ax^2 + Bx + C) e^x \, x$ (R*) ✓

---

(7) $y'' = 3x^2$

$y_h(x) \Rightarrow y'' = 0$

$r^2 = 0$

$r_1 = r_2 = 0 \quad \Rightarrow y_1(x) = 1 \ , \ y_2(x) = x$

$y_h(x) = c_1 y_1(x) + c_2 y_2(x)$

$= c_1 + c_2 x$

$y_p(x) = (Ax^2 + Bx + C) x^2$ (R*) ✓

$= Ax^4 + Bx^3 + Cx^2$

Note that we can solve Exp (7) as follows:

$y'' = 3x^2 \quad \Rightarrow \quad y' = x^3 + c_2$

The gen. sol. is $y(x) = \frac{x^4}{4} + c_2 x + c_1 = y_h + y_p$

we can find A, B, C

and conclude that $A = \frac{1}{4}$, $B = 0$, $C = 0$

Th Assume Y_1 and Y_2 are solution for the nonhomogenous DE

$$\boxed{y'' + p(t)y' + q(t)y = g(t)} \quad \text{①}$$

Then $Y_1 - Y_2$ is solution for the homogenous DE

$$\boxed{y'' + p(t)y' + q(t)y = 0} \quad \text{②}$$

Furthermore, if y_1 and y_2 form Fundemental set of solutions for the homogenous DE ②, then $\exists$ constants c_1 and c_2 s.t

$$Y_1 - Y_2 = c_1 y_1 + c_2 y_2$$

Proof Since Y_1 and Y_2 solutions for ① $\Rightarrow$

$$Y_1'' + p(t)\, Y_1' + q(t)\, Y_1 = g(t) \longrightarrow A$$
$$Y_2'' + p(t)\, Y_2' + q(t)\, Y_2 = g(t) \longrightarrow B$$

$A - B \Rightarrow$

$$(Y_1 - Y_2)'' + p(t)(Y_1 - Y_2)' + q(t)(Y_1 - Y_2) = 0$$

Thus, $Y_1 - Y_2$ is solution for ②.

Remark Ⓐ If Y_1 and Y_2 solutions for ① then $cY_1 - cY_2$ is solution for ②

The second part of the proof follows trivially $\Rightarrow$ since if y_1 and y_2 are indep. $\Rightarrow \exists$ c_1 and c_2 s.t the linear combination is solution so $c_1 y_1 + c_2 y_2$ is sol. for ② but $Y_1 - Y_2$ is solution for ② $\Rightarrow Y_1 - Y_2 = c_1 y_1 + c_2 y_2$

by Principle of Superposition Theorem

3.6 Variation of Parameters Method

Recall that we can find the particular solution $y_p(t)$ for 2^{nd} order linear homogenous DE:

$$\boxed{y'' + p(t)y' + q(t)y = g(t)} \quad * \quad \text{using}$$

[A] the method of undetermined coefficients (3.5) when $p(t)$ and $q(t)$ are constants and $g(t)$ is sin / cos / poly. / exp

[B] the method of Variation of Parameters when $p(t)$ and $q(t)$ are other than constants and $g(t)$ is other than sin / cos / poly. / exp

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Th 3.6.1 • Assume $p(t)$, $q(t)$, $g(t)$ are cont. functions on an open interval I for the nonhomogenous DE $*$.

• If y_1 and y_2 form fundemental solutions for the corresponding homogenous DE:

$$y'' + p(t)y' + q(t)y = 0$$

then the particular solution $y_p(t)$ is given by

$$y_p(t) = v_1(t)y_1(t) + v_2(t)y_2(t)$$

where

$$v_1(t) = -\int \frac{y_2(t)\,g(t)}{W(y_1,y_2)(t)}\,dt \quad \text{and} \quad v_2(t) = \int \frac{y_1(t)\,g(t)\,dt}{W(y_1,y_2)(t)}$$

• Furthermore, the gen. sol. of the nonhomogenous DE $*$ is given by

$$y(t) = y_h(t) + y_p(t)$$
$$= c_1y_1(t) + c_2y_2(t) + v_1(t)y_1(t) + v_2(t)y_2(t)$$

Exp solve the DE : $y'' - 4y' + 4y = \frac{e^{2x}}{x}$, $x>0$

The gen. sol. is $y(x) = y_h(x) + y_p(x)$

nonhomogenous
$g(x) = \frac{1}{x} e^{2x}$
3.6 ✓

• $y_h(x)$: $y'' - 4y' + 4y = 0$

$$r^2 - 4r + 4 = 0$$
$$(r-2)(r-2) = 0$$
$$r_1 = r_2 = 2 \Rightarrow y_1(x) = e^{2x} , \quad y_2(x) = x e^{2x}$$

$$y_h(x) = c_1 y_1(x) + c_2 y_2(x) = c_1 e^{2x} + c_2 x e^{2x}$$

• $y_p(x) = v_1(x) y_1(x) + v_2(x) y_2(x)$

$$W(y_1,y_2)(x) = \begin{vmatrix} e^{2x} & xe^{2x} \\ 2e^{2x} & 2xe^{2x} + e^{2x} \end{vmatrix} = \cancel{2xe^{4x}} + e^{4x} - \cancel{2xe^{4x}} = e^{4x}$$

$$v_1(x) = -\int \frac{y_2 g}{W} dx = -\int \frac{xe^{2x} \frac{1}{x} e^{2x}}{e^{4x}} dx = -\int dx = -x + k_1$$

$$v_2(x) = \int \frac{y_1 g}{W} dx = \int \frac{e^{2x} \frac{1}{x} e^{2x}}{e^{4x}} dx = \int \frac{dx}{x} = \ln x + k_2$$

$$y_p(x) = v_1(x) y_1(x) + v_2(x) y_2(x) = (-x + k_1) e^{2x} + (\ln x + k_2) x e^{2x}$$

• gen. sol. $y(x) = y_h(x) + y_p(x)$

$$= c_1 e^{2x} + c_2 x e^{2x} + (-x + k_1) e^{2x} + (\ln x + k_2) x e^{2x}$$

$d_1 = c_1 + k_1$
$d_2 = c_2 + k_2 - 1$

$$= d_1 e^{2x} + d_2 x e^{2x} + x \ln x \, e^{2x} \rightarrow y_p$$

Exp Find $y_p(t)$ for the DE:

$$t^2 y'' - 3t y' + 3y = 12 t^4 \quad , \; t > 0$$

nonhomogenous 3.6

$g(t) = 12t^2$

• $y_h(t) \Rightarrow \quad t^2 y'' - 3t y' + 3y = 0$

Euler DE with $\alpha = -3$ and $\beta = 3$

$$r^2 + (\alpha - 1) r + \beta = 0$$
$$r^2 - 4r + 3 = 0$$
$$(r-3)(r-1) = 0$$
$$r_1 = 3 \; , \; r_2 = 1 \quad \Rightarrow y_1(t) = t^3 \; , \; y_2(t) = t$$

$$y_h(t) = c_1 y_1(t) + c_2 y_2(t)$$
$$= c_1 t^3 + c_2 t$$

• $W(t^3, t)(t) = \begin{vmatrix} t^3 & t \\ 3t^2 & 1 \end{vmatrix} = t^3 - 3t^3 = -2t^3$

• $y_p(t) = v_1(t) y_1(t) + v_2(t) y_2(t)$

$$v_1(t) = -\int \frac{y_2 g}{W} dt = -\int \frac{t(12t^2)}{-2t^3} dt = \int 6 dt = 6t + k_1$$

$$v_2(t) = \int \frac{y_1 g}{W} dt = \int \frac{t^3 (12t^2)}{-2t^3} dt = -\int 6t^2 dt = -2t^3 + k_2$$

$$y_p(t) = v_1(t) y_1(t) + v_2(t) y_2(t)$$
$$= (6t + k_1) t^3 + (-2t^3 + k_2) t$$
$$= \underbrace{4t^4}_{\to y_p} + \underbrace{k_1 t^3 + k_2 t}_{\to y_h}$$

we can use 3.5 also

$y_p(t) = (At^2 + Bt + C) t^4$

substitute y_p, y_p', y_p''

above $\Rightarrow A = B \cancel{= C} = 0$

and $C = 4 \Rightarrow y_p(t) = 4t^4$

Exp Solve the DE: $y'' + y = \tan t$

nonhomogeneous
$g(t) = \tan t$
3.6 ✓

- $y_h(t) \Rightarrow y'' + y = 0$

$r^2 + 1 = 0$

$r_{1,2} = \pm i \Rightarrow y_1(t) = \cos t \;,\; y_2(t) = \sin t$

$\lambda = 0, \mu = 1$

$y_h(t) = c_1 y_1(t) + c_2 y_2(t)$
$= c_1 \cos t + c_2 \sin t$

- $W(\cos t, \sin t)(t) = \begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix} = \cos^2 t - -\sin^2 t = 1$

- $v_1(t) = -\int \frac{y_2 g}{W} dt = -\int \frac{\sin t \tan t}{1} dt = -\int \frac{\sin^2 t}{\cos t} dt$

$= -\int \frac{1 - \cos^2 t}{\cos t} dt = \int (\cos t - \sec t)\, dt$

$= \sin t - \ln|\sec t + \tan t| + k_1$

- $v_2(t) = \int \frac{y_1 g}{W} dt = \int \frac{\cos t \tan t}{1} dt = \int \sin t\, dt = -\cos t + k_2$

- $y_p(t) = v_1(t)\, y_1(t) + v_2(t)\, y_2(t)$

$= (\sin t - \ln|\sec t + \tan t| + k_1)\cos t + (-\cos t + k_2)\sin t$
$= (-\ln|\sec t + \tan t| + k_1)\cos t + k_2 \sin t$

- gen. sol. $y(t) = y_h(t) + y_p(t)$

$= c_1 y_1 + c_2 y_2 + v_1 y_1 + v_2 y_2$

$d_1 = c_1 + k_1$
$d_2 = c_2 + k_2$

$y(t) = c_1 \cos t + c_2 \sin t + (-\ln|\sec t + \tan t| + k_1)\cos t + k_2 \sin t$
$= d_1 \cos t + d_2 \sin t - \cos t \ln|\sec t + \tan t|$

Exercises :

Solve the DE's:

A $x^2 y'' + x y' - y = x \ln x \quad , x > 0$

B $x^2 y'' - 3x y' + 4y = x^2 , \quad x > 0$

C If $y_1(t) = e^t$ is solution for the DE

$$t y'' - (1+t) y' + y = t^2 e^{2t} , \quad t > 0$$

Find $y_p(t)$.