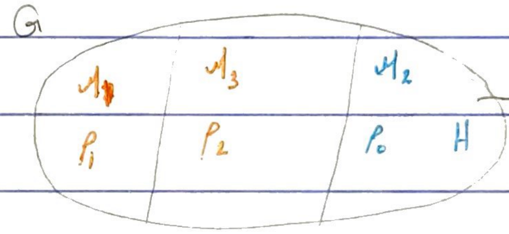
Chapter 7: Cosets and Lagrange's Theorem

Def: cosets of H in G.

let G be a group and let H be a subset of G. For any $a \in G$ the set $\{ah : h \in H\}$ is denoted By aH. Analogously, $Ha = \{ha : h \in H\}$ and $aHa^{-1} = \{aha^{-1} : h \in H\}$. When H is a subgroup of G, the set aH is called the left coset of H in G containing a, whereas Ha is the coset representative of aH (or Ha). We use $|aH|$ to denote the number of elements in the set aH, and $|Ha|$ to denote the number of elements in Ha.

exp1 → (on Book)



→ $\rho_0 H = \mu_1 H = \{\rho_0, \mu_3\}$
$= \{\rho_0, (13)\}$

exp2: $G = \mathbb{Z}_9 = \{0, 1, 2, \ldots, 8\}$

$H = \{0, 3, 6\}$ subgroup. Find the all cosets of G!!

→ $0H = 0 + H = 0+0, 0+3, 0+6 = \{0, 3, 6\}$

→ $1H = 1 + H = 1+0, 1+3, 1+6 = \{1, 4, 7\}$

→ $2H = 2 + H = 2+0, 2+3, 2+6 = \{2, 5, 8\}$

→ $3H = 3 + H = \ldots = \{3, 6, 9\}$

H = 0,3,6	1 + H	2 + H
3 + H	4 + H	7 + H

3 cosets طلع إلها مختلفات

H نفسها

واحد من ال cosets بيكون هو ال subgroup H.

exp: $(\mathbb{Z}, +)$, $H = \{0, \pm 4, \pm 8, \pm 12, \dots\}$

→ $0 + H = H$

→ $1 + H = \{\dots, -11, -7, -3, 1, 5, 9, 13, \dots\} = 4 + H$

→ $2 + H = \{\dots, -10, -6, -2, 2, 6, 10, 14, \dots\} = 5 + H$

→ $3 + H = \{\dots, -9, -5, -1, 3, 7, 11, 15, \dots\} = 6 + H$

→ $4 + H = \{\dots, -8, -4, 0, 4, 8, 12, 16, \dots\} = H$

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lemma : properties of cosets. proof ✓

let H be a subgroup of G, and let a and b belong to G. Then,

1. $a \in aH$
2. $aH = H$ iff $a \in H$.
3. $aH = bH$ iff $a \in bH$.
4. $aH = bH$ or $aH \cap bH = \emptyset$
5. $aH = bH$ iff $a^{-1}b \in H$.
6. $|aH| = |bH|$
7. aH (left coset) $= Ha$ (right coset) iff $H = aHa^{-1}$.
8. aH is a subgroup of G iff $a \in H$.

★ اذا كانت الجروب Abelian
ف $aH = Ha$

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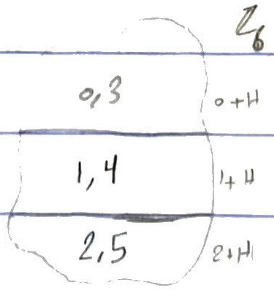
exp: $\mathbb{Z}_6 = \{0, 1, 2, 3, 4, 5\}$

$H = \{0, 3\}$

$0 + H = \{0, 3\} = 3 + H$

$1 + H = \{1, 4\} = 4 + H$

$2 + H = \{2, 5\} = 5 + H$.



Exp 4:

$U(32) = \{1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23, 25, 27, 29, 31\}$

$H = \{1, 15\}$

$\#$ of cosets $= \frac{16}{2} = 8$

$1.H = \{1, 15\}$ $\quad$ $9.H = \{9, 7\}$

$3.H = \{3, 13\}$ $\quad$ $11.H = \{11, 5\}$

$5.H = \{5, 11\}$ $\quad$ $13.H = \{13, 3\}$

$7.H = \{7, 9\}$ $\quad$ $15.H = \{15, 1\}$

Theorem 7.1: Lagrange's Theorem: $|H|$ divides $|G|$

subgroup

If G is a finite group and H is a subgroup of G, then $|H|$ divides $|G|$. Moreover, the number of distinct left (right) cosets of H in G is $|G| \setminus |H|$.

same exp 4: $\#$ of cosets $= \frac{|G|}{|H|} = \frac{16}{2} = 8$ لازم يكون عدد صحيح

→ $\#$ of subgroup divides $\#$ of Group.

exp: a. $|G| = 20$, $H \leq G$.

$|H| = 1, 2, 4, 5, 10, 20$.

b. $G = \langle a \rangle$, $|G| = 20$, $H \leq G$. Find subgroups of G.

$H_1 = \langle a^{\frac{20}{1}} \rangle = \{e\}$

$H_2 = \langle a^{\frac{20}{2}} \rangle = \{e, a^{10}\}$

$H_4 = \langle a^{\frac{20}{4}} \rangle = \{e, a^5, a^{10}, a^{15}\}$

$H_5 = \langle a^{\frac{20}{5}} \rangle = \{e, a^4, a^8, a^{12}, a^{16}\}$

$H_{10} = \langle a^2 \rangle = \{e, a^2, a^4, a^6, a^8, a^{10}, a^{12}, a^{14}, a^{16}, a^{18}\}$

$H_{20} = \langle a^1 \rangle = G$

→ # of cosets of H in G.

Corollary 1: $|G:H| = |G| \backslash |H|$.

If G is a finite group and H is a subgroup of G, then $|G:H| = \frac{|G|}{|H|}$.

Corollary 2: $|a|$ divides $|G|$.

In a finite group, the order of each element of the group divides the order of the group.

If G is a group, $a \in G$, $\langle a \rangle = \{e, a, a^2, \dots\}$ if $|a| = n \Rightarrow |\langle a \rangle| = |a|$.
$\langle a \rangle$ ↓ $H \leq G$.

$\rightarrow |a| \backslash |G|$.

$|G| = 20$, $a \in G$.

$|a| = 1, 2, 4, 5, 10, 20$.

Corollary 3:

a group of prime order is cyclic.

if $a \neq e$, $a \in G$

$\Rightarrow |a| \backslash |G| = P$

$\Rightarrow |a| = P = |G| \rightarrow G$ cyclic.

Corollary 4: $a^{|G|} = e$.

$a^{|a|} = a^{|G|/k} = (a^{|G|})^k = e^k = e$

let G be a finite group, and let $a \in G$. Then $a^{|G|} = e$.

Corollary 5: X.

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