Chapter 10 : Infinite Sequences & series:

10.1 Sequences:

Sequence: is a List of numbers: $a_1, a_2, \ldots, a_n, \ldots$

for $n \geq 1$, where:

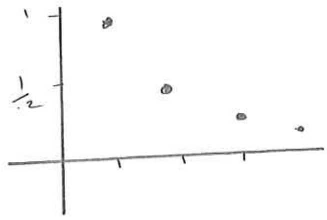
a_1 : is the first term.

a_2 : is the 2nd term.

$\vdots$

a_n : is the nth term.

Example: ① $a_n = \frac{1}{n}, \; n \geq 1$



$a_1 = 1, \; a_2 = \frac{1}{2}, \; a_3 = \frac{1}{3}, \ldots$

$$\lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{1}{n} = 0 \quad [\text{converge}].$$

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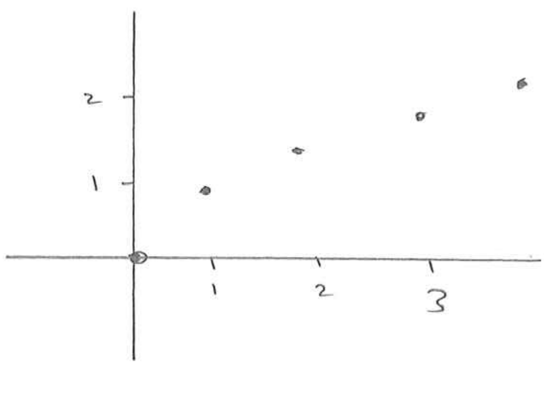
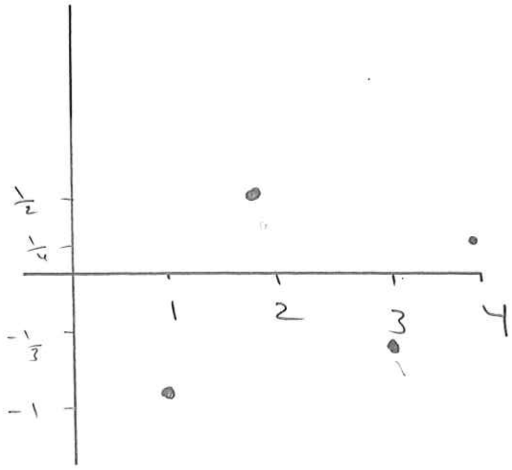
② $b_n = \sqrt{n}, \; n \geq 0.$

$$\left.\begin{aligned} b_0 &= 0 \\ b_1 &= 1 \\ b_2 &= \sqrt{2} \\ b_3 &= \sqrt{3} \\ &\vdots \end{aligned}\right\} \Rightarrow \lim_{n \to \infty} b_n = \infty$$

[diverges]



③ $C_n = (-1)^n \frac{1}{n} \quad , \quad n \geq 1$



$C_1 = -1 \quad , \quad C_2 = \frac{1}{2}$

$C_3 = -\frac{1}{3} \quad , \quad C_4 = \frac{1}{4}$

$\lim_{n \to \infty} C_n = 0$

Theorem: Let $\{a_n\}$ & $\{b_n\}$ be sequences of real numbers, and let A & $B \in \mathbb{R}$ (s.t)

$\lim_{n \to \infty} a_n = A$ & $\lim_{n \to \infty} b_n = B$, then:

1) $\lim_{n \to \infty} (a_n \pm b_n) = A \pm B$

2) $\lim_{n \to \infty} k \, b_n = k \cdot B \quad , \quad (k \in \mathbb{R})$

3) $\lim_{n \to \infty} a_n \cdot b_n = A \cdot B$

4) $\lim_{n \to \infty} \frac{a_n}{b_n} = \frac{A}{B} \quad , \quad B \neq 0$

Example: ① $\lim_{n\to\infty} \frac{-\sqrt{3}}{n} = -\sqrt{3} \lim_{n\to\infty} \frac{1}{n} = \boxed{0}$.

② $\lim_{n\to\infty} \frac{2n+5}{3n} = \lim_{n\to\infty} \frac{2}{3} + \lim_{n\to\infty} \frac{5}{3n} = \boxed{\frac{2}{3}}$

③ $\lim_{n\to\infty} \frac{n-2n^3}{n^3} = \boxed{-2}$.

Example: Find a formula for the nth term of:

① $1, -4, 9, -16, 25, \dots$

$a_n = (-1)^{n+1} n^2 \quad , \quad n \geq 1$

② $0, 3, 8, 15, 24, \dots$

$a_n = n^2 - 1 \quad , \quad n \geq 1$.

Theorem: Sandwich Theorem for Sequences

If $a_n \leq b_n \leq c_n$, $\forall n$ and

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$

Example: Let $A_n = \frac{\sin n}{n}$, $n \geq 1$

then $-1 \leq \sin n \leq 1$

$$\frac{-1}{n} \leq \frac{\sin n}{n} \leq \frac{1}{n}$$

$$0 = \lim_{n\to\infty} \frac{-1}{n} \leq \lim_{n\to\infty} \frac{\sin n}{n} \leq \lim_{n\to\infty} \frac{1}{n} = 0$$

Therefore $\lim_{n\to\infty} A_n = \lim_{n\to\infty} \frac{\sin n}{n} = 0$

Example: $B_n = \frac{(-1)^n}{n}$.

$$(-1) \leq (-1)^n \leq 1$$

$$\frac{-1}{n} \leq \frac{(-1)^n}{n} \leq \frac{1}{n}$$

Taking the limit as $n \to \infty$ and using Sandwich thm.

then $\lim_{n\to\infty} \frac{(-1)^n}{n} = 0$.

Example: $C_n = \frac{1}{2^n}$

$$0 \leq \frac{1}{2^n} \leq \frac{1}{n} \Rightarrow \lim_{n\to\infty} 0 \leq \lim_{n\to\infty} \frac{1}{2^n} \leq \lim_{n\to\infty} \frac{1}{n}$$

$$\Rightarrow \lim_{n\to\infty} \frac{1}{2^n} = 0.$$

Thm: The Continuous Function Theorem for Sequences:

Let $\{a_n\}$ be a sequence of real numbers.

If $a_n \longrightarrow L$ and if f is a Continuous function at L

then $f(a_n) \longrightarrow f(L)$.

Example: show that $\sqrt{\frac{n+1}{n}} \longrightarrow 1$.

We know that $\frac{n+1}{n} \longrightarrow 1$

Let $f(x) = \sqrt{x}$ and $L = 1$, then

$f\left(\frac{n+1}{n}\right) = \sqrt{\frac{n+1}{n}} \longrightarrow f(1) = \sqrt{1} = 1$

Example: show that $2^{\left(\frac{1}{n}\right)}$ converges to 1.

Let $a_n = \frac{1}{n}$ and $f(x) = 2^x$ and $L = 0$

then $2^{\frac{1}{n}} = f\left(\frac{1}{n}\right) \longrightarrow f(0) = 2^0 = 1$

Therefore $2^{\frac{1}{n}} \longrightarrow 1$

Using L'Hopital's Rule:

Theorem: Suppose that $f(x)$ is a function defined for all $x \geq n_0$, and $\{a_n\}$ is a sequence of real numbers (s.t) $a_n = f(n)$, $\forall n \geq n_0$.

Then $\lim\limits_{x\to\infty} f(x) = L \Rightarrow \lim\limits_{n\to\infty} a_n = L$.

Example: show that $\lim\limits_{n\to\infty} \frac{\ln n}{n} = 0$

Let $f(x) = \frac{\ln x}{x}$, which is defined $\forall x \geq 1$.

$\lim\limits_{x\to\infty} f(x) = \lim\limits_{n\to\infty} \frac{\ln n}{n}$ (Using the above thm).

$$\lim_{x\to\infty} \frac{\ln x}{x} \overset{L.H}{=} \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = 0$$

Then $\lim\limits_{n\to\infty} \frac{\ln n}{n} = 0$

Example: Does the sequence whose nth term is

$$a_n = \left(\frac{n+1}{n-1}\right)^n \rightarrow 1^{\infty}$$

converge? If so, find $\lim_{n\to\infty} a_n$

we will apply L. Hopital's Rule.

$f(x) = (x+1)^x$

$$\ln a_n = n \ln\left(\frac{n+1}{n-1}\right)$$

$e^{\ln f(x)} = e^{x \ln(x+1)}$

$$\lim_{n\to\infty} \ln a_n = \lim_{n\to\infty} n \ln\left(\frac{n+1}{n-1}\right) \quad (\infty \cdot 0)$$

$$= \lim_{n\to\infty} \frac{\ln\left(\frac{n+1}{n-1}\right)}{\frac{1}{n}} \quad \left(\frac{0}{0}\right)$$

$$\overset{L.H}{=} \lim_{n\to\infty} \frac{-2/(n^2-1)}{-1/n^2} = \lim_{n\to\infty} \frac{2n^2}{n^2-1} = \boxed{2}$$

Therefore $\ln a_n \longrightarrow 2$

$$\Rightarrow \quad a_n \longrightarrow e^2$$

$$\lim_{n\to\infty} a_n = e^2$$

Theorem (*):

1. $$\lim_{n \to \infty} \frac{\ln n}{n} = 0.$$

2. $$\lim_{n \to \infty} \sqrt[n]{n} = 1.$$

3. $$\lim_{n \to \infty} x^{\frac{1}{n}} = 1, \quad (x > 0).$$

4. $$\lim_{n \to \infty} x^{n} = 0, \quad |x| < 1$$

5. $$\lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^{n} = e^{x}, \quad (\text{for any } x).$$

6. $$\lim_{n \to \infty} \frac{x^{n}}{n!} = 0, \quad (\text{for any } x).$$

② Let $f(x) = x^{\frac{1}{x}}$

$$\Rightarrow e^{\ln f(x)} = e^{\ln x^{\left(\frac{1}{x}\right)}} = e^{\frac{1}{x}\ln x} = e^{\frac{\ln x}{x}}$$

$$\Rightarrow \lim_{x \to \infty} f(x) = e^{\lim\limits_{x \to \infty} \frac{\ln x}{x}} = e^{0} = 1$$

proof: (1) $\lim\limits_{n\to\infty} \frac{\frac{1}{n}}{1} = 0$

(2) $\lim\limits_{n\to\infty} n^{\frac{1}{n}} = \lim\limits_{n\to\infty} e^{\ln n^{\frac{1}{n}}} =$

$= \lim\limits_{n\to\infty} e^{\frac{1}{n}\ln n} \overset{(1)}{=} e^{0} = 1$

(3) $\lim\limits_{n\to\infty} e^{\ln x^{\frac{1}{n}}} = \lim\limits_{n\to\infty} e^{\frac{\ln x}{n}} = 1$

(5) $\lim\limits_{n\to\infty} e^{\ln\left(1+\frac{x}{n}\right)^{n}} = \lim\limits_{n\to\infty} e^{\frac{\ln\left(1+\frac{x}{n}\right)}{\frac{1}{n}}}$

$= \lim\limits_{n\to\infty} e^{\left(\frac{\frac{-x}{n^2}}{1+\frac{x}{n}}\right) / -\frac{1}{n^2}} = \lim\limits_{n\to\infty} e^{\frac{x}{1+\frac{x}{n}}} = e^{x}$

Example: Find:

(1) $\lim\limits_{n\to\infty} \frac{\ln n^{3}}{3n} = \lim\limits_{n\to\infty} \frac{\ln n}{n} = 0$

(2) $\lim\limits_{n\to\infty} \sqrt[n]{n^{3}} = \lim\limits_{n\to\infty} \left(\sqrt[n]{n}\right)^{3} = (1)^{3} = 1$

(3) $\lim\limits_{n\to\infty} \sqrt[n]{\pi n} = \lim\limits_{n\to\infty} (\pi)^{\frac{1}{n}} \sqrt[n]{n} = (1)(1) = 1$

(4) $\lim_{n\to\infty} \left(\frac{\pi}{e}\right)^{-n} = \lim_{n\to\infty} \left(\frac{e}{\pi}\right)^{n} = 0$, $|x|<1$

(5) $\lim_{n\to\infty} \left(\frac{n+1}{n-1}\right)^{n} = \lim_{n\to\infty} \left(\frac{n-1+2}{n-1}\right)^{n}$

$= \lim_{n\to\infty} \left(1 + \frac{2}{n-1}\right)^{n}$, [Let $u = n-1$]

$\Rightarrow \lim_{u\to\infty} \left(1 + \frac{2}{u}\right)^{u+1} = \lim_{u\to\infty} \left(1+\frac{2}{u}\right)^{u} \lim_{u\to\infty}\left(1+\frac{2}{u}\right)$

$= e^{2}(1) = \boxed{e^{2}}$

Recursive Sequence:

Assume the following recursive sequences:

$a_1 = 1$, $\underbrace{a_{n+1} = \frac{1}{2} a_n}_{\text{Recursion formula}}$, Find $\lim_{n\to\infty} a_n$

$a_1 = 1$

$a_2 = a_{1+1} = \frac{1}{2} a_1 = \frac{1}{2}$

$a_3 = a_{2+1} = \frac{1}{2} a_2 = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{4} = \left(\frac{1}{2}\right)^{2}$

$a_4 = a_{3+1} = \frac{1}{2}\left(\frac{1}{2}\right)^{2} = \left(\frac{1}{2}\right)^{3}$

$\Rightarrow a_n = \left(\frac{1}{2}\right)^{n-1} \Rightarrow \lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(\frac{1}{2}\right)^{n}\left(\frac{1}{2}\right)^{-1} = \boxed{0}$

Def: $\boxed{1}$ A sequence $\{a_n\}$, $n \geq 1$ is called bounded from above, if there exist a number M such that $a_n \leq M$, $\forall n$.

- M is called an upper bound for $\{a_n\}$.

$\boxed{2}$ A sequence $\{a_n\}$, $n \geq 1$ is called bounded from below, if $\exists$ a number m such that $a_n \geq m$, $\forall n$

- m is called a Lower bound for $\{a_n\}$.

$\boxed{3}$ If a sequence $\{a_n\}$ is bounded from above and bounded from below, we say that $\{a_n\}$ is bounded.

$\boxed{4}$ If $\{a_n\}$ is not bounded, then its unbounded.

Note: In $\boxed{1}$ if There is no number Less than M that is an upper bound of $\{a_n\}$, then M is called Least upper bound for $\{a_n\}$

In $\boxed{2}$ if $\exists$ no number greater than m that is a Lower bound of $\{a_n\}$, then m is the greatest lower bound.

Example: 1 $1, 2, 3, 4, \ldots$ is bound from below by:

$$\left.\begin{array}{l} m_1 = 0 \\ m_2 = -1 \\ m_3 = -2 \end{array}\right\} \text{ all are Lower bounds}$$

but the greatest Lower bound is $\boxed{m = 1}$

2 $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$ is bounded above.

$M_1 = 1$, $M_2 = 0.9$, ...

but $M = \frac{1}{2}$ is the ~~greatest~~ smallest Upper bound.

moreover, its bounded below by 0 (G. Lower. B)

$\Longrightarrow$ The sequence is bounded.

Def: A sequence $\{a_n\}$ is non decreasing if:

$a_{n+1} \geq a_n$, $\forall n$. That is $a_1 \leq a_2 \leq a_3 \leq \ldots$

Def: A sequence $\{a_n\}$ is non increasing if:

$a_n \geq a_{n+1}$, $\forall n$

Def: The sequence $\{a_n\}$ is monotonic if it is either nondecreasing or nonincreasing.

Example: $\boxed{1}$ $1, 2, 3, \ldots, n, \ldots$ is nondecreasing.

$\boxed{2}$ $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots, \frac{n}{n+1}, \ldots$ is nondecreasing

$\boxed{3}$ $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots, \frac{1}{2^n}, \ldots$ is nonincreasing.

$\boxed{4}$ $5, 5, 5, 5, \ldots, 5, \ldots$ is both nondecreasing and nonincreasing.

$\boxed{5}$ $1, -1, 1, -1, 1, \ldots,$ is not monotnic.

Theorem: The monotonic sequence theorem:

If a sequence $\{a_n\}$ is both bounded and monotonic then the sequence converges.

Example: $1, \frac{1}{2}, \ldots, \frac{1}{2^n}, \ldots$ is monotonic & bounded, Hence the sequence $\frac{1}{2^n}$ converges.

Example 114 page 28

(10.1) 46 $\lim_{n \to \infty} \frac{\sin^2 n}{2^n}$

$-1 \le \sin n \le 1 \iff 0 \le \sin^2 n \le 1 \iff 0 \le \frac{\sin^2 n}{2^n} \le \frac{1}{2^n}$

$\lim_{n \to \infty} 0 \le \lim_{n \to \infty} \frac{\sin^2 n}{2^n} \le \lim_{n \to \infty} \frac{1}{2^n}$ (Using Sandwich Thm).

then $\lim_{n \to \infty} \frac{\sin^2 n}{2^n} = 0$. (Converges)

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59 $\lim_{n \to \infty} \frac{\ln n}{n^{\frac{1}{n}}} = \frac{\lim_{n \to \infty} \ln n}{\lim_{n \to \infty} n^{1/n}} = \frac{\infty}{1} = \infty$ (Diverges)

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68 $\lim_{n \to \infty} \ln\left(1 + \frac{1}{n}\right)^n = \ln\left(\lim_{n \to \infty}\left(1 + \frac{1}{n}\right)^n\right) = \ln e = \boxed{1}$

(Converges)

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84 $\lim_{n \to \infty} \sqrt[n]{n^2+n} = \lim_{n \to \infty} \exp\left[\frac{\ln(n^2+n)}{n}\right]$

$\overset{L.H}{=} \lim_{n \to \infty} \exp\left[\frac{2n+1}{n^2+n}\right] \overset{L.H}{=} \lim_{n \to \infty} \exp\left[\frac{2}{2n+1}\right]$

$= e^0 = 1$ (Converges)

which of the sequences converges & which diverge?

(30) $a_n = \dfrac{2n+1}{1-3\sqrt{n}}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{\frac{2n+1}{\sqrt{n}}}{\frac{1-3\sqrt{n}}{\sqrt{n}}} = \lim_{n\to\infty} \frac{2\sqrt{n} + \frac{1}{\sqrt{n}}}{\frac{1}{\sqrt{n}} - 3} = -\infty$$

[diverges].

(36) $a_n = (-1)^n \left(1 - \frac{1}{n}\right)$

$\lim_{n\to\infty} a_n$ D.N.E [1 or -1] then diverages.

(38) $a_n = \left(2 - \frac{1}{2^n}\right)\left(3 + \frac{1}{2^n}\right)$.

$\lim_{n\to\infty} a_n = (2)(3) = 6 \Rightarrow$ Converges.

(42) $a_n = \dfrac{1}{(0.9)^n} \Rightarrow \lim_{n\to\infty} a_n = \lim_{n\to\infty} \left(\frac{10}{9}\right)^n = \infty \Rightarrow$ diverges

(47) $a_n = \dfrac{n}{2^n} \Rightarrow \lim_{n\to\infty} \dfrac{n}{2^n} \overset{L.H}{=} \lim \dfrac{1}{2^n \ln 2} = 0 \Rightarrow$ converges

(50) $a_n = \dfrac{\ln n}{\ln 2n} \Rightarrow \lim_{n\to\infty} a_n \overset{L.H}{=} \lim_{n\to\infty} \dfrac{\frac{1}{n}}{\frac{2}{2n}} = 1 \Rightarrow$ Converges

(54) $a_n = \left(1 - \frac{1}{n}\right)^n$ Thm (#5)

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \left(1 + \frac{(-1)}{n}\right)^n = e^{-1} \Rightarrow \text{converges}$$

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(62) $a_n = \sqrt[n]{3^{2n+1}} \Rightarrow$

Thm (#3)

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} 3^{\frac{(2n+1)}{n}} = \lim_{n\to\infty} 3^{2+\frac{1}{n}} = \boxed{9} \Rightarrow \text{converges}$$

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(63) $a_n = \frac{n!}{n^n}$

D.C.T

$$0 \leq \lim_{n\to\infty} a_n = \frac{1(2)(3)\cdots(n-1)n}{n^n} \leq \lim_{n\to\infty} \frac{1}{n} = 0$$

$$\Rightarrow \lim_{n\to\infty} a_n = 0 \quad [\text{converges}]$$

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(66) $a_n = \frac{n!}{2^n \cdot 3^n}$

(63)

$\frac{1}{0}$

$$\lim_{n\to\infty} a_n = \lim \frac{1}{\frac{(6)^n}{n!}} = \infty. \quad [\text{diverges}]$$

(69) $a_n = \left(\frac{3n+1}{3n-1}\right)^n$.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} e^{n \ln\left(\frac{3n+1}{3n-1}\right)}$$

$$= \lim_{n\to\infty} e^{\left[\frac{\ln(3n+1) - \ln(3n-1)}{\frac{1}{n}}\right]}$$

$$\underline{\underline{L.H}} \lim_{n\to\infty} e^{\left[\frac{3}{3n+1} - \frac{3}{3n-1}\right] / \left(-\frac{1}{n^2}\right)}$$

$$= \lim_{n\to\infty} e^{\left[\frac{6n^2}{(3n+1)(3n-1)}\right]} = e^{\frac{6}{9}} \Rightarrow [\text{converges}].$$

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(72) $a_n = \left(1 - \frac{1}{n^2}\right)^n$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \left(1 - \frac{1}{n^2}\right)^n = \lim_{n\to\infty} e^{n\left(\ln\left(1-\frac{1}{n^2}\right)\right)}$$

$$= \lim_{n\to\infty} e^{\frac{\ln\left(1-\frac{1}{n^2}\right)}{\frac{1}{n}}} = \lim_{n\to\infty} e^{\left[\frac{\frac{2}{n^3}}{\left(1-\frac{1}{n^2}\right)}\right] \Big| \left(\frac{-1}{n^2}\right)}$$

$$= \lim_{n\to\infty} e^{\left(\frac{-2}{n}\right) \Big| \left(1-\frac{1}{n^2}\right)} = \lim_{n\to\infty} e^{\left(\frac{-2n}{n^2-1}\right)} = e^0 = \boxed{1}$$

[converges]

(76) $\lim_{n\to\infty} \sinh(\ln n) = \lim_{n\to\infty} \frac{e^{\ln n} - e^{-\ln n}}{2}$

$= \lim_{n\to\infty} \frac{n - \frac{1}{n}}{2} = \infty$, [diverges].

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(82) $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{\sqrt{n}} \tan^{-1} n = 0 \cdot \frac{\pi}{2} = 0$, [converges]

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(92) Find the limit of a_n: where $\lim a_n = L$ [converges].

$a_1 = -1$, $a_{n+1} = \frac{a_n + 6}{a_n + 2}$.

$\lim_{n\to\infty} a_n = L \implies \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \frac{a_n + 6}{a_n + 2} = L$

$\implies L = \frac{L+6}{L+2} \implies L(L+2) = L+6$

$\implies L^2 + L - 6 = 0 \implies \boxed{L = -3}$ or $\boxed{L = 2}$

$a_n > 0$, $\forall\ n \geq 2 \implies \boxed{L = 2}$

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(111) $a_n = \frac{3n+1}{n+1}$

Determine if it monotonic or bounded or both.

Take $a_{n+1} = \frac{3(n+1)+1}{(n+1)+1}$, $a_n = \frac{3n+1}{n+1}$

Notice that $a_{n+1} = \frac{3n+4}{n+2} > \frac{3n+1}{n+1}$

$\Rightarrow \cancel{3n^2} + \underbrace{3n+4n} + 4 > \cancel{3n^2} + \underbrace{6n+n} + 2 \Rightarrow \boxed{4 > 2}$

The steps are reversible, so the sequence is Nondecreasing $\nearrow$

Now: $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{3n+1}{n+1} = 3$.

The sequence is bounded above by $\boxed{3}$.

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114 $a_n = 2 - \frac{2}{n} - \left(\frac{1}{2^n}\right)$

$a_{n+1} \overset{??}{\geq} a_n \Rightarrow \cancel{2} - \frac{2}{n+1} - \frac{1}{2^{n+1}} \geq \cancel{2} - \frac{2}{n} - \frac{1}{2^n}$

$\Rightarrow \frac{2}{n} - \frac{2}{n+1} \geq \frac{1}{2^{n+1}} - \frac{1}{2^n}$

$\Rightarrow \frac{2(n+1) - 2n}{n(n+1)} \geq \frac{2^n - 2^{n+1}}{2^n(2^{n+1})} \Rightarrow \frac{2}{n(n+1)} \geq \frac{\cancel{2^n}(1-2)}{\cancel{2^n}(2^{n+1})}$

$\Rightarrow \frac{2}{n(n+1)} \geq \frac{-1}{2^{n+1}} \Rightarrow$ The sequence is Nondecreasing.

Now $\lim_{n\to\infty} a_n = 2 \Rightarrow$ The sequence is bounded above by $\boxed{2}$.