2.3] Impulse Response

$$x(t) \longrightarrow \boxed{LTI} \longrightarrow y(t)$$

$$\underbrace{\sum_{i=0}^{n} \frac{d y^{(i)}(t)}{dt^{(i)}} = \sum_{j=0}^{m} \frac{d x^{(j)}(t)}{dt^{(j)}}}_{\text{standard differential equation of LTI system}}$$

solution $\longleftarrow$

$$y(t) = y_{ZI}(t) + y_{ZS}(t)$$

$y_{ZI}(t) \downarrow$ zero input solution $(x(t) = 0)$

$y_{ZS}(t) \downarrow$ zero state solution $(y(0) = 0,\ y'(0) = 0, \ldots)$

$$\downarrow$$

$$y_{ZS}(t) = \underbrace{h(t)}_{\text{impulse response}} * x(t)$$

$\xrightarrow{\mathcal{L}}$ Laplace Transform

$$Y_{ZS}(s) = \underbrace{H(s)}_{\text{Transfer Function}} X(s)$$

Laplace transform $\downarrow$

$$\mathcal{L}[x(t)] = X(s)$$

$$\mathcal{L}[y_{ZS}(t)] = Y_{ZS}(s)$$

$$\mathcal{L}[h(t)] = H(s)$$

$$y_{zs}(t) = h(t) * x(t)$$

$$\text{If } x(t) = \delta(t) \Rightarrow \underbrace{y_{zs}(t) = h(t) * \delta(t) = h(t)}$$

Proof

$$h(t) * \delta(t) = \int_{-\infty}^{\infty} h(t-\tau)\,\delta(\tau)\,d\tau$$

$$= \int_{-\infty}^{\infty} h(t)\,\delta(\tau)\,d\tau$$

$$= h(t) \int_{-\infty}^{\infty} \delta(\tau)\,d\tau = h(t)$$

$$\Rightarrow h(t) * \delta(t) = h(t)$$

⇒ The impulse response model for LTI system is given by:

$$h(t) = g(t)u(t) + \sum_{n=1} \frac{d\delta^{(n)}(t)}{dt^{n}}$$

where $g(t)$ can be found using the idea of zero input response.

$$h(t) = g(t)u(t) + \alpha_0 \delta(t) + \alpha_1 \dot{\delta}(t) + \ldots$$

Ex :- Determine the impulse response of the system $\frac{dy(t)}{dt} + 5y(t) = 2x(t)$

$$sY_{zs}(s) + 5Y_{zs}(s) = 2X(s)$$

$$H(s) = \frac{Y_{zs}(s)}{X(s)} = \frac{2}{s+5}$$

$$h(t) = 2e^{-5t}u(t)$$

Ex :- Determine the impulse response of the system $\frac{dy(t)}{dt} + 5y(t) = 2\frac{dx(t)}{dt}$

$$sY_{zs}(s) + 5Y_{zs}(s) = 2sX(s)$$

$$H(s) = \frac{Y_{zs}(s)}{X(s)} = \frac{2s}{s+5} = 2\left[\frac{s+5-5}{s+5}\right]$$

$$H(s) = 2\left[1 - \frac{5}{s+5}\right] = 2 - \frac{10}{s+5}$$

$$h(t) = 2\delta(t) - 10e^{-5t}u(t)$$

Ex :- Determine the impulse response of the system $y''(t) + 3y'(t) + 2y(t) = x''(t)$

$$(s^2 + 3s + 2)\, Y_{zs}(t) = s^2 X(s)$$

$$H(s) = \frac{Y_{zs}(s)}{X(s)} = \frac{s^2}{s^2 + 3s + 2} = 1 - \left[\frac{3s+2}{s^2 + 3s + 2} \right]$$

$$H(s) = 1 - \left[\frac{3s+2}{(s+2)(s+1)} \right] = 1 - \left[\frac{K_1}{s+2} + \frac{K_2}{s+1} \right]$$

$$K_1 = \left. \frac{3s+2}{s+1} \right|_{s=-2} = -4 \quad , \quad K_2 = \left. \frac{3s+2}{s+2} \right|_{s=-1} = 1$$

$$H(s) = 1 - \left[\frac{-4}{s+2} + \frac{1}{s+1} \right] \Rightarrow h(t) = \delta(t) - (e^{-t} - e^{-4t})\, u(t)$$

EX:- Determine the impulse response of the system $\frac{dy(t)}{dt} + 5y(t) = 2x(t)$

$x(t) = \delta(t) \Rightarrow \frac{dh(t)}{dt} + 5h(t) = 2\delta(t)$

$h(t) = g(t)u(t)$

$\frac{dg(t)}{dt} + 5g(t) = 0 \Rightarrow g(t) = g(0)e^{-5t}$

$\frac{d}{dt}(g(t)u(t)) + 5(g(t)u(t)) = 2\delta(t)$

$g(0)\delta(t) + \frac{dg(t)}{dt}u(t) + 5g(t)u(t) = 2\delta(t) \Rightarrow g(0) = 2$

$\Rightarrow g(t) = 2e^{-5t} \Rightarrow h(t) = 2e^{-5t}$

$\rightarrow$ Determine $y_{zs}(t)$ for $x(t) = \delta(2t-8)$

Ex:- Determine the impulse response of the system

$$y'(t)+5y(t)=2x'(t)$$

$$x(t)=\delta(t) \Rightarrow y_{zs}(t)=h(t) \Rightarrow h'(t)+5h(t)=2\dot{\delta}(t)$$

$$h(t)=g(t)u(t)+\alpha_0\,\delta(t)$$

$$\delta(t) \rightarrow \boxed{LTI} \rightarrow h(t)$$

$$\dot{\delta}(t) \rightarrow \boxed{LTI} \rightarrow h'(t)$$

$$g'(t)+5g(t)=0 \Rightarrow g(t)=g(0)\,e^{-5t}$$

$$h'(t)=g(0)\delta(t)+g'(t)u(t)+\alpha_0\,\dot{\delta}(t)$$

$$\left(g(0)\delta(t)+g'(t)u(t)+\alpha_0\,\dot{\delta}(t)\right)+5\left(g(t)u(t)+\alpha_0\,\delta(t)\right)=2\dot{\delta}(t)$$

$$\left.\begin{aligned} g(0)&=-5\alpha_0 \\ \alpha_0&=2 \end{aligned}\right\} \Rightarrow g(0)=-10$$

$$\boxed{h(t)=-10\,e^{-5t}u(t)+2\,\delta(t)}$$

EX :- Determine the impulse response of the system $y''(t)+3y'(t)+2y(t)=x''(t)$

$$x(t)=\delta(t) \Rightarrow y_{zs}(t)=h(t) \Rightarrow h''(t)+3h'(t)+2h(t)=\ddot{\delta}(t)$$

$$h(t)=g(t)u(t)+\alpha_0\delta(t)$$

$$g''(t)+3g'(t)+2g(t)=0$$

ch. equation $s^2+3s+2=0 \Rightarrow s_1=-1$ & $s_2=-2$

$$\Rightarrow g(t)=Ae^{-t}+Be^{-2t}$$

$$h'(t)=g(0)\delta(t)+\dot{g}(t)u(t)+\alpha_0\dot{\delta}(t)$$

$$h''(t)=g(0)\dot{\delta}(t)+\dot{g}(0)\delta(t)+g''(t)u(t)+\alpha_0\ddot{\delta}(t)$$

Balance $\ddot{\delta}(t) \Rightarrow \alpha_0 = 1$

Balance $\dot{\delta}(t) \Rightarrow g(0) + 3\alpha_0 = 0 \Rightarrow g(0) = -3$

Balance $\delta(t) \Rightarrow g'(0) + 3g(0) + 2\alpha_0 = 0 \Rightarrow g'(0) = 7$

$$\left.\begin{aligned} g(0) &= A + B = -3 \\ g'(0) &= -A - 2B = 7 \end{aligned}\right) \Rightarrow A = 1 \;+\; B = -4$$

$$\Rightarrow \boxed{h(t) = \left(e^{-t} - 4e^{-2t}\right)u(t) + \delta(t)}$$