10.9 Part 1

Monday, April 26, 2021 2:13 PM

f has all derivatives on $[a,b]$, $c \in (a,b)$

Taylor Series of f about $x=c$

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots$$

Maclurine Series = Taylor at $c=0$

Maclurine Series for

① $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

② $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$

③ $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$

Exp ㉒ Find Maclurine series ($c=0$) of $f(x) = \frac{2}{(1-x)^3}$

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots$$

$\Rightarrow f(0) = 2$

$f = 2(1-x)^{-3}$

$f = 2(1-x)^{-3}$ $\Rightarrow f(0) = 2$

$f' = -6(1-x)^{-4}(-1) = 6(1-x)^{-4}$ $\Rightarrow f'(0) = 6$

$f'' = -24(1-x)^{-5}(-1) = 24(1-x)^{-5}$ $\Rightarrow f''(0) = 24$

$f''' = -120(1-x)^{-6}(-1) = 120(1-x)^{-6}$ $\Rightarrow f'''(0) = 120$

$$2 + 6x + \frac{24}{2!}x^2 + \frac{120}{3!}x^3 + \cdots$$

$$\boxed{2 + 6x + 12x^2 + 20x^3 + \cdots} = \sum_{n=0}^{\infty}(n+2)(n+1)x^n$$

Or

$$f(x) = \frac{2}{(1-x)^3} = \left(\frac{1}{1-x}\right)''$$

$\left(\frac{1}{1-x}\right)' = \left[(1-x)^{-1}\right]' = -(1-x)^{-2}$

$a = 1$

$r = x$

$\frac{a}{1-r} = \frac{1}{1-x}$

$$= \left(1 + x + x^2 + x^3 + x^4 + x^5 + \cdots\right)''$$

$$= \left(0 + 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \cdots\right)'$$

$$= \boxed{\left(0 + 0 + 2 + 6x + 12x^2 + 20x^3 + \cdots\right)}$$

$\left(\frac{1}{1-x}\right)'' = 2(1-x)^{-3} = \frac{2}{(1-x)^3} = f(x)$

(40) Assume $\sqrt{1+x} \approx 1 + \frac{x}{2}$ Estimate the error if $|x| < 0.01$

if $|x| < 0.01$

" Use Alternating Series Estimation Theorem "

$-0.01 < x < 0.01$

Maclurine Series $\sqrt{1+x}$ $\boxed{c=0}$

$$f(x) = \sqrt{1+x} = (1+x)^{\frac{1}{2}} \Rightarrow f(0) = 1$$

$$f' = \frac{1}{2}(1+x)^{-\frac{1}{2}} \Rightarrow f'(0) = \frac{1}{2}$$

$$f'' = -\frac{1}{4}(1+x)^{-\frac{3}{2}} \Rightarrow f''(0) = -\frac{1}{4}$$

$$f''' = \frac{3}{8}(1+x)^{-2.5} \Rightarrow f'''(0) = \frac{3}{8}$$

$$\boxed{\sqrt{1+x}} = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots$$

$$= \boxed{1 + \frac{x}{2}} - \frac{1}{8}x^2 + \frac{x^3}{16} + \cdots$$

$|x| < 0.01$

$$\text{Error} < \left|-\frac{1}{8}x^2\right| = \frac{x^2}{8} < \frac{(0.01)^2}{8} = 1.25 \times 10^{-5}$$

by ASET

Taylor Th

$f, f', f'', \ldots, f^{(n+1)}$ cont. on $[a,b]$

Then, $\exists$ a number $\boxed{c \in (a,b)}$ s.t

$$f(b) = f(a) + f'(a)(b-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(b-a)^n + \underbrace{\frac{f^{(n+1)}(c)}{(n+1)!}(b-a)}_{\text{Remainder}}$$

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Note MVT: $\left.\begin{array}{l} f \text{ cont. on } [a,b] \\ f \text{ diff on } (a,b) \end{array}\right\} \Rightarrow \exists\, c \in (a,b)$ s.t. $f'(c) = \dfrac{f(b)-f(a)}{b-a}$

MVT is special case from Taylor theorem

$$f(b) = f(a) + f'(c)(b-a)$$

$$f(b) - f(a) = f'(c)(b-a)$$

$$\frac{f(b)-f(a)}{b-a} = f'(c)$$

Replace b by x $\Rightarrow$

$$f(x) = \underbrace{f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n}_{P_n(x)} + \underbrace{\frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}}_{R_n(x)}$$

$c \in (a,b)$

$\boxed{c \in (a,x)}$ center

$P_n(x)$ Poly. of degree n

$R_n(x)$ Remainder

TF

$$f(x) = P_n(x) + R_n(x) \longrightarrow \text{Taylor formula}$$

$$\boxed{f(x) = P_n(x) + R_n(x)} \to \text{laylor formula}$$

$$P_n(x) \approx f(x) \quad \text{with} \quad \text{error} = |R_n(x)|$$

Remark (Convergence of Taylor Series)

If $\boxed{\lim_{n\to\infty} R_n(x) = 0}$ $\forall x \in I$, then

Taylor Series generated by f at $x=a$ converges to f on I

That is

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Exp Show that Taylor Series generated by $f(x) = e^x$ at $x=0$ converges to $f(x)$ $\forall x$

Taylor Series of $f(x) = e^x$ at $x=0$ is Maclurine Series

TF $f(x) = P_n(x) + R_n(x)$

$$f(x) = \underbrace{1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}}_{P_n(x)} + R_n(x)$$

$$e^x = 1 + x + \frac{x^2}{} + \cdots + \frac{x^n}{} + \frac{f^{(n+1)}(c)}{} x^{n+1}$$

$f = e^x$

$f' = e^x$

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \frac{f^{(n+1)}(c)}{(n+1)!}\,x^{n+1}$$

$f = e^x$

$f' = e^x$

$\vdots$

$f^{(n+1)}(x) = e^x$

$f^{(n+1)}(c) = e^c$

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}\,x^{n+1} = \frac{e^c\,x^{n+1}}{(n+1)!}$$

$$\lim_{n\to\infty} R_n(x) = \lim_{n\to\infty} \frac{e^c\,x^{n+1}}{(n+1)!} = e^c \boxed{\lim_{n\to\infty} \frac{x^{n+1}}{(n+1)!}}$$

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$$= e^c\,(0) = 0 \quad \forall x$$

Hence, Maclurine Series converges to e^x $\Longrightarrow$

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$e^1 = 1 + 1 + \frac{1^2}{2!} + \cdots = \sum_{n=0}^{\infty} \frac{1^n}{n!}$$

$$e = \sum_{n=0}^{\infty} \frac{1}{n!}$$

$$= \underbrace{\boxed{1 + 1 + \frac{1}{2}}}_{2.5} + \overbrace{\boxed{\frac{1}{6} + \frac{1}{24} + \cdots}}^{0.2}$$

$e \approx 2.718$

Th (The Remainder Estimation Th)

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IF $\quad f(x) = P_n(x) + \underbrace{R_n(x)}$ $\quad c \in (a, x)$

$$\frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

Assume $\left| f^{(n+1)}(t) \right| \le M$ for all $t \in (a, x)$.

$\uparrow$ c ✓ $\qquad$ أعلى قيمة upper bound for $f^{(n+1)}$

Then, Remainder $= \boxed{\left| R_n(x) \right| \le \dfrac{M}{(n+1)!} \left| x-a \right|^{n+1}}$ $\rightarrow *$

If $*$ holds for all n, then the Taylor Series generated by $f(x)$ converges to $f(x)$

Exp Show that the Maclurine Series for $\sin x$ converges to $\sin x$ $\forall x$

IF $\quad f(x) = P_{2n+1}(x) + R_{2n+1}(x)$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + \frac{(-1)^n x^{2n+1}}{(2n+1)!} + \frac{f^{(2n+2)}(c)}{(2n+2)!} x^{2n+2}$$

$f = \sin x$

$f' = \cos x$

$f'' = -\sin$

$\vdots$

$\left| f^{2n+2} \right| \le \boxed{1}$ $\quad$ M

$$\left| R_{2n+1}(x) \right| = \left| \frac{f^{(2n+2)}(c)}{(2n+2)!} x^{2n+2} \right| \le \frac{\boxed{1}}{(2n+2)!} \left| x^{2n+2} \right|$$

$$\left| R(x) \right| \le \left| x^{2n+2} \right|$$

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$$0 \le \left|R_{2n+1}(x)\right| \le \left|\frac{x^{2n+2}}{(2n+2)!}\right|$$

$$\lim_{n\to\infty} 0 \le \lim_{n\to\infty} \left|R_{2n+1}(x)\right| \le \boxed{\lim_{n\to\infty} \left|\frac{x^{2n+2}}{(2n+2)!}\right|} \rightarrow \text{10.1 Th}$$

$$0 \le \lim_{n\to\infty} \left|R_{2n+1}(x)\right| \le 0$$

by S.T $\Rightarrow \lim_{n\to\infty} \left|R_{2n+1}(x)\right| = 0$

Hence, Maclurine Series generated by sinx converges to sinx $\forall x \Rightarrow$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2n+1}}{(2n+1)!}$$