12

Introduction to the Laplace Transform

Assessment Problems

AP 12.1 **[a]** $\cosh\beta t = \dfrac{e^{\beta t}+e^{-\beta t}}{2}$

Therefore,

$$\mathcal{L}\{\cosh\beta t\} = \frac{1}{2}\int_{0^-}^{\infty}[e^{(s-\beta)t}+e^{-(s-\beta)t}]dt$$

$$= \frac{1}{2}\left[\frac{e^{-(s-\beta)t}}{-(s-\beta)}\Big|_{0^-}^{\infty} + \frac{e^{-(s+\beta)t}}{-(s+\beta)}\Big|_{0^-}^{\infty}\right]$$

$$= \frac{1}{2}\left(\frac{1}{s-\beta}+\frac{1}{s+\beta}\right) = \frac{s}{s^2-\beta^2}$$

[b] $\sinh\beta t = \dfrac{e^{\beta t}-e^{-\beta t}}{2}$

Therefore,

$$\mathcal{L}\{\sinh\beta t\} = \frac{1}{2}\int_{0^-}^{\infty}\left[e^{-(s-\beta)t}-e^{-(s+\beta)t}\right]\,dt$$

$$= \frac{1}{2}\left[\frac{e^{-(s-\beta)t}}{-(s-\beta)}\right]_{0^-}^{\infty} - \frac{1}{2}\left[\frac{e^{-(s+\beta)t}}{-(s+\beta)}\right]_{0^-}^{\infty}$$

$$= \frac{1}{2}\left(\frac{1}{s-\beta}-\frac{1}{s+\beta}\right) = \frac{\beta}{(s^2-\beta^2)}$$

AP 12.2 **[a]** Let $f(t) = te^{-at}$:

$$F(s) = \mathcal{L}\{te^{-at}\} = \frac{1}{(s+a)^2}$$

Now, $\quad \mathcal{L}\{tf(t)\} = -\dfrac{dF(s)}{ds}$

$$\text{So,}\quad \mathcal{L}\{t\cdot te^{-at}\} = -\frac{d}{ds}\left[\frac{1}{(s+a)^2}\right] = \frac{2}{(s+a)^3}$$

[b] Let $f(t) = e^{-at}\sinh\beta t$, then

$$\mathcal{L}\{f(t)\} = F(s) = \frac{\beta}{(s+a)^2-\beta^2}$$

$$\mathcal{L}\left\{\frac{df(t)}{dt}\right\} = sF(s) - f(0^-) = \frac{s(\beta)}{(s+a)^2-\beta^2} - 0 = \frac{\beta s}{(s+a)^2-\beta^2}$$

[c] Let $f(t) = \cos\omega t$. Then

$$F(s) = \frac{s}{(s^2+\omega^2)} \quad\text{and}\quad \frac{dF(s)}{ds} = \frac{-(s^2-\omega^2)}{(s^2+\omega^2)^2}$$

$$\text{Therefore}\quad \mathcal{L}\{t\cos\omega t\} = -\frac{dF(s)}{ds} = \frac{s^2-\omega^2}{(s^2+\omega^2)^2}$$

AP 12.3

$$F(s) = \frac{6s^2+26s+26}{(s+1)(s+2)(s+3)} = \frac{K_1}{s+1} + \frac{K_2}{s+2} + \frac{K_3}{s+3}$$

$$K_1 = \frac{6-26+26}{(1)(2)} = 3; \qquad K_2 = \frac{24-52+26}{(-1)(1)} = 2$$

$$K_3 = \frac{54-78+26}{(-2)(-1)} = 1$$

Therefore $f(t) = [3e^{-t} + 2e^{-2t} + e^{-3t}]\,u(t)$

AP 12.4

$$F(s) = \frac{7s^2+63s+134}{(s+3)(s+4)(s+5)} = \frac{K_1}{s+3} + \frac{K_2}{s+4} + \frac{K_3}{s+5}$$

$$K_1 = \frac{63-189-134}{1(2)} = 4; \qquad K_2 = \frac{112-252+134}{(-1)(1)} = 6$$

$$K_3 = \frac{175-315+134}{(-2)(-1)} = -3$$

$$f(t) = [4e^{-3t} + 6e^{-4t} - 3e^{-5t}]u(t)$$

AP 12.5

$$F(s) = \frac{10(s^2+119)}{(s+5)(s^2+10s+169)}$$

$$s_{1,2} = -5 \pm \sqrt{25-169} = -5 \pm j12$$

$$F(s) = \frac{K_1}{s+5} + \frac{K_2}{s+5-j12} + \frac{K_2^*}{s+5+j12}$$

$$K_1 = \frac{10(25+119)}{25-50+169} = 10$$

$$K_2 = \frac{10[(-5+j12)^2+119]}{(j12)(j24)} = j4.17 = 4.17\underline{/90^\circ}$$

Therefore

$$\begin{aligned} f(t) &= [10e^{-5t} + 8.33e^{-5t}\cos(12t+90^\circ)]\,u(t) \\ &= [10e^{-5t} - 8.33e^{-5t}\sin 12t]\,u(t) \end{aligned}$$

AP 12.6

$$F(s) = \frac{4s^2+7s+1}{s(s+1)^2} = \frac{K_0}{s} + \frac{K_1}{(s+1)^2} + \frac{K_2}{s+1}$$

$$K_0 = \frac{1}{(1)^2} = 1; \qquad K_1 = \frac{4-7+1}{-1} = 2$$

$$\begin{aligned} K_2 &= \frac{d}{ds}\left[\frac{4s^2+7s+1}{s}\right]_{s=-1} = \left.\frac{s(8s+7)-(4s^2+7s+1)}{s^2}\right|_{s=-1} \\ &= \frac{1+2}{1} = 3 \end{aligned}$$

Therefore $\quad f(t) = [1 + 2te^{-t} + 3e^{-t}]\,u(t)$

AP 12.7

$$\begin{aligned} F(s) &= \frac{40}{(s^2+4s+5)^2} = \frac{40}{(s+2-j1)^2(s+2+j1)^2} \\ &= \frac{K_1}{(s+2-j1)^2} + \frac{K_2}{(s+2-j1)} + \frac{K_1^*}{(s+2+j1)^2} \\ &\quad + \frac{K_2^*}{(s+2+j1)} \end{aligned}$$

$$K_1 = \frac{40}{(j2)^2} = -10 = 10\underline{/180^\circ} \quad \text{and} \quad K_1^* = -10$$

$$K_2 = \frac{d}{ds}\left[\frac{40}{(s+2+j1)^2}\right]_{s=-2+j1} = \frac{-80(j2)}{(j2)^4} = -j10 = 10\underline{/-90^\circ}$$

$$K_2^* = j10$$

Therefore

$$\begin{aligned} f(t) &= [20te^{-2t}\cos(t+180^\circ) + 20e^{-2t}\cos(t-90^\circ)]\,u(t) \\ &= 20e^{-2t}[\sin t - t\cos t]\,u(t) \end{aligned}$$

AP 12.8

$$F(s) = \frac{5s^2+29s+32}{(s+2)(s+4)} = \frac{5s^2+29s+32}{s^2+6s+8} = 5 - \frac{s+8}{(s+2)(s+4)}$$

$$\frac{s+8}{(s+2)(s+4)} = \frac{K_1}{s+2} + \frac{K_2}{s+4}$$

$$K_1 = \frac{-2+8}{2} = 3; \qquad K_2 = \frac{-4+8}{-2} = -2$$

Therefore,

$$F(s) = 5 - \frac{3}{s+2} + \frac{2}{s+4}$$

$$f(t) = 5\delta(t) + [-3e^{-2t} + 2e^{-4t}]u(t)$$

AP 12.9

$$F(s) = \frac{2s^3+8s^2+2s-4}{s^2+5s+4} = 2s - 2 + \frac{4(s+1)}{(s+1)(s+4)} = 2s - 2 + \frac{4}{s+4}$$

$$f(t) = 2\frac{d\delta(t)}{dt} - 2\delta(t) + 4e^{-4t}\,u(t)$$

AP 12.10

$$\lim_{s\to\infty} sF(s) = \lim_{s\to\infty}\left[\frac{7s^3[1+(9/s)+(134/(7s^2))]}{s^3[1+(3/s)][1+(4/s)][1+(5/s)]}\right] = 7$$

$$\therefore \quad f(0^+) = 7$$

$$\lim_{s\to 0} sF(s) = \lim_{s\to 0}\left[\frac{7s^3+63s^2+134s}{(s+3)(s+4)(s+5)}\right] = 0$$

$$\therefore \quad f(\infty) = 0$$

$$\lim_{s\to\infty} sF(s) = \lim_{s\to\infty}\left[\frac{s^3[4+(7/s)+(1/s)^2]}{s^3[1+(1/s)]^2}\right] = 4$$

$$\therefore \quad f(0^+) = 4$$

$$\lim_{s\to 0} sF(s) = \lim_{s\to 0}\left[\frac{4s^2+7s+1}{(s+1)^2}\right] = 1$$

$$\therefore \quad f(\infty) = 1$$

$$\lim_{s\to\infty} sF(s) = \lim_{s\to\infty}\left[\frac{40s}{s^4[1+(4/s)+(5/s^2)]^2}\right] = 0$$

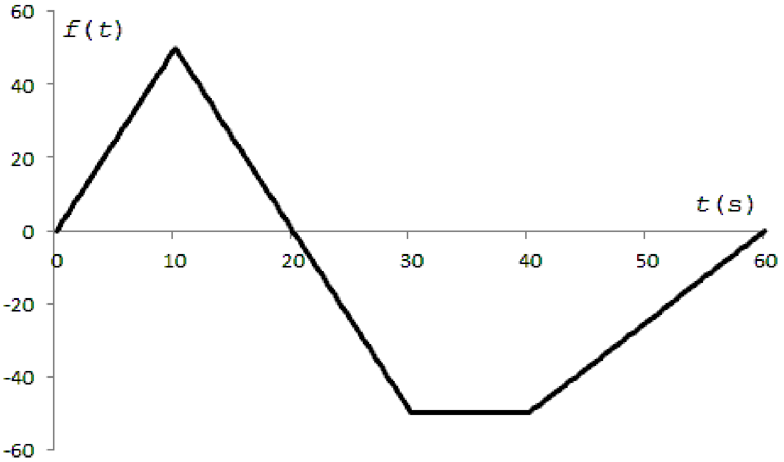
$$\therefore \quad f(0^+) = 0$$

$$\lim_{s\to 0} sF(s) = \lim_{s\to 0}\left[\frac{40s}{(s^2+4s+5)^2}\right] = 0$$

$$\therefore \quad f(\infty) = 0$$

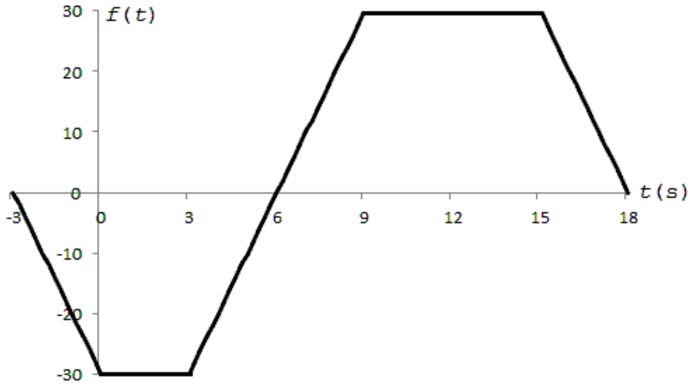
Problems

P 12.1 **[a]**



[b] $$f(t) = 5t[u(t) - u(t-10)] + (100 - 5t)[u(t-10) - u(t-30)]$$
$$-50[u(t-30) - u(t-40)]$$
$$+(2.5t - 150)[u(t-40) - u(t-60)]$$

P 12.2



P 12.3 **[a]** $$(-3t - 15)[u(t+5) - u(t)] + (-3t + 15)[u(t) - u(t-5)]$$
$$= -3(t+5)u(t+5) + 30u(t) + 3(t-5)u(t-5)$$

[b] $$(5t + 20)[u(t+4) - u(t+2)] - 5t[u(t+2) - u(t-2)]$$
$$+(5t - 20)[u(t-2) - u(t-4)]$$
$$= 5(t+4)u(t+4) - 10(t+2)u(t+2) + 10(t-2)u(t-2)$$
$$-5(t-4)u(t-4)$$

P 12.4 **[a]** $6.25t[u(t)-u(t-8)]+50[u(t-8)-u(t-12)]$

$$+(125-6.25t)[u(t-12)-u(t-20)]$$

[b] $25e^{-t}[u(t)-u(t-2)]$

[c] $(30-2t)[u(t)-u(t-5)]+20[u(t-5)-u(t-15)]$

$$+(50-2t)[u(t-15)-u(t-25)]$$

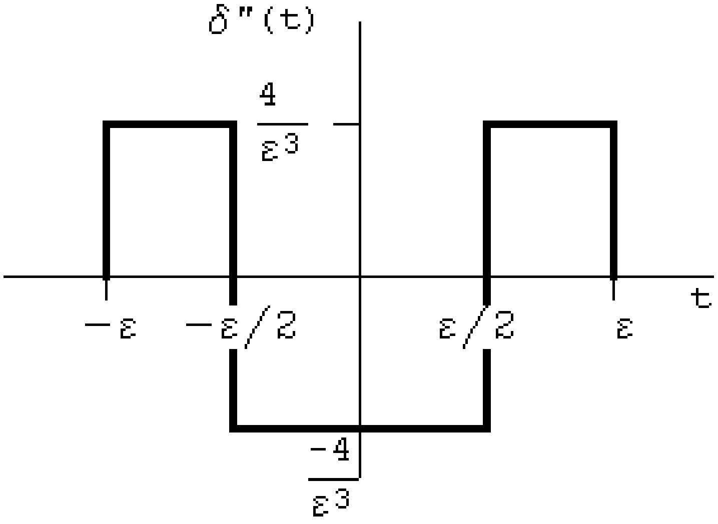
P 12.5 As $\varepsilon \to 0$ the amplitude $\to \infty$; the duration $\to 0$; and the area is independent of ε, i.e.,

$$A=\int_{-\infty}^{\infty}\frac{\varepsilon}{\pi}\frac{1}{\varepsilon^2+t^2}\,dt=1$$

P 12.6 **[a]** $A=\left(\frac{1}{2}\right)bh=\left(\frac{1}{2}\right)(2\varepsilon)\left(\frac{1}{\varepsilon}\right)=1$

[b] 0; **[c]** ∞

P 12.7



$$F(s)=\int_{-\varepsilon}^{-\varepsilon/2}\frac{4}{\varepsilon^3}e^{-st}\,dt+\int_{-\varepsilon/2}^{\varepsilon/2}\left(\frac{-4}{\varepsilon^3}\right)e^{-st}\,dt+\int_{\varepsilon/2}^{\varepsilon}\frac{4}{\varepsilon^3}e^{-st}\,dt$$

Therefore $F(s)=\frac{4}{s\varepsilon^3}[e^{s\varepsilon}-2e^{s\varepsilon/2}+2e^{-s\varepsilon/2}-e^{-s\varepsilon}]$

$$\mathcal{L}\{\delta''(t)\}=\lim_{\varepsilon\to 0}F(s)$$

After applying L'Hopital's rule three times, we have

$$\lim_{\varepsilon\to 0}\frac{2s}{3}\left[se^{s\varepsilon}-\frac{s}{4}e^{s\varepsilon/2}-\frac{s}{4}e^{-s\varepsilon/2}+se^{-s\varepsilon}\right]=\frac{2s}{3}\left(\frac{3s}{2}\right)$$

Therefore $\mathcal{L}\{\delta''(t)\}=s^2$

P 12.8 **[a]** $I = \int_{-1}^{3}(t^3+2)\delta(t)\,dt + \int_{-1}^{3} 8(t^3+2)\delta(t-1)\,dt$

$$= (0^3+2) + 8(1^3+2) = 2 + 8(3) = 26$$

[b] $I = \int_{-2}^{2} t^2\delta(t)\,dt + \int_{-2}^{2} t^2\delta(t+1.5)\,dt + \int_{-2}^{2}\delta(t-3)\,dt$

$$= 0^2 + (-1.5)^2 + 0 = 2.25$$

P 12.9 $F(s) = \int_{-\varepsilon}^{\varepsilon}\frac{1}{2\varepsilon}e^{-st}\,dt = \frac{e^{s\varepsilon}-e^{-s\varepsilon}}{2\varepsilon s}$

$$F(s) = \frac{1}{2s}\lim_{\varepsilon\to 0}\left[\frac{se^{s\varepsilon}+se^{-s\varepsilon}}{1}\right] = \frac{1}{2s}\cdot\frac{2s}{1} = 1$$

P 12.10 $f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{(4+j\omega)}{(9+j\omega)}\cdot\pi\delta(\omega)\cdot e^{jt\omega}\,d\omega = \left(\frac{1}{2\pi}\right)\left(\frac{4+j0}{9+j0}\pi e^{-jt0}\right) = \frac{2}{9}$

P 12.11 $\mathcal{L}\left\{\frac{d^n f(t)}{dt^n}\right\} = s^nF(s) - s^{n-1}f(0^-) - s^{n-2}f'(0^-) - \cdots,$

Therefore

$$\mathcal{L}\{\delta^n(t)\} = s^n(1) - s^{n-1}\delta(0^-) - s^{n-2}\delta'(0^-) - s^{n-3}\delta''(0^-) - \cdots = s^n$$

P 12.12 **[a]** Let $dv = \delta'(t-a)\,dt, \qquad v = \delta(t-a)$

$u = f(t), \qquad du = f'(t)\,dt$

Therefore

$$\int_{-\infty}^{\infty} f(t)\delta'(t-a)\,dt = f(t)\delta(t-a)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty}\delta(t-a)f'(t)\,dt$$

$$= 0 - f'(a)$$

[b] $\mathcal{L}\{\delta'(t)\} = \int_{0^-}^{\infty}\delta'(t)e^{-st}\,dt = -\left[\frac{d(e^{-st})}{dt}\right]_{t=0} = -\left[-se^{-st}\right]_{t=0} = s$

P 12.13 **[a]** $\mathcal{L}\{20e^{-500(t-10)}u(t-10)\} = \frac{20e^{-10s}}{(s+500)}$

[b] First rewrite $f(t)$ as

$$\begin{aligned} f(t) &= (5t+20)u(t+4) - (10t+20)u(t+2) \\ &\quad +(10t-20)u(t-2) - (5t-20)u(t-4) \\ &= 5(t+4)u(t+4) - 10(t+2)u(t+2) \\ &\quad +10(t-2)u(t-2) - 5(t-4)u(t-4) \end{aligned}$$

$$\therefore \quad F(s) = \frac{5[e^{4s} - 2e^{2s} + 2e^{-2s} - e^{-4s}]}{s^2}$$

P 12.14 **[a]** $f(t) = (-8t - 80)[u(t+10) - u(t+5)]$

$$+8t[u(t+5) - u(t-5)]$$

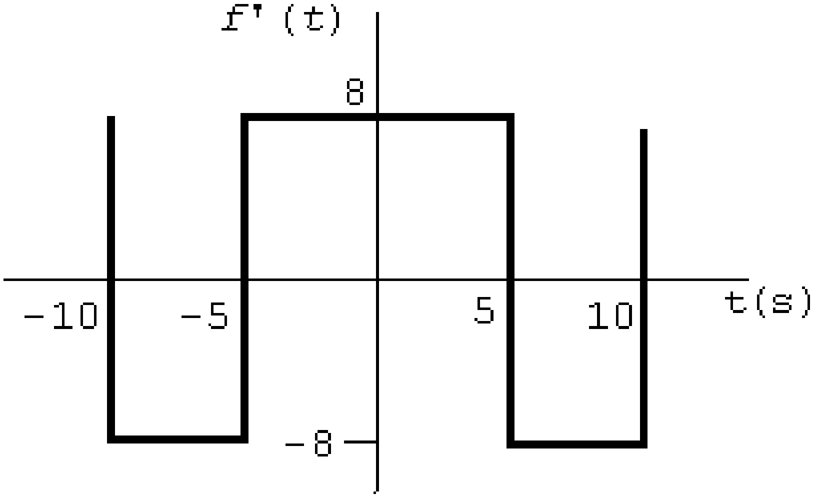
$$+(-8t+80)[u(t-5) - u(t-10)]$$

$$= -8(t+10)u(t+10) + 16(t+5)u(t+5)$$

$$-16(t-5)u(t-5) + 8(t-10)u(t-10)$$

$$\therefore \quad F(s) = \frac{8[-e^{10s} + 2e^{5s} - 2e^{-5s} + e^{-10s}]}{s^2}$$

[b]



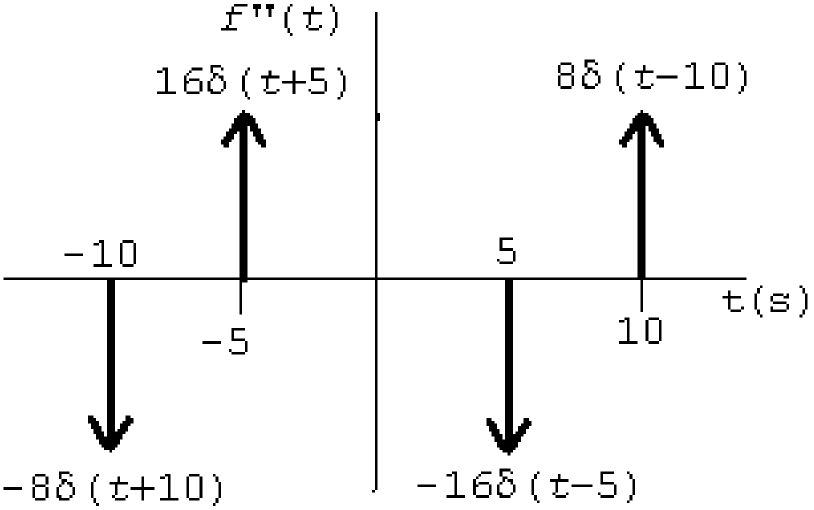
$$f'(t) = -8[u(t+10) - u(t+5)] + 8[u(t+5) - u(t-5)]$$

$$+(-8)[u(t-5) - u(t-10)]$$

$$= -8u(t+10) + 16u(t+5) - 16u(t-5) + 8u(t-10)$$

$$\mathcal{L}\{f'(t)\} = \frac{8[-e^{10s} + 2e^{5s} - 2e^{-5s} + e^{-10s}]}{s}$$

[c]



$$f''(t) = -8\delta(t+10) + 16\delta(t+5) - 16\delta(t-5) + 8\delta(t-10)$$

$$\mathcal{L}\{f''(t)\} = 8[-e^{10s} + 2e^{5s} - 2e^{-5s} + e^{-10s}]$$

P 12.15 $\mathcal{L}\{e^{-at}f(t)\} = \int_{0^-}^{\infty} [e^{-at}f(t)]e^{-st}\,dt = \int_{0^-}^{\infty} f(t)e^{-(s+a)t}\,dt = F(s+a)$

P 12.16 $\mathcal{L}\{f(at)\} = \int_{0^-}^{\infty} f(at)e^{-st}\,dt$

Let $u = at, \qquad du = a\,dt, \qquad u = 0^-$ when $t = 0^-$

and $u = \infty$ when $t = \infty$

Therefore $\mathcal{L}\{f(at)\} = \int_{0^-}^{\infty} f(u)e^{-(u/a)s}\frac{du}{a} = \frac{1}{a}F(s/a)$

P 12.17 **[a]** $\mathcal{L}\{te^{-at}\} = \int_{0^-}^{\infty} te^{-(s+a)t}\,dt$

$$= \frac{e^{-(s+a)t}}{(s+a)^2}\Big[\; -(s+a)t - 1\Big]_{0^-}^{\infty}$$

$$= 0 + \frac{1}{(s+a)^2}$$

$$\therefore\ \mathcal{L}\{te^{-at}\} = \frac{1}{(s+a)^2}$$

[b] $\mathcal{L}\left\{\frac{d}{dt}(te^{-at})u(t)\right\} = \frac{s}{(s+a)^2} - 0$

$$\mathcal{L}\left\{\frac{d}{dt}(te^{-at})u(t)\right\} = \frac{s}{(s+a)^2}$$

[c] $\frac{d}{dt}(te^{-at}) = -ate^{-at} + e^{-at}$

$$\mathcal{L}\{-ate^{-at} + e^{-at}\} = \frac{-a}{(s+a)^2} + \frac{1}{(s+a)} = \frac{-a}{(s+a)^2} + \frac{s+a}{(s+a)^2}$$

$$\therefore \quad \mathcal{L}\left\{\frac{d}{dt}(te^{-at})\right\} = \frac{s}{(s+a)^2} \qquad \text{CHECKS}$$

P 12.18 **[a]** $\mathcal{L}\left\{\int_{0^-}^{t} e^{-ax}\,dx\right\} = \frac{F(s)}{s} = \frac{1}{s(s+a)}$

[b] $\int_{0^-}^{t} e^{-ax}\,dx = \frac{1}{a} - \frac{e^{-at}}{a}$

$$\mathcal{L}\left\{\frac{1}{a} - \frac{e^{-at}}{a}\right\} = \frac{1}{a}\left[\frac{1}{s} - \frac{1}{s+a}\right] = \frac{1}{s(s+a)}$$

P 12.19 **[a]** $\int_{0^-}^{t} x\,dx = \frac{t^2}{2}$

$$\mathcal{L}\left\{\frac{t^2}{2}\right\} = \frac{1}{2}\int_{0^-}^{\infty} t^2 e^{-st}\,dt$$

$$= \frac{1}{2}\left[\frac{e^{-st}}{-s^3}(s^2t^2 + 2st + 2)\Big|_{0^-}^{\infty}\right]$$

$$= \frac{1}{2s^3}(2) = \frac{1}{s^3}$$

$$\therefore \quad \mathcal{L}\left\{\int_{0^-}^{t} x\,dx\right\} = \frac{1}{s^3}$$

[b] $\mathcal{L}\left\{\int_{0^-}^{t} x\,dx\right\} = \frac{\mathcal{L}\{t\}}{s} = \frac{1/s^2}{s} = \frac{1}{s^3}$

$$\therefore \quad \mathcal{L}\left\{\int_{0^-}^{t} x\,dx\right\} = \frac{1}{s^3} \qquad \text{CHECKS}$$

P 12.20 **[a]** $\mathcal{L}\{t\} = \frac{1}{s^2}$; therefore $\mathcal{L}\{te^{-at}\} = \frac{1}{(s+a)^2}$

[b] $\sin\omega t = \dfrac{e^{j\omega t} - e^{-j\omega t}}{j2}$

Therefore

$$\mathcal{L}\{\sin\omega t\} = \left(\frac{1}{j2}\right)\left(\frac{1}{s-j\omega} - \frac{1}{s+j\omega}\right) = \left(\frac{1}{j2}\right)\left(\frac{2j\omega}{s^2+\omega^2}\right)$$
$$= \frac{\omega}{s^2+\omega^2}$$

[c] $\sin(\omega t+\theta) = (\sin\omega t\cos\theta + \cos\omega t\sin\theta)$

Therefore

$$\mathcal{L}\{\sin(\omega t+\theta)\} = \cos\theta\mathcal{L}\{\sin\omega t\} + \sin\theta\mathcal{L}\{\cos\omega t\}$$
$$= \frac{\omega\cos\theta + s\sin\theta}{s^2+\omega^2}$$

[d] $\mathcal{L}\{t\} = \displaystyle\int_0^\infty te^{-st}\,dt = \frac{e^{-st}}{s^2}(-st-1)\Big|_0^\infty = 0 - \frac{1}{s^2}(0-1) = \frac{1}{s^2}$

[e] $f(t) = \cosh t\cosh\theta + \sinh t\sinh\theta$

From Assessment Problem 12.1(a)

$$\mathcal{L}\{\cosh t\} = \frac{s}{s^2-1}$$

From Assessment Problem 12.1(b)

$$\mathcal{L}\{\sinh t\} = \frac{1}{s^2-1}$$

$$\therefore\quad \mathcal{L}\{\cosh(t+\theta)\} = \cosh\theta\left[\frac{s}{(s^2-1)}\right] + \sinh\theta\left[\frac{1}{s^2-1}\right]$$
$$= \frac{\sinh\theta + s[\cosh\theta]}{(s^2-1)}$$

P 12.21 **[a]** $\dfrac{dF(s)}{ds} = \dfrac{d}{ds}\left[\displaystyle\int_{0^-}^\infty f(t)e^{-st}\,dt\right] = -\displaystyle\int_{0^-}^\infty tf(t)e^{-st}\,dt$

Therefore $\quad\mathcal{L}\{tf(t)\} = -\dfrac{dF(s)}{ds}$

[b] $\dfrac{d^2F(s)}{ds^2} = \displaystyle\int_{0^-}^\infty t^2f(t)e^{-st}\,dt; \qquad \dfrac{d^3F(s)}{ds^3} = \displaystyle\int_{0^-}^\infty -t^3f(t)e^{-st}\,dt$

Therefore $\quad\dfrac{d^nF(s)}{ds^n} = (-1)^n\displaystyle\int_{0^-}^\infty t^nf(t)e^{-st}\,dt = (-1)^n\mathcal{L}\{t^nf(t)\}$

[c] $\mathcal{L}\{t^5\} = \mathcal{L}\{t^4t\} = (-1)^4\dfrac{d^4}{ds^4}\left(\dfrac{1}{s^2}\right) = \dfrac{120}{s^6}$

$$\mathcal{L}\{t\sin\beta t\} = (-1)^1\frac{d}{ds}\left(\frac{\beta}{s^2+\beta^2}\right) = \frac{2\beta s}{(s^2+\beta^2)^2}$$

$\mathcal{L}\{te^{-t}\cosh t\}$:

From Assessment Problem 12.1(a),

$$F(s) = \mathcal{L}\{\cosh t\} = \frac{s}{s^2-1}$$

$$\frac{dF}{ds} = \frac{(s^2-1)1 - s(2s)}{(s^2-1)^2} = -\frac{s^2+1}{(s^2-1)^2}$$

Therefore $\quad -\dfrac{dF}{ds} = \dfrac{s^2+1}{(s^2-1)^2}$

Thus

$$\mathcal{L}\{t\cosh t\} = \frac{s^2+1}{(s^2-1)^2}$$

$$\mathcal{L}\{e^{-t}t\cosh t\} = \frac{(s+1)^2+1}{[(s+1)^2-1]^2} = \frac{s^2+2s+2}{s^2(s+2)^2}$$

P 12.22 **[a]** $\mathcal{L}\left\{\dfrac{d\sin\omega t}{dt}u(t)\right\} = \dfrac{s\omega}{s^2+\omega^2} - \sin(0) = \dfrac{s\omega}{s^2+\omega^2}$

[b] $\mathcal{L}\left\{\dfrac{d\cos\omega t}{dt}u(t)\right\} = \dfrac{s^2}{s^2+\omega^2} - \cos(0) = \dfrac{s^2}{s^2+\omega^2} - 1 = \dfrac{-\omega^2}{s^2+\omega^2}$

[c] $\mathcal{L}\left\{\dfrac{d^3(t^2)}{dt^3}u(t)\right\} = s^3\left(\dfrac{2}{s^3}\right) - s^2(0) - s(0) - 2(0) = 2$

[d] $\dfrac{d\sin\omega t}{dt} = (\cos\omega t)\cdot\omega, \qquad \mathcal{L}\{\omega\cos\omega t\} = \dfrac{\omega s}{s^2+\omega^2}$

$$\frac{d\cos\omega t}{dt} = -\omega\sin\omega t$$

$$\mathcal{L}\{-\omega\sin\omega t\} = -\frac{\omega^2}{s^2+\omega^2}$$

$$\frac{d^3(t^2u(t))}{dt^3} = 2\delta(t); \qquad \mathcal{L}\{2\delta(t)\} = 2$$

P 12.23 **[a]** $\mathcal{L}\{f'(t)\} = \displaystyle\int_{-\varepsilon}^{\varepsilon} \frac{e^{-st}}{2\varepsilon}\,dt + \int_{\varepsilon}^{\infty} - ae^{-a(t-\varepsilon)}e^{-st}\,dt$

$$= \frac{1}{2s\varepsilon}(e^{s\varepsilon} - e^{-s\varepsilon}) - \left(\frac{a}{s+a}\right)e^{-s\varepsilon} = F(s)$$

$$\lim_{\varepsilon\to 0} F(s) = 1 - \frac{a}{s+a} = \frac{s}{s+a}$$

[b] $\mathcal{L}\{e^{-at}\} = \dfrac{1}{s+a}$

Therefore $\mathcal{L}\{f'(t)\} = sF(s) - f(0^-) = \dfrac{s}{s+a} - 0 = \dfrac{s}{s+a}$

P 12.24 **[a]** $f_1(t) = e^{-at}\cos\omega t; \qquad F_1(s) = \dfrac{s+a}{(s+a)^2+\omega^2}$

$$F(s) = sF_1(s) - f_1(0^-) = \frac{s(s+a)}{(s+a)^2+\omega^2} - 1 = \frac{-a^2 - sa - \omega^2}{(s+a)^2+\omega^2}$$

[b] $f_1(t) = e^{-at}\sin\omega t; \qquad F_1(s) = \dfrac{\omega}{(s+a)^2+\omega^2}$

$$F(s) = \frac{F_1(s)}{s} = \frac{\omega}{s[(s+a)^2+\omega^2]}$$

[c] $\dfrac{d}{dt}[e^{-at}\cos\omega t] = -\omega e^{-at}\sin\omega t - ae^{-at}\cos\omega t$

Therefore $F(s) = \dfrac{-\omega^2 - a(s+a)}{(s+a)^2+\omega^2} = \dfrac{-a^2 - sa - \omega^2}{(s+a)^2+\omega^2}$

$$\int_{0^-}^{t} e^{-ax}\sin\omega x\,dx = \frac{-ae^{-at}\sin\omega t - \omega e^{-at}\cos\omega t + \omega}{a^2+\omega^2}$$

Therefore

$$F(s) = \frac{1}{a^2+\omega^2}\left[\frac{-a\omega}{(s+a)^2+\omega^2} - \frac{\omega(s+a)}{(s+a)^2+\omega^2} + \frac{\omega}{s}\right]$$

$$= \frac{\omega}{s[(s+a)^2+\omega^2]}$$

P 12.25 **[a]** $\displaystyle\int_s^\infty F(u)du = \int_s^\infty \left[\int_{0^-}^\infty f(t)e^{-ut}\,dt\right]du = \int_{0^-}^\infty \left[\int_s^\infty f(t)e^{-ut}\,du\right]dt$

$$= \int_{0^-}^\infty f(t)\int_s^\infty e^{-ut}\,du\,dt = \int_{0^-}^\infty f(t)\left[\frac{e^{-tu}}{-t}\Big|_s^\infty\right]dt$$

$$= \int_{0^-}^\infty f(t)\left[\frac{-e^{-st}}{-t}\right]dt = \mathcal{L}\left\{\frac{f(t)}{t}\right\}$$

[b] $\mathcal{L}\{t\sin\beta t\} = \dfrac{2\beta s}{(s^2+\beta^2)^2}$

therefore $\mathcal{L}\left\{\dfrac{t\sin\beta t}{t}\right\} = \displaystyle\int_s^\infty \left[\frac{2\beta u}{(u^2+\beta^2)^2}\right]du$

Let $\omega = u^2+\beta^2$, then $\omega = s^2+\beta^2$ when $u = s$, and $\omega = \infty$ when $u = \infty$; also $d\omega = 2u\,du$, thus

$$\mathcal{L}\left\{\frac{t\sin\beta t}{t}\right\} = \beta\int_{s^2+\beta^2}^\infty \left[\frac{d\omega}{\omega^2}\right] = \beta\left(\frac{-1}{\omega}\right)\Big|_{s^2+\beta^2}^\infty = \frac{\beta}{s^2+\beta^2}$$

P 12.26 $I_g(s) = \dfrac{5s}{s^2+400}; \qquad \dfrac{1}{RC} = 16; \qquad \dfrac{1}{LC} = 100; \qquad \dfrac{1}{C} = 20$

$$\frac{V(s)}{R} + \frac{1}{L}\frac{V(s)}{s} + C[sV(s) - v(0^-)] = I_g(s)$$

$$V(s)\left[\frac{1}{R} + \frac{1}{Ls} + sC\right] = I_g(s)$$

$$V(s) = \frac{I_g(s)}{\frac{1}{R} + \frac{1}{Ls} + sC} = \frac{LsI_g(s)}{\frac{sL}{R} + 1 + s^2LC} = \frac{\frac{1}{C}sI_g(s)}{s^2 + \frac{1}{RC}s + \frac{1}{LC}}$$

$$= \frac{(20)(5)s^2}{(s^2+16s+100)(s^2+400)} = \frac{100s^2}{(s^2+16s+100)(s^2+400)}$$

P 12.27 **[a]** $I_{\text{dc}} = \dfrac{1}{L}\displaystyle\int_0^t v_o\,dx + \dfrac{v_o}{R} + C\dfrac{dv_o}{dt}$

[b] $\dfrac{I_{\text{dc}}}{s} = \dfrac{V_o(s)}{sL} + \dfrac{V_o(s)}{R} + sCV_o(s)$

$$\therefore \quad V_o(s) = \frac{I_{\text{dc}}/C}{s^2 + (1/RC)s + (1/LC)}$$

[c] $i_o = C\dfrac{dv_o}{dt}$

$$\therefore \quad I_o(s) = sCV_o(s) = \frac{sI_{\text{dc}}}{s^2 + (1/RC)s + (1/LC)}$$

P 12.28 **[a]** For $t \geq 0^+$:

$$\frac{v_o}{R} + C\frac{dv_o}{dt} + i_o = 0$$

$$v_o = L\frac{di_o}{dt}; \qquad \frac{dv_o}{dt} = L\frac{d^2i_o}{dt^2}$$

$$\therefore \quad \frac{L}{R}\frac{di_o}{dt} + LC\frac{d^2i_o}{dt^2} + i_o = 0$$

$$\text{or} \quad \frac{d^2i_o}{dt^2} + \frac{1}{RC}\frac{di_o}{dt} + \frac{1}{LC}i_o = 0$$

[b] $s^2I_o(s) - sI_{\text{dc}} - 0 + \dfrac{1}{RC}[sI_o(s) - I_{\text{dc}}] + \dfrac{1}{LC}I_o(s) = 0$

$$I_o(s)\left[s^2 + \frac{1}{RC}s + \frac{1}{LC}\right] = I_{\text{dc}}(s + 1/RC)$$

$$I_o(s) = \frac{I_{\text{dc}}[s + (1/RC)]}{[s^2 + (1/RC)s + (1/LC)]}$$

P 12.29 **[a]** For $t \geq 0^+$:

$$Ri_o + L\frac{di_o}{dt} + v_o = 0$$

$$i_o = C\frac{dv_o}{dt} \qquad \frac{di_o}{dt} = C\frac{d^2v_o}{dt^2}$$

$$\therefore \quad RC\frac{dv_o}{dt} + LC\frac{d^2v_o}{dt^2} + v_o = 0$$

or

$$\frac{d^2v_o}{dt^2} + \frac{R}{L}\frac{dv_o}{dt} + \frac{1}{LC}v_o = 0$$

[b] $s^2V_o(s) - sV_{\rm dc} - 0 + \dfrac{R}{L}[sV_o(s) - V_{\rm dc}] + \dfrac{1}{LC}V_o(s) = 0$

$$V_o(s)\left[s^2 + \frac{R}{L}s + \frac{1}{LC}\right] = V_{\rm dc}(s + R/L)$$

$$V_o(s) = \frac{V_{\rm dc}[s + (R/L)]}{[s^2 + (R/L)s + (1/LC)]}$$

P 12.30 **[a]** $\dfrac{v_o - V_{\rm dc}}{R} + \dfrac{1}{L}\displaystyle\int_0^t v_o\,dx + C\dfrac{dv_o}{dt} = 0$

$$\therefore \qquad v_o + \frac{R}{L}\int_0^t v_o\,dx + RC\frac{dv_o}{dt} = V_{\rm dc}$$

[b] $V_o + \dfrac{R}{L}\dfrac{V_o}{s} + RCsV_o = \dfrac{V_{\rm dc}}{s}$

$$\therefore \qquad sLV_o + RV_o + RCLs^2V_o = LV_{\rm dc}$$

$$\therefore \qquad V_o(s) = \frac{(1/RC)V_{\rm dc}}{s^2 + (1/RC)s + (1/LC)}$$

[c] $i_o = \dfrac{1}{L}\displaystyle\int_0^t v_o\,dx$

$$I_o(s) = \frac{V_o}{sL} = \frac{V_{\rm dc}/RLC}{s[s^2 + (1/RC)s + (1/LC)]}$$

P 12.31 **[a]** $\dfrac{1}{L}\displaystyle\int_0^t v_1\,d\tau + \dfrac{v_1 - v_2}{R} = i_g u(t)$

and

$$C\frac{dv_2}{dt} + \frac{v_2}{R} - \frac{v_1}{R} = 0$$

[b] $\frac{V_1}{sL} + \frac{V_1 - V_2}{R} = I_g$

$$\frac{V_2 - V_1}{R} + sCV_2 = 0$$

or

$$(R + sL)V_1(s) - sLV_2(s) = RLsI_g(s)$$

$$-V_1(s) + (RCs + 1)V_2(s) = 0$$

Solving,

$$V_2(s) = \frac{sI_g(s)}{C[s^2 + (R/L)s + (1/LC)]}$$

P 12.32 **[a]** $625 = 150i_1 + 62.5\frac{di_1}{dt} + 25\frac{d}{dt}(i_2 - i_1) + 12.5\frac{d}{dt}(i_1 - i_2) - 25\frac{di_1}{dt}$

$$0 = 12.5\frac{d}{dt}(i_2 - i_1) + 25\frac{di_1}{dt} + 100i_2$$

Simplifying the above equations gives:

$$625 = 150i_1 + 25\frac{di_1}{dt} + 12.5\frac{di_2}{dt}$$

$$0 = 100i_2 + 12.5\frac{di_1}{dt} + 12.5\frac{di_2}{dt}$$

[b] $\frac{625}{s} = (25s + 150)I_1(s) + 12.5sI_2(s)$

$$0 = 12.5sI_1(s) + (12.5s + 100)I_2(s)$$

[c] Solving the equations in (b),

$$I_1(s) = \frac{50(s + 8)}{s(s + 4)(s + 24)}$$

$$I_2(s) = \frac{-50}{(s + 4)(s + 24)}$$

P 12.33 From Problem 12.26:

$$V(s) = \frac{100s^2}{(s^2 + 16s + 100)(s^2 + 400)}$$

$$s^2 + 16s + 100 = (s + 8 + j6)(s + 8 - j6); \qquad s^2 + 400 = (s - j20)(s + j20)$$

Therefore

$$V(s) = \frac{100s^2}{(s+8+j6)(s+8-j6)(s-j20)(s+j20)}$$

$$= \frac{K_1}{s+8-j6} + \frac{K_1^*}{s+8+j6} + \frac{K_2}{s-j20} + \frac{K_2^*}{s+j20}$$

$$K_1 = \frac{100s^2}{(s+8+j6)(s^2+400)}\bigg|_{s=-8+j6} = 1.9\underline{/-151.1^\circ}$$

$$K_2 = \frac{100s^2}{(s+j20)(s^2+16s+100)}\bigg|_{s=j20} = 2.28\underline{/-43.15^\circ}$$

Therefore

$$v(t) = [3.8e^{-8t}\cos(6t-151.1^\circ) + 4.56\cos(20t-43.15^\circ)]u(t)\text{ V}$$

P 12.34 **[a]** $\frac{1}{RC} = \frac{1}{(20)(20\times10^{-6})} = 2500$

$$\frac{1}{LC} = \frac{1}{(0.05)(20\times10^{-6})} = 10^6; \qquad \frac{1}{C} = 50{,}000$$

$$V_o(s) = \frac{50{,}000I_{\text{dc}}}{s+2500s+10^6}$$

$$= \frac{50{,}000I_{\text{dc}}}{(s+500)(s+2000)}$$

$$= \frac{3750}{(s+500)(s+2000)}$$

$$= \frac{K_1}{s+500} + \frac{K_2}{s+2000}$$

$$K_1 = \frac{3750}{1500} = 2.5; \qquad K_2 = \frac{3750}{-1500} = -2.5$$

$$V_o(s) = \frac{2.5}{s+500} - \frac{2.5}{s+2000}$$

$$v_o(t) = [2.5e^{-500t} - 2.5e^{-2000t}]u(t)\text{ V}$$

[b] $I_o(s) = \frac{0.075s}{(s+500)(s+2000)}$

$$= \frac{K_1}{s+500} + \frac{K_2}{s+2000}$$

$$K_1 = \frac{0.075(-500)}{1500} = -0.025$$

$$K_2 = \frac{0.075(-2000)}{-1500} = 0.1$$

$$I_o(s) = \frac{-0.025}{s+500} + \frac{0.1}{s+2000}$$

$$i_o(t) = (-25e^{-500t} + 100e^{-2000t})u(t)\,\text{mA}$$

[c] $i_o(0) = 100 - 25 = 75\,\text{mA}$

Yes. The initial inductor current is zero by hypothesis, the initial resistor current is zero because the initial capacitor voltage is zero by hypothesis. Thus at $t = 0$ the source current appears in the capacitor.

P 12.35 $\frac{1}{RC} = 200{,}000; \qquad \frac{1}{LC} = 10^{10}$

$$I_o(s) = \frac{0.04(s+200{,}000)}{s^2 + 200{,}000s + 10^{10}}$$

$$s_{1,2} = -100{,}000$$

$$I_o(s) = \frac{0.04(s+200{,}000)}{(s+100{,}000)^2} = \frac{K_1}{(s+100{,}000)^2} + \frac{K_2}{s+100{,}000}$$

$$K_1 = 0.04(s+200{,}000)\Big|_{s=-100{,}000} = 4000$$

$$K_2 = \frac{d}{ds}\left[0.04(s+200{,}000)\right]_{s=-100{,}000} = 0.04$$

$$I_o(s) = \frac{4000}{(s+100{,}000)^2} + \frac{0.04}{s+100{,}000}$$

$$i_o(t) = [4000te^{-100{,}000t} + 0.04e^{-100{,}000t}]u(t)\,\text{A}$$

P 12.36 $\frac{R}{L} = 10{,}000; \qquad \frac{1}{LC} = 16 \times 10^6$

$$V_o(s) = \frac{120(s+10{,}000)}{s^2 + 10{,}000s + 16\times 10^6}$$

$$= \frac{120(s+10{,}000)}{(s+2000)(s+8000)} = \frac{K_1}{s+2000} + \frac{K_2}{s+8000}$$

$$K_1 = \frac{120(8000)}{6000} = 160\,\text{V}; \qquad K_2 = \frac{120(2000)}{-6000} = -40\,\text{V}$$

$$V_o(s) = \frac{160}{s+2000} - \frac{40}{s+8000}$$

$$v_o(t) = [160e^{-2000t} - 40e^{-8000t}]u(t)\,\text{V}$$

P 12.37 **[a]** $$\frac{1}{LC} = \frac{1}{(1.6)(5\times10^{-6})} = 125\times10^3$$

$$\frac{1}{RC} = \frac{1}{(2000)(5\times10^{-6})} = 100$$

$$V_o(s) = \frac{5600}{s^2 + 100s + 125\times10^3}$$

$$s_{1,2} = -50 \pm j350 \text{ rad/s}$$

$$V_o(s) = \frac{5600}{(s+50-j350)(s+50+j350)}$$

$$= \frac{K_1}{s+50-j350} + \frac{K_1^*}{s+50+j350}$$

$$K_1 = \frac{5600}{j700} = 8\underline{/-90^\circ}$$

$$v_o(t) = 16e^{-50t}\cos(350t-90^\circ)]u(t)\,\text{V}$$

$$= [16e^{-50t}\sin 350t]u(t)\,\text{V}$$

[b] $$I_o(s) = \frac{56(62.5)}{s(s+50-j350)(s+50+j350)}$$

$$= \frac{K_1}{s} + \frac{K_2}{s+50-j350} + \frac{K_2^*}{s+50+j350}$$

$$K_1 = \frac{3500}{125\times10^3} = 28\,\text{mA}$$

$$K_2 = \frac{3500}{(-50+j350)(j700)} = 14.14\underline{/171.87^\circ}\,\text{mA}$$

$$i_o(t) = [28 + 28.28e^{-50t}\cos(350t+171.87^\circ)]u(t)\,\text{mA}$$

P 12.38 $$\frac{1}{C} = 64\times10^6; \qquad \frac{1}{LC} = 1600\times10^6; \qquad \frac{R}{L} = 100{,}000; \qquad I_g = \frac{0.15}{s}$$

$$V_2(s) = \frac{96\times10^5}{s^2 + 10^5 s + 1600\times10^6}$$

$$s_1 = -20{,}000; \qquad s_2 = -80{,}000$$

$$V_2(s) = \frac{96\times10^5}{(s+20{,}000)(s+80{,}000)}$$

$$= \frac{160}{s+20{,}000} - \frac{160}{s+80{,}000}$$

$$v_2(t) = [160e^{-20{,}000t} - 160e^{-80{,}000t}]u(t)\,\text{V}$$

P 12.39 **[a]** $I_1(s) = \dfrac{K_1}{s} + \dfrac{K_2}{s+4} + \dfrac{K_3}{s+24}$

$$K_1 = \frac{(50)(8)}{(4)(24)} = 4.167; \qquad K_2 = \frac{(50)(4)}{(-4)(20)} = -2.5$$

$$K_3 = \frac{(50)(-16)}{(-24)(-20)} = -1.667$$

$$I_1(s) = \left(\frac{4.167}{s} - \frac{2.5}{s+4} - \frac{1.667}{s+24}\right)$$

$$i_1(t) = (4.167 - 2.5e^{-4t} - 1.667e^{-24t})u(t)\,\text{A}$$

$$I_2(s) = \frac{K_1}{s+4} + \frac{K_2}{s+24}$$

$$K_1 = \frac{-50}{20} = -2.5; \qquad K_2 = \frac{-50}{-20} = 2.5$$

$$I_2(s) = \left(\frac{-2.5}{s+4} + \frac{2.5}{s+24}\right)$$

$$i_2(t) = (2.5e^{-24t} - 2.5e^{-4t})u(t)\,\text{A}$$

[b] $i_1(\infty) = 4.167\,\text{A}; \qquad i_2(\infty) = 0\,\text{A}$

[c] Yes, at $t = \infty$

$$i_1 = \frac{625}{150} = 4.167\,\text{A}$$

Since i_1 is a dc current at $t = \infty$ there is no voltage induced in the 12.5 H inductor; hence, $i_2 = 0$. Also note that $i_1(0) = 0$ and $i_2(0) = 0$. Thus our solutions satisfy the condition of no initial energy stored in the circuit.

P 12.40 **[a]** $F(s) = \dfrac{K_1}{s+5} + \dfrac{K_2}{s+8}$

$$K_1 = \frac{6(s+10)}{(s+8)}\bigg|_{s=-5} = 10$$

$$K_2 = \frac{6(s+10)}{(s+5)}\bigg|_{s=-8} = -4$$

$$f(t) = [10e^{-5t} - 4e^{-8t}]u(t)$$

[b] $F(s) = \dfrac{K_1}{s} + \dfrac{K_2}{s+3} + \dfrac{K_3}{s+7}$

$$K_1 = \frac{20s^2 + 141s + 315}{(s+3)(s+7)}\bigg|_{s=0} = 15$$

$$K_2 = \frac{20s^2 + 141s + 315}{s(s+7)}\Big|_{s=-3} = -6$$

$$K_3 = \frac{20s^2 + 141s + 315}{s(s+3)}\Big|_{s=-7} = 11$$

$$f(t) = [15 - 6e^{-3t} + 11e^{-7t}]u(t)$$

[c] $F(s) = \frac{K_1}{s+2} + \frac{K_2}{s+4} + \frac{K_3}{s+6}$

$$K_1 = \frac{15s^2 + 112s + 228}{(s+4)(s+6)}\Big|_{s=-2} = 8$$

$$K_2 = \frac{15s^2 + 112s + 228}{(s+2)(s+6)}\Big|_{s=-4} = -5$$

$$K_3 = \frac{15s^2 + 112s + 228}{(s+2)(s+4)}\Big|_{s=-6} = 12$$

$$f(t) = [8e^{-2t} - 5e^{-4t} + 12e^{-6t}]u(t)$$

[d] $F(s) = \frac{K_1}{s} + \frac{K_2}{s+1} + \frac{K_3}{s+2} + \frac{K_3}{s+3}$

$$K_1 = \frac{2s^3 + 33s^2 + 93s + 54}{(s+1)(s+2)(s+3)}\Big|_{s=0} = 9$$

$$K_2 = \frac{2s^3 + 33s^2 + 93s + 54}{s(s+2)(s+3)}\Big|_{s=-1} = 4$$

$$K_3 = \frac{2s^3 + 33s^2 + 93s + 54}{s(s+1)(s+3)}\Big|_{s=-2} = -8$$

$$K_4 = \frac{2s^3 + 33s^2 + 93s + 54}{s(s+1)(s+2)}\Big|_{s=-3} = -3$$

$$f(t) = [9 + 4e^{-t} - 8e^{-2t} - 3e^{-3t}]u(t)$$

P 12.41 **[a]** $F(s) = \frac{K_1}{s+7-j14} + \frac{K_1^*}{s+7+j14}$

$$K_1 = \frac{280}{s+7+j14}\Big|_{s=-7+j14} = -j10 = 10\underline{/-90^\circ}$$

$$f(t) = [20e^{-7t}\cos(14t - 90^\circ)]u(t) = [20e^{-7t}\sin 14t]u(t)$$

[b] $F(s) = \dfrac{K_1}{s} + \dfrac{K_2}{s+5-j8} + \dfrac{K_2^*}{s+5+j8}$

$$K_1 = \frac{-s^2+52s+445}{s^2+10s+89}\bigg|_{s=0} = 5$$

$$K_2 = \frac{-s^2+52s+445}{s(s+5+j8)}\bigg|_{s=-5+j8} = -3-j2 = 3.6\underline{/-146.31^\circ}$$

$$f(t) = [5+7.2e^{-5t}\cos(8t-146.31^\circ)]u(t)$$

[c] $F(s) = \dfrac{K_1}{s+6} + \dfrac{K_2}{s+2-j4} + \dfrac{K_2^*}{s+2+j4}$

$$K_1 = \frac{14s^2+56s+152}{s^2+4s+20}\bigg|_{s=-6} = 10$$

$$K_2 = \frac{14s^2+56s+152}{(s+6)(s+2+j4)}\bigg|_{s=-2+j4} = 2+j2 = 2.83\underline{/45^\circ}$$

$$f(t) = [10e^{-6t}+5.66e^{-2t}\cos(4t+45^\circ)]u(t)$$

[d] $F(s) = \dfrac{K_1}{s+5-j3} + \dfrac{K_1^*}{s+5+j3} + \dfrac{K_2}{s+4-j2} + \dfrac{K_2^*}{s+4+j2}$

$$K_1 = \frac{8(s+1)^2}{(s+5+j3)(s^2+8s+20)}\bigg|_{s=-5+j3} = 4.62\underline{/-40.04^\circ}$$

$$K_2 = \frac{8(s+1)^2}{(s+4+j2)(s^2+10s+34)}\bigg|_{s=-4+j2} = 3.61\underline{/168.93^\circ}$$

$$f(t) = [9.25e^{-5t}\cos(3t-40.05^\circ)+7.21e^{-4t}\cos(2t+168.93^\circ)]u(t)$$

P 12.42 **[a]** $F(s) = \dfrac{K_1}{s^2} + \dfrac{K_2}{s} + \dfrac{K_3}{s+8}$

$$K_1 = \frac{320}{s+8}\bigg|_{s=0} = 40$$

$$K_2 = \frac{d}{ds}\left[\frac{320}{s+8}\right] = \left[\frac{-320}{(s+8)^2}\right]_{s=0} = -5$$

$$K_3 = \frac{320}{s^2}\bigg|_{s=-8} = 5$$

$$f(t) = [40t-5+5e^{-8t}]u(t)$$

[b] $F(s) = \dfrac{K_1}{s} + \dfrac{K_2}{(s+2)^2} + \dfrac{K_3}{s+2}$

$$K_1 = \frac{80(s+3)}{(s+2)^2}\bigg|_{s=0} = 60$$

$$K_2 = \frac{80(s+3)}{s}\bigg|_{s=-2} = -40$$

$$K_3 = \frac{d}{ds}\left[\frac{80(s+3)}{s}\right] = \left[\frac{80}{s} - \frac{80(s+3)}{s^2}\right]_{s=-2} = -60$$

$$f(t) = [60 - 40te^{-2t} - 60e^{-2t}]u(t)$$

[c] $F(s) = \dfrac{K_1}{(s+1)^2} + \dfrac{K_2}{s+1} + \dfrac{K_3}{s+3-j4} + \dfrac{K_3^*}{s+3+j4}$

$$K_1 = \frac{60(s+5)}{s^2+6s+25}\bigg|_{s=-1} = 12$$

$$K_2 = \frac{d}{ds}\left[\frac{60(s+5)}{s^2+6s+25}\right] = \left[\frac{60}{s^2+6s+25} - \frac{60(s+5)(2s+6)}{(s^2+6s+25)^2}\right]_{s=-1} = 0.6$$

$$K_3 = \frac{60(s+5)}{(s+1)^2(s+3+j4)}\bigg|_{s=-3+j4} = 1.68\underline{/100.305^\circ}$$

$$f(t) = [12te^{-t} + 0.6e^{-t} + 3.35e^{-3t}\cos(4t + 100.305^\circ)]u(t)$$

[d] $F(s) = \dfrac{K_1}{s^2} + \dfrac{K_2}{s} + \dfrac{K_3}{(s+5)^2} + \dfrac{K_4}{s+5}$

$$K_1 = \frac{25(s+4)^2}{(s+5)^2}\bigg|_{s=0} = 16$$

$$K_2 = \frac{d}{ds}\left[\frac{25(s+4)^2}{(s+5)^2}\right] = \left[\frac{25(2)(s+4)}{(s+5)^2} - \frac{25(2)(s+4)^2}{(s+5)^3}\right]_{s=0} = 1.6$$

$$K_3 = \frac{25(s+4)^2}{s^2}\bigg|_{s=-5} = 1$$

$$K_4 = \frac{d}{ds}\left[\frac{25(s+4)^2}{s^2}\right] = \left[\frac{25(2)(s+4)}{s^2} - \frac{25(2)(s+4)^2}{s^3}\right]_{s=-5} = -1.6$$

$$f(t) = [16t + 1.6 + te^{-5t} - 1.6e^{-5t}]u(t)$$

P 12.43 **[a]** $F(s) = \dfrac{K_1}{s} + \dfrac{K_2}{(s+3)^3} + \dfrac{K_3}{(s+3)^2} + \dfrac{K_4}{s+3}$

$$K_1 = \frac{135}{(s+3)^3}\bigg|_{s=0} = 5$$

$$K_2 = \frac{135}{s}\bigg|_{s=-3} = -45$$

$$K_3 = \frac{d}{ds}\left[\frac{135}{s}\right] = \left[\frac{-135}{s^2}\right]_{s=-3} = -15$$

$$K_4 = \frac{1}{2}\frac{d}{ds}\left[\frac{-135}{s^2}\right] = \left[\frac{1}{2}(-2)\left(\frac{-135}{s^3}\right)\right]_{s=-3} = -5$$

$$f(t) = [5 - 22.5t^2e^{-3t} - 15te^{-3t} - 5e^{-3t}]u(t)$$

[b] $F(s) = \dfrac{K_1}{(s+1-j1)^2} + \dfrac{K_1^*}{(s+1+j1)^2} + \dfrac{K_2}{s+1-j1} + \dfrac{K_2^*}{s+1+j1}$

$$K_1 = \frac{10(s+2)^2}{(s+1+j1)^2}\bigg|_{s=-1+j1} = -j5 = 5\underline{/-90^\circ}$$

$$K_2 = \frac{d}{ds}\left[\frac{10(s+2)^2}{(s+1+j1)^2}\right] = \left[\frac{10(2)(s+2)}{(s+1+j1)^2} - \frac{10(2)(s+2)^2}{(s+1+j1)^3}\right]_{s=0}$$

$$= -j5 = 5\underline{/-90^\circ}$$

$$f(t) = [10te^{-t}\cos(t-90^\circ) + 10e^{-t}\cos(t-90^\circ)]u(t)$$

[c]

$$F(s) = \begin{array}{r|l} & \qquad\qquad\qquad\quad 25 \\ \cline{2-2} s^2+15s+54 & \quad 25s^2+395s+1494 \\ \cline{1-1} & \quad 25s^2+375s+1350 \\ \cline{2-2} & \qquad\qquad\quad 20s+144 \end{array}$$

$$F(s) = 25 + \frac{20s+144}{s^2+15s+54} = 25 + \frac{K_1}{s+6} + \frac{K_2}{s+9}$$

$$K_1 = \frac{20s+144}{s+9}\bigg|_{s=-6} = 8$$

$$K_2 = \frac{20s+144}{s+6}\bigg|_{s=-9} = 12$$

$$f(t) = 25\delta(t) + [8e^{-6t} + 12e^{-9t}]u(t)$$

[d]

$$\begin{array}{rr}
 & 5s-15 \\
F(s) = \underline{s^2+7s+10} & \overline{\big)\, 5s^3+20s^2-49s-108} \\
 & \underline{5s^3+35s^2+50s} \\
 & -15s^2-99s-108 \\
 & \underline{-15s^2-105s-150} \\
 & 6s+42
\end{array}$$

$$F(s) = 5s - 15 + \frac{K_1}{s+2} + \frac{K_2}{s+5}$$

$$K_1 = \frac{6s+42}{s+5}\bigg|_{s=-2} = 10$$

$$K_2 = \frac{6s+42}{s+2}\bigg|_{s=-5} = -4$$

$$f(t) = 5\delta'(t) - 15\delta(t) + [10e^{-2t} - 4e^{-5t}]u(t)$$

P 12.44 $f(t) = \mathcal{L}^{-1}\left\{\frac{K}{s+\alpha-j\beta} + \frac{K^*}{s+\alpha+j\beta}\right\}$

$$= Ke^{-\alpha t}e^{j\beta t} + K^*e^{-\alpha t}e^{-j\beta t}$$

$$= |K|e^{-\alpha t}[e^{j\theta}e^{j\beta t} + e^{-j\theta}e^{-j\beta t}]$$

$$= |K|e^{-\alpha t}[e^{j(\beta t+\theta)} + e^{-j(\beta t+\theta)}]$$

$$= 2|K|e^{-\alpha t}\cos(\beta t+\theta)$$

P 12.45 **[a]** $\mathcal{L}\{t^n f(t)\} = (-1)^n\left[\frac{d^nF(s)}{ds^n}\right]$

Let $f(t) = 1$, then $F(s) = \frac{1}{s}$, thus $\frac{d^nF(s)}{ds^n} = \frac{(-1)^n n!}{s^{(n+1)}}$

Therefore $\mathcal{L}\{t^n\} = (-1)^n\left[\frac{(-1)^n n!}{s^{(n+1)}}\right] = \frac{n!}{s^{(n+1)}}$

It follows that $\mathcal{L}\{t^{(r-1)}\} = \frac{(r-1)!}{s^r}$

and $\mathcal{L}\{t^{(r-1)}e^{-at}\} = \frac{(r-1)!}{(s+a)^r}$

Therefore $\frac{K}{(r-1)!}\mathcal{L}\{t^{r-1}e^{-at}\} = \frac{K}{(s+a)^r} = \mathcal{L}\left\{\frac{Kt^{r-1}e^{-at}}{(r-1)!}\right\}$

[b] $f(t) = \mathcal{L}^{-1}\left\{\frac{K}{(s+\alpha-j\beta)^r} + \frac{K^*}{(s+\alpha+j\beta)^r}\right\}$

Therefore

$$\begin{aligned} f(t) &= \frac{Kt^{r-1}}{(r-1)!}e^{-(\alpha-j\beta)t} + \frac{K^*t^{r-1}}{(r-1)!}e^{-(\alpha+j\beta)t} \\ &= \frac{|K|t^{r-1}e^{-\alpha t}}{(r-1)!}\left[e^{j\theta}e^{j\beta t} + e^{-j\theta}e^{-j\beta t}\right] \\ &= \left[\frac{2|K|t^{r-1}e^{-\alpha t}}{(r-1)!}\right]\cos(\beta t+\theta) \end{aligned}$$

P 12.46 **[a]** $\lim_{s\to\infty} sV(s) = \lim_{s\to\infty}\left[\frac{1.92s^3}{s^4[1+(1.6/s)+(1/s^2)][1+(1/s^2)]}\right] = 0$

Therefore $v(0^+) = 0$

[b] No, V has a pair of poles on the imaginary axis.

P 12.47 $sV_o(s) = \frac{(I_{\rm dc}/C)s}{s^2+(1/RC)s+(1/LC)}$

$\lim_{s\to 0} sV_o(s) = 0, \qquad \therefore \quad v_o(\infty) = 0$

$\lim_{s\to\infty} sV_o(s) = 0, \qquad \therefore \quad v_o(0^+) = 0$

$sI_o(s) = \frac{s^2 I_{\rm dc}}{s^2+(1/RC)s+(1/LC)}$

$\lim_{s\to 0} sI_o(s) = 0, \qquad \therefore \quad i_o(\infty) = 0$

$\lim_{s\to\infty} sI_o(s) = I_{\rm dc}, \qquad \therefore \quad i_o(0^+) = I_{\rm dc}$

P 12.48 $sI_o(s) = \frac{I_{\rm dc}s[s+(1/RC)]}{s^2+(1/RC)s+(1/LC)}$

$\lim_{s\to 0} sI_o(s) = 0, \qquad \therefore \quad i_o(\infty) = 0$

$\lim_{s\to\infty} sI_o(s) = I_{\rm dc}, \qquad \therefore \quad i_o(0^+) = I_{\rm dc}$

P 12.49 $$sV_o(s) = \frac{sV_{\rm dc}/RC}{s^2 + (1/RC)s + (1/LC)}$$

$$\lim_{s\to 0} sV_o(s) = 0, \qquad \therefore \quad v_o(\infty) = 0$$

$$\lim_{s\to\infty} sV_o(s) = 0, \qquad \therefore \quad v_o(0^+) = 0$$

$$sI_o(s) = \frac{V_{\rm dc}/RC)}{s^2 + (1/RC)s + (1/LC)}$$

$$\lim_{s\to 0} sI_o(s) = \frac{V_{\rm dc}/RLC}{1/LC} = \frac{V_{\rm dc}}{R}, \qquad \therefore \quad i_o(\infty) = \frac{V_{\rm dc}}{R}$$

$$\lim_{s\to\infty} sI_o(s) = 0, \qquad \therefore \quad i_o(0^+) = 0$$

P 12.50 **[a]** $$sF(s) = \frac{6s^2 + 60s}{(s+5)(s+8)}$$

$$\lim_{s\to 0} sF(s) = 0, \qquad \therefore \quad f(\infty) = 0$$

$$\lim_{s\to\infty} sF(s) = 6, \qquad \therefore \quad f(0^+) = 6$$

[b] $$sF(s) = \frac{20s^2 + 141s + 315}{(s^2 + 10s + 21)}$$

$$\lim_{s\to 0} sF(s) = 15; \qquad \therefore \quad f(\infty) = 15$$

$$\lim_{s\to\infty} sF(s) = 20, \qquad \therefore \quad f(0^+) = 20$$

[c] $$sF(s) = \frac{15s^3 + 112s^2 + 228s}{(s+2)(s+4)(s+6)}$$

$$\lim_{s\to 0} sF(s) = 0, \qquad \therefore \quad f(\infty) = 0$$

$$\lim_{s\to\infty} sF(s) = 15, \qquad \therefore \quad f(0^+) = 15$$

[d] $$sF(s) = \frac{2s^3 + 33s^2 + 93s + 54}{(s+1)(s^2 + 5s + 6)}$$

$$\lim_{s\to 0} sF(s) = 9, \qquad \therefore \quad f(\infty) = 9$$

$$\lim_{s\to\infty} sF(s) = 2, \qquad \therefore \quad f(0^+) = 2$$

P 12.51 **[a]** $sF(s) = \dfrac{280s}{s^2 + 14s + 245}$

$$\lim_{s\to 0} sF(s) = 0, \qquad \therefore \quad f(\infty) = 0$$

$$\lim_{s\to \infty} sF(s) = 0, \qquad \therefore \quad f(0^+) = 0$$

[b] $sF(s) = \dfrac{-s^2 + 52s + 445}{s^2 + 10s + 89}$

$$\lim_{s\to 0} sF(s) = 5, \qquad \therefore \quad f(\infty) = 5$$

$$\lim_{s\to \infty} sF(s) = -1, \qquad \therefore \quad f(0^+) = -1$$

[c] $sF(s) = \dfrac{14s^3 + 56s^2 + 152s}{(s+6)(s^2 + 4s + 20)}$

$$\lim_{s\to 0} sF(s) = 0, \qquad \therefore \quad f(\infty) = 0$$

$$\lim_{s\to \infty} sF(s) = 14, \qquad \therefore \quad f(0^+) = 14$$

[d] $sF(s) = \dfrac{8s(s+1)^2}{(s^2 + 10s + 34)(s^2 + 8s + 20)}$

$$\lim_{s\to 0} sF(s) = 0, \qquad \therefore \quad f(\infty) = 0$$

$$\lim_{s\to \infty} sF(s) = 0, \qquad \therefore \quad f(0^+) = 0$$

P 12.52 **[a]** $sF(s) = \dfrac{320}{s(s+8)}$

$F(s)$ has a second-order pole at the origin so we cannot use the final value theorem here.

$$\lim_{s\to \infty} sF(s) = 0, \qquad \therefore \quad f(0^+) = 0$$

[b] $sF(s) = \dfrac{80(s+3)}{(s+2)^2}$

$$\lim_{s\to 0} sF(s) = 60, \qquad \therefore \quad f(\infty) = 60$$

$$\lim_{s\to \infty} sF(s) = 0, \qquad \therefore \quad f(0^+) = 0$$

[c] $sF(s) = \dfrac{60s(s+5)}{(s+1)^2(s^2 + 6s + 25)}$

$$\lim_{s\to 0} sF(s) = 0, \qquad \therefore \quad f(\infty) = 0$$

$$\lim_{s\to \infty} sF(s) = 0, \qquad \therefore \quad f(0^+) = 0$$

[d] $sF(s) = \dfrac{25(s+4)^2}{s(s+5)^2}$

$F(s)$ has a second-order pole at the origin so we cannot use the final value theorem here.

$\lim_{s\to\infty} sF(s) = 0, \qquad \therefore \quad f(0^+) = 0$

P 12.53 **[a]** $sF(s) = \dfrac{135}{(s+3)^3}$

$\lim_{s\to 0} sF(s) = 5, \qquad \therefore \quad f(\infty) = 5$

$\lim_{s\to\infty} sF(s) = 0, \qquad \therefore \quad f(0^+) = 0$

[b] $sF(s) = \dfrac{10s(s+2)^2}{(s^2+2s+2)^2}$

$\lim_{s\to 0} sF(s) = 0, \qquad \therefore \quad f(\infty) = 0$

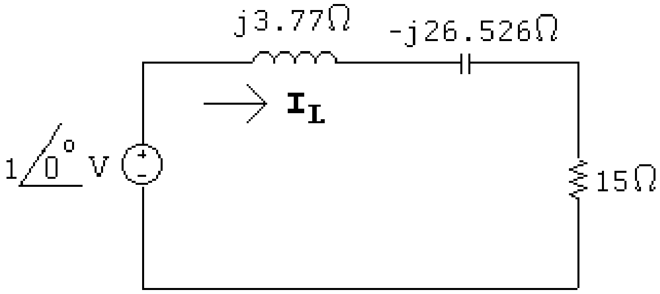
$\lim_{s\to\infty} sF(s) = 0, \qquad \therefore \quad f(0^+) = 0$

[c] This $F(s)$ function is an improper rational function, and thus the corresponding $f(t)$ function contains impulses $(\delta(t))$. Neither the initial value theorem nor the final value theorem may be applied to this $F(s)$ function!

[d] This $F(s)$ function is an improper rational function, and thus the corresponding $f(t)$ function contains impulses $(\delta(t))$. Neither the initial value theorem nor the final value theorem may be applied to this $F(s)$ function!

P 12.54 **[a]** $Z_L = j120\pi(0.01) = j3.77\,\Omega; \qquad Z_C = \dfrac{-j}{120\pi(100\times 10^{-6})} = -j26.526\,\Omega$

The phasor-transformed circuit is



$\mathbf{I}_L = \dfrac{1}{15 + j3.77 - j26.526} = 36.69\underline{/56.61^\circ}\,\text{mA}$

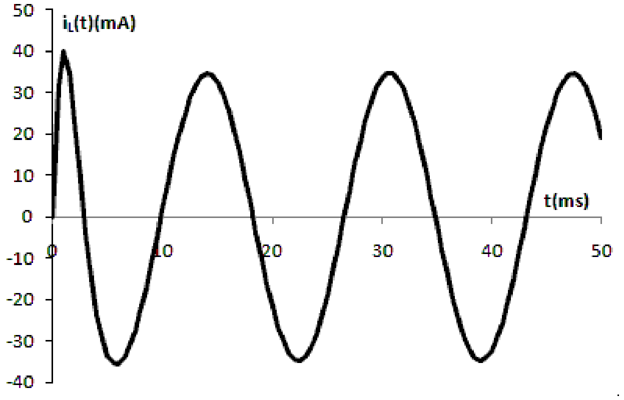
$\therefore \quad i_{L-ss}(t) = 36.69\cos(120\pi t + 56.61^\circ)\,\text{mA}$

[b] The steady-state response is the second term in Eq. 12.109, which matches the steady-state response just derived in part (a).

P 12.55 The transient and steady-state components are both proportional to the magnitude of the input voltage. Therefore,

$$K = \frac{40}{42.26} = 0.947$$

So if we make the amplitude of the sinusoidal source 0.947 instead of 1, the current will not exceed the 40 mA limit. A plot of the current through the inductor is shown below with the amplitude of the sinusoidal source set at 0.947.



P 12.56 We begin by using Eq. 12.105, and changing the right-hand side so it is the Laplace transform of Kte^{-100t}:

$$15I_L(s) + 0.01sI_L(s) + 10^4\frac{I_L(s)}{s} = \frac{A}{(s+100)^2}$$

Solving for $I_L(s)$,

$$I_L(s) = \frac{100Ks}{(s^2+1500s+10^6)(s+100)^2} = \frac{K_1}{s+750-j661.44} + \frac{K_1^*}{s+750+j661.44}$$

$$+\frac{K_2}{(s+100)^2} + \frac{K_3}{s+100}$$

$$K_1 = \frac{100Ks}{(s+750+j661.44)(s+100)^2}\bigg|_{s=-750+j661.44} = 87.9K\underline{/139.59^\circ}\,\mu\text{A}$$

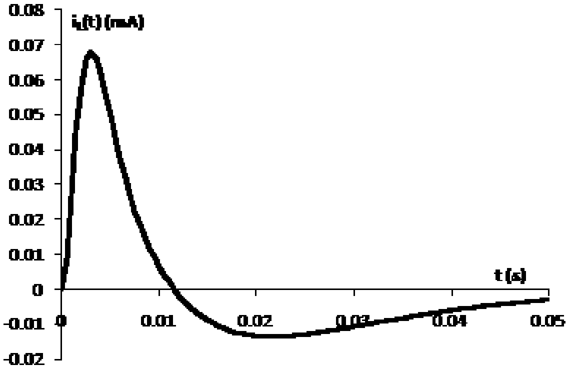
$$K_2 = \frac{100Ks}{(s^2+1500s+10^6)}\bigg|_{s=-100} = -11.63K\text{ mA}$$

$$K_3 = \frac{d}{ds}\left[\frac{100Ks}{(s^2+1500s+10^6)}\right]_{s=-100} = 133.86K\,\mu\text{A}$$

Therefore,

$$i_L(t) = K[0.176e^{-750t}\cos(661.44t + 139.59^\circ) - 11.63te^{-100t} + 0.134e^{-100t}]u(t)\,\text{mA}$$

Plot the expression above with $K = 1$:



The maximum value of the inductor current is $0.068K$ mA. Therefore,

$$K = \frac{40}{0.068} = 588$$

So the inductor current rating will not be exceeded if the input to the RLC circuit is $588te^{-100t}$ V.