Root Locus

Dr. Jama Siam

Root Locus:

The root locus is the geometric loci (plot) of all the closed loop system poles that can be obtained by a variable gain (proportional controller). That is the roots of the characteristic equation (Denominator) of the closed loop transfer function:

$D(s) = 1 + kG(s)H(s) = 0 \quad k \in R$

Observation1: the solution of this equation is obtained by substituting values of k and then computing and plotting these roots in the complex plane.

Observation2: The control engineering procedure opts to obtain an approximated root locus plot based on the geometric equation of the closed loop characteristic equation rewritten in the form that employs the open loop transfer function instead the closed loop form. That is the closed loop characteristic equation written in the form:

$G(s)H(s) = \frac{-1}{k} \quad k \in R$

Geometric interpretation of the closed loop characteristic equation based on the open loop transfer function:

To reach the objectives of the open l this interpretation, the open loop transfer function is expressed in the pole-zero form that is $G(s)H(s) = \frac{\prod_{i=1}^{m}(s+z_i)}{\prod_{i=1}^{n}(s+p_i)} = \frac{-1}{k}$.

Magnitude characteristic:

Let $|s+z_i| = \rho_{z_i}$ and $|s+p_i| = \rho_{p_i}$, then the point of the complex plane $s^* \epsilon L$ (root locus) $\leftrightarrow \exists k\ so\ that$

$|G(s^*)H(s^*)| = \frac{\prod_{i=1}^{m}\rho_{z_i}}{\prod_{i=1}^{n}\rho_{p_i}} = \frac{1}{|k|}$.

However, the existence is verified since $k \in R$ and this condition is transformed to determine the value of $|k|$ at a point $s^* \epsilon L$, that is $|k| = \frac{\prod_{i=1}^{m}\rho_{p_i}}{\prod_{i=1}^{n}\rho_{z_i}}$

Phase Characteristic:

The phase condition is used to determine the points of the complex plane that belong to the locus. That is, a point

Of the plane $s^* \epsilon L \leftrightarrow < G(s^*)H(s^*) = < \frac{-1}{k}$ *that is* $\begin{cases} (2\nu+1)\pi \; for \; k > 0 \\ 2\nu\pi \qquad for \; k < 0 \end{cases}$

Observation: using two plots one to determine the locus for $k > 0$ and the other for $k < 0$ hides the direct role of the parameter k in the search process.

Root Locus: approximation rules:

Observation: The conditions will be set for the case $k > 0$ *and* $n > m$, the changes for the case $k < 0$ are obtained by substituting $(2\nu + 1)\pi$ by $2\nu\pi$. Some related changes will be included in parenthesis when necessary.

Moreover, a simple example based on $G(s)H(s) = \frac{1}{(s+1)(s+3)}$ will be initially used to illustrate the rules.

- The root locus has a number of branches that equals to the system order.
  In our case $n = 2$, therefore the locus has two branches.
- The root locus is always symmetric with respect to the real axe.
- A point $s^* \in L$ if it leaves an odd number of open loop poles and zeros to the right (its left for $k < 0$).
  In our case $s^* \in L \leftrightarrow s^* \in]-3, -1[$.
- The root locus departs branches from poles at finite (for $n \geq m$) or finite and infinite (for $n < m$) and terminate in zeros at finite and infinite (for $n > m$) or at finite (for $n \leq m$).
  In our case the branches depart from $p = -1$ and $p = -3$ and terminates at infinite zeros.
- The branches terminate at infinite zeros or depart from infinite pole according to a uniform star of rays with:
  Centroid $\sigma = \frac{\sum_{i=1}^{n} Re(p_i) - \sum_{i=1}^{m} Re(z_i)}{n-m}$ and phases $\vartheta_\nu = \frac{(2\nu+1)\pi}{n-m} \nu = 0,1, \dots n-(m-1)$. (note: $\vartheta_\nu = \frac{2\nu\pi}{n-m}$ *for* $k < 0$)
  In our case $\sigma = \frac{-1-3}{2-0} = -2$ phases $\vartheta_0 = \frac{(2 \cdot 0+1)\pi}{2-0} = \frac{\pi}{2}$ and $\vartheta_1 = \frac{(2 \cdot 1+1)\pi}{2-0} = \frac{3\pi}{2}$ or $\frac{-\pi}{2}$ by symmetry.

Observation:
The centroid is not a point of the root locus since it is a virtual point that represents the intersection of the virtual asymptotes lines that set the direction of convergence to infinity.

- Angles of departure from finite poles:

The root locus departs from the pole P_j with angle that satisfies:

$$-\boldsymbol{\vartheta}_{\boldsymbol{p}_j} - \sum_{\substack{i=1 \\ i \neq j}}^{n} \boldsymbol{\vartheta}_{\boldsymbol{p}_{j,i}} + \sum_{i=1}^{m} \boldsymbol{\vartheta}_{\boldsymbol{z}_{j,i}} = (\mathbf{2} \cdot \boldsymbol{\nu} + \mathbf{1})\boldsymbol{\pi}. \quad (for\ k < 0\ becomes\ 2\nu\pi)$$

Note: the angle is measured as $\boldsymbol{\vartheta}_{\boldsymbol{p}_{j,i}} = tan^{-1}(\frac{im(P_j - P_i)}{Re(P_j - P_i)})$ with the tail of the vector at P_i and the head of the vector arrow at P_j.

In our case, the angle of departure from $p = -1$: $-\vartheta_{p_{=-1}} - 0 + (0)_{no_{zeros}} = (2 \cdot \nu + 1)\pi$, for first cycle with $\nu = 0$ we get $\vartheta_{p_{=-1}} = -\pi$ which is coherent with the root locus real axe region and the symmetry condition.

The angle of departure from $p = -3$: $-\vartheta_{p_{=-3}} - \pi + (0)_{no_{zeros}} = (2 \cdot \nu + 1)\pi$, for first cycle with $\nu = 0$ we get $\vartheta_{p_{=-3}} = -2\pi$ which is coherent with the root locus real axe region and the symmetry condition.

Angles of arrival finite zeros :
The root locus arrives to the zero Z_j with angle that satisfies:

$$\boldsymbol{\vartheta}_{\boldsymbol{z}_j} + \sum_{\substack{i=1 \\ i \neq j}}^{m} \boldsymbol{\vartheta}_{\boldsymbol{z}_{j,i}} - \sum_{i=1}^{n} \boldsymbol{\vartheta}_{\boldsymbol{z}_{j,i}} = (\mathbf{2} \cdot \boldsymbol{\nu} + \mathbf{1})\boldsymbol{\pi}. \quad (for\ k < 0\ becomes\ 2\nu\pi)$$

In our case there is no finite zeros and the branches terminate asymptotically in the infinite zeros as seen previously.

Repeated roots:

- If r branches meet in r-points, this point is a system pole with multiplicity r (repeated r-times). The input and output tangent lines at this point form a uniform star with centroid at this point and angles that divide the plane into 2r sectors. Each sector ha angle $\frac{\pi}{r}$.
- The repeated pole can be determined by solving the set of equations:

$$\begin{cases} D\left(s_{rep}\right) = 1 + kG\left(s_{rep}\right)H\left(s_{rep}\right) = 0 \\ \dfrac{dD(s_{rep})}{ds} = \dfrac{dG\left(s_{rep}\right)H\left(s_{rep}\right)}{ds} = 0 \\ \dots \\ \dfrac{d^{r-1}(G\left(s_{rep}\right)H\left(s_{rep}\right))}{ds^{r-1}} = 0 \end{cases}$$

Or $\sum_{i=1}^{n} \frac{1}{(s+p_i)} - \sum_{i=1}^{m} \frac{1}{(s+z_i)} = 0$

In our case using the second equation we get $\frac{1}{(s+1)} + \frac{1}{(s+3)} = 0 \leftrightarrow s + 3 + s + 1 = 2s + 4 = 0 \rightarrow s = -2$.

Therefore, the two branches meet at $s_{rep} = -2$ and divide the plane into 4 sectors with angles $\frac{\pi}{2}$.

Intersection with the imaginary axe:

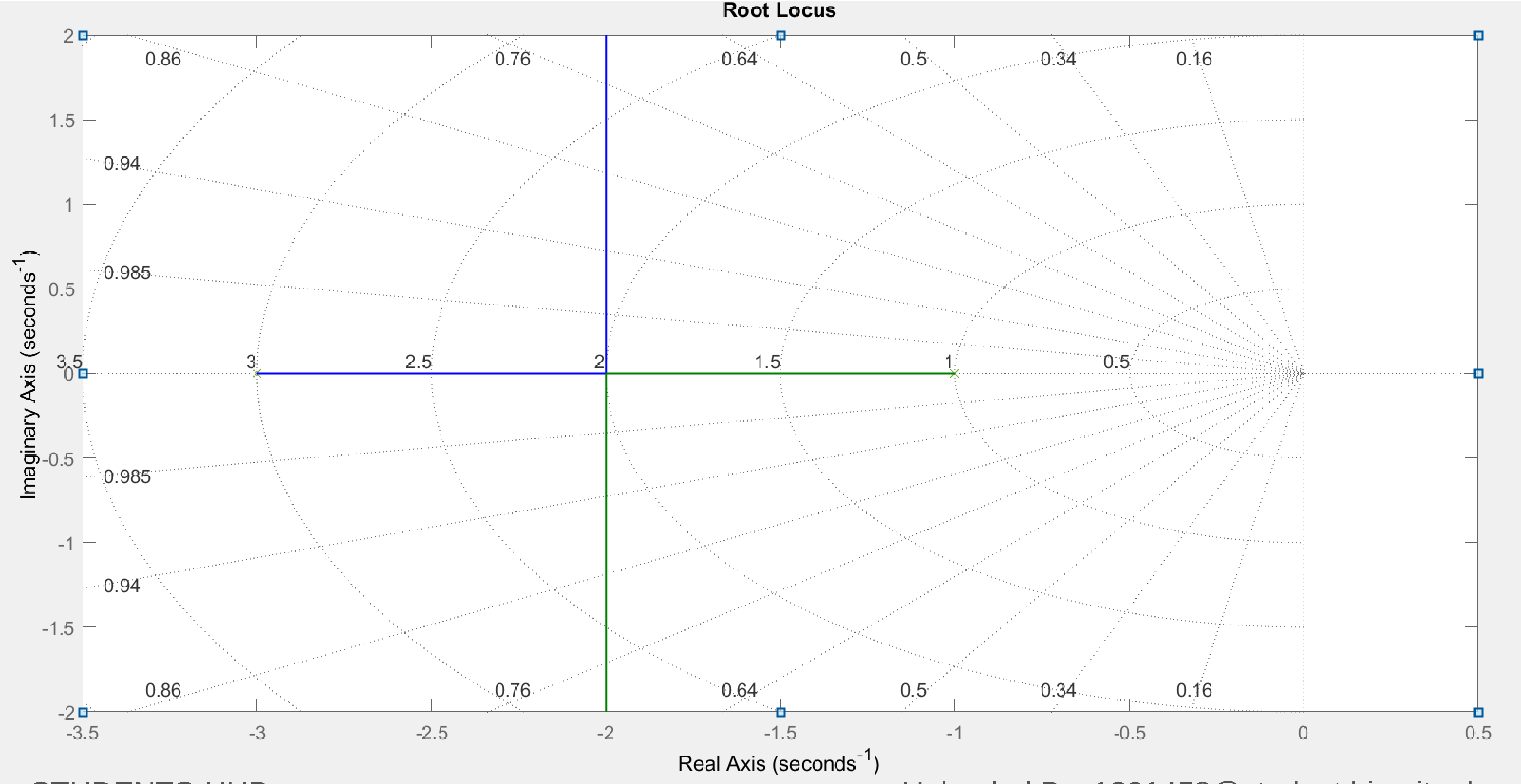
To find the intersection with the imaginary axe we apply the Routh-Hurwitz criteria to the characteristic equation of the closed loop system. That is to $D(s) = 1 + kG(s)H(s) = 0$. The crossing point existence is obtained at the $k - values$ that create the rows of zeros.

In our case $D(s) = 1 + k\frac{1}{(s+1)(s+3)} = 0 \rightarrow s^2 + 4s + 3 + k = 0$

Row number		
2	1	$3+k$
1	4	0
0	$3+k$	

For rows of zero: $3+k=0$ which is obtained for $k=-3$. This value is not acceptable because we are plotting the root locus for $k>0. Thus\ there\ is\ no\ crossing\ to\ the\ imaginary\ axe$.

The plot of the root locus : next slide


Root Locus
Imaginary Axis (seconds^-1)
Real Axis (seconds^-1)
0.86
0.76
0.64
0.5
0.34
0.16
0.94
0.985
3.5
3
2.5
2
1.5
1
0.5
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-3.5
-3
-2.5
-2
-1.5
-1
-0.5
0
0.5

Example 2:

$$G(s)H(s) = \frac{(s+5)}{(s+1)(s+3)}$$

- The root locus has 2 branches.
- The root locus is symmetric with respect to the real axe.
- The points of the real axe that belong to the locus are those who leave an odd number of open-loop poles and zeros to their right. $s \in]-\infty, -5[\ \cup \ ]-3, -1[$
- There is one infinite zero because $n - m = 1$, therefore we have one asymptote with centroid $\sigma = \frac{-3-1-(-5)}{2-1} = 1$ and angle $\vartheta_0 = \frac{(2\cdot 0+1)\pi}{2-1} = \pi$
- Angles of departure to finite poles

  Angle od departure for $P = -1$: $-\boldsymbol{\vartheta_{p=-1}} - (\mathbf{0})_{\boldsymbol{from\ pole\ at\ -3}} + (\mathbf{0})_{\boldsymbol{from\ zero\ at\ z=-5}} = (\mathbf{2}\cdot\boldsymbol{\nu}+\mathbf{1})\boldsymbol{\pi}$

  for $\boldsymbol{\nu} = \mathbf{0}$: $\boldsymbol{\vartheta_{p=-1}} = -\boldsymbol{\pi}$.

  Angle od departure for $P = -3$: $-\boldsymbol{\vartheta_{p=-3}} - (\boldsymbol{\pi}\ )_{\boldsymbol{from\ pole\ at\ -3}} + (\mathbf{0})_{\boldsymbol{from\ zero\ at\ z=-5}} = (\mathbf{2}\cdot\boldsymbol{\nu}+\mathbf{1})\boldsymbol{\pi}$

  for $\boldsymbol{\nu} = \mathbf{0}$: $\boldsymbol{\vartheta_{p=-3}} = -\mathbf{2}\boldsymbol{\pi}$

  Angle of arrival to the finite zero at $Z = -5$: $\boldsymbol{\vartheta_{z=-5}} - (\boldsymbol{\pi})_{\boldsymbol{from\ pole\ at\ -3}} - (\boldsymbol{\pi})_{\boldsymbol{from\ pole\ at\ -1}} = (\mathbf{2}\cdot\boldsymbol{\nu}+\mathbf{1})\boldsymbol{\pi}$

  for $\boldsymbol{\nu} = \mathbf{0}$: $\boldsymbol{\vartheta_{z=-5}} = \mathbf{3}\boldsymbol{\pi}$.

Repeated poles:

$$\frac{1}{(s+1)} + \frac{1}{(s+3)} - \frac{1}{s+5} = 0 \rightarrow (s+3)(s+5) + (s+1)(s+5) - (s+1)(s+3) = 0$$

$$s^2 + 8s + 15 + s^2 + 6s + 5 - (s^2 + 4s + 3) = 0 \rightarrow s^2 + 10s + 17 = 0\ with\ solutions\ s_1 = -2.17\ and\ s_2 = -7.83$$

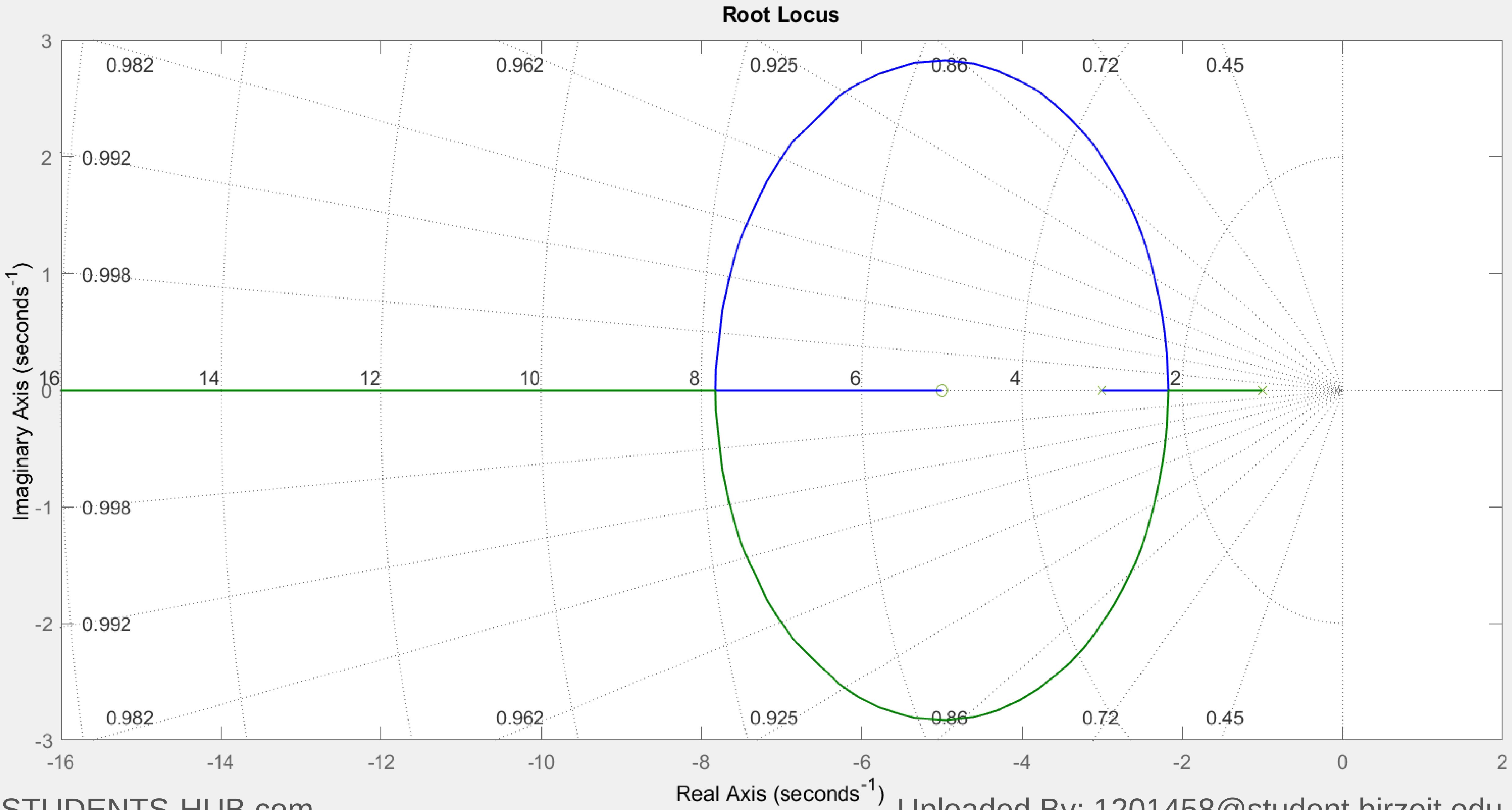
Observation: Two repeated poles imply that the two branches departing from the open loop poles meet at a repeated pole the leave then leave the real axis with direction $\frac{\pi}{2}$ $and \frac{-\pi}{2}$ dividing the plane into four sectors of angles $\frac{\pi}{2}$. The branches then return to meet at the second repeated pole at $s_2 = -7.83$ to terminate at the finite and infinite zeros according to the computed directions.

Intersection with the imaginary axe:

$$\boldsymbol{D(s) = 1 + k\frac{(s+5)}{(s+1)(s+3)} = 0 \rightarrow s^2 + (4+k)s + 3 + 5k = 0}$$

Row number		
2	1	$3+5k$
1	$4+k$	0
0	$3+5k$	

For rows of zero: $4 + k = 0$ which is obtained for $k = -4$ and $3 + 5k = 0$ which is obtained for $k = -\frac{3}{5}$. These values are not acceptable because we are plotting the root locus for $k > 0. Thus\ there\ is\ no\ crossing\ to\ the\ imaginary\ axe$.



Root Locus
Imaginary Axis (seconds⁻¹)
Real Axis (seconds⁻¹)
0.982
0.962
0.925
0.86
0.72
0.45
0.992
0.998
16
14
12
10
8
6
4
2

Example 3:

$$G(s)H(s) = \frac{1}{(s+1)(s+3)(s+5)}$$

- The root locus has 3 branches.
- The root locus is symmetric with respect to the real axe.
- The points of the real axe that belong to the locus are those who leave an odd number of open-loop poles and zeros to their right. $s \in]-\infty, -5[\ \cup\ ]-3, -1[$
- There are three infinite zeros because $n - m = 3$, therefore we have three asymptotes with centroid $\sigma = \frac{-3-1-5}{3-0} = -3$ and angle $\vartheta_0 = \frac{(2\cdot 0+1)\pi}{3-0} = \frac{\pi}{3}$, by symmetry $\vartheta_{-1} = \frac{(2\cdot -1+1)\pi}{3-0} = -\frac{\pi}{3}$, $\vartheta_1 = \frac{(2\cdot 1+1)\pi}{3-0} = \pi$

  of departure to finite poles

  Angle of departure for $P = -1$: $-\boldsymbol{\vartheta}_{p=-1} - (\mathbf{0})_{from\ pole\ at\ -3} - (\mathbf{0})_{from\ pole\ at\ p=-5} = (\mathbf{2}\cdot\boldsymbol{\nu}+\mathbf{1})\boldsymbol{\pi}$

  for $\boldsymbol{\nu} = \mathbf{0}$: $\boldsymbol{\vartheta}_{p=-1} = -\boldsymbol{\pi}$.

  Angle of departure for $P = -3$: $-\boldsymbol{\vartheta}_{p=-3} - (\boldsymbol{\pi})_{from\ pole\ at\ -1} - (\mathbf{0})_{from\ pole\ at\ p=-5} = (\mathbf{2}\cdot\boldsymbol{\nu}+\mathbf{1})\boldsymbol{\pi}$

  for $\boldsymbol{\nu} = \mathbf{0}$: $\boldsymbol{\vartheta}_{p=-3} = -\mathbf{2}\boldsymbol{\pi}$

  Angle of departure for $P = -5$: $-\boldsymbol{\vartheta}_{p=-5} - (\boldsymbol{\pi})_{from\ pole\ at\ -1} - (\boldsymbol{\pi})_{from\ pole\ at\ p=-5} = (\mathbf{2}\cdot\boldsymbol{\nu}+\mathbf{1})\boldsymbol{\pi}$

  for $\boldsymbol{\nu} = \mathbf{0}$: $\boldsymbol{\vartheta}_{p=-5} = -\mathbf{3}\boldsymbol{\pi}$

Repeated poles:

$$\frac{1}{(s+1)} + \frac{1}{(s+3)} + \frac{1}{s+5} = 0 \rightarrow (s+3)(s+5) + (s+1)(s+5) + (s+1)(s+3) = 0$$

$$s^2 + 8s + 15 + s^2 + 6s + 5 + (s^2 + 4s + 3) = 0 \rightarrow 3s^2 + 18s + 23 = 0\ with\ solutions\ s_1 = -1.85\ and\ s_2 = -4.15$$

Observation: We obtained two solutions but only s_1 belongs to the real axe set of points of the locus. Thus, the two branches departing from the open loop poles at $p = -1 \ and \ p = -3$ meet at s_1 and then leave the real axis with direction $\frac{\pi}{2} \ and \frac{-\pi}{2}$ dividing the plane into four sectors of angles $\frac{\pi}{2}$. The branches then go to cross the imaginary axe and terminate at the infinite zeros according to the computed asymptotic directions $\frac{\pi}{3}, \frac{-\pi}{3}, -\pi$.

Intersection with the imaginary axe:

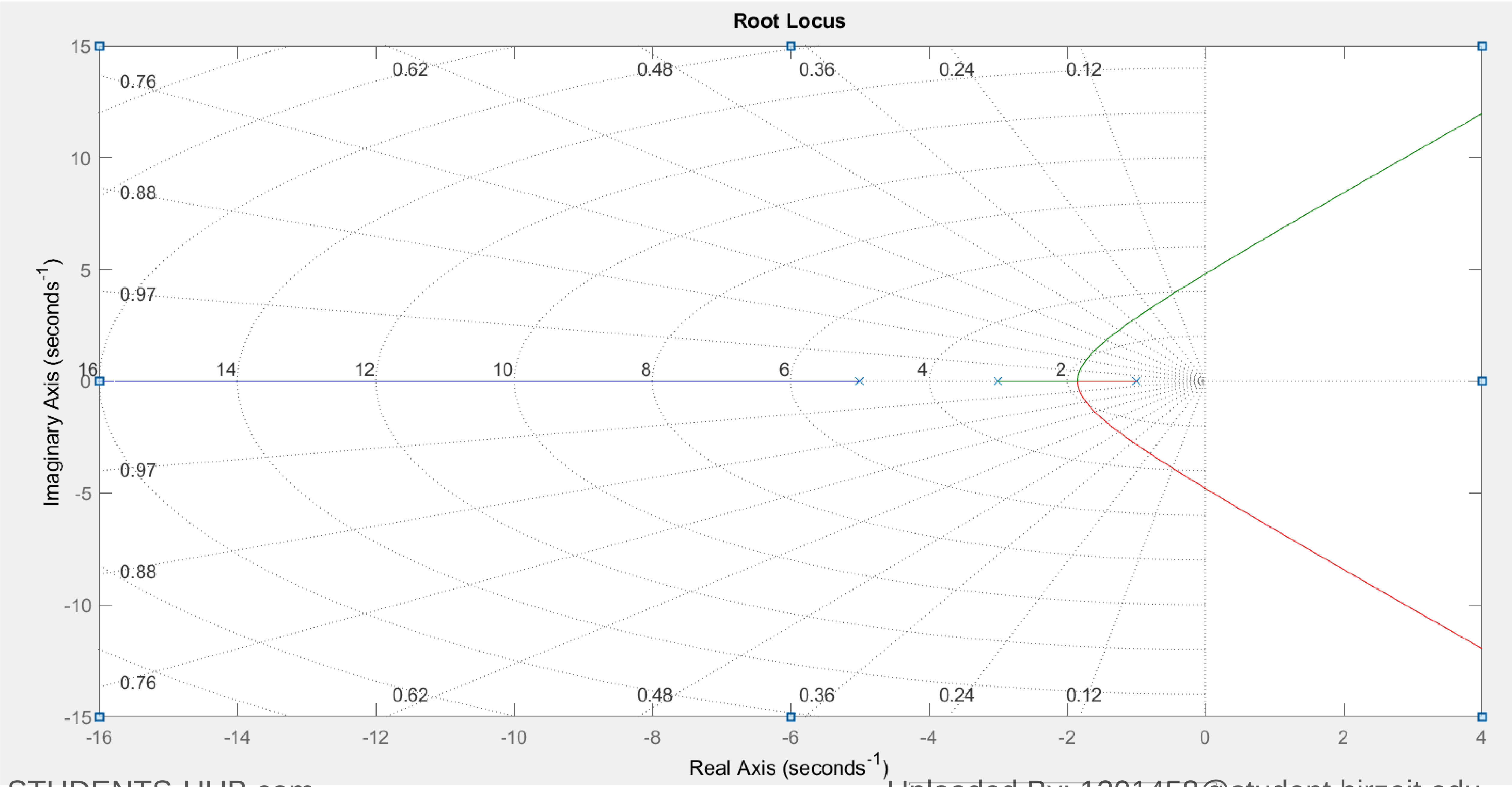
$$\boldsymbol{D(s) = 1 + k\frac{1}{(s+1)(s+3)(s+5)} = 0 \rightarrow s^3 + 9s^2 + 23s + 15 + k = 0}$$

Row number		
3	1	23
2	9	$15 + k$
1	$192 - k$	0
0	$15 + k$	0

For rows of zero: $192 - k = 0$ which is obtained for $k = -192$ and $15 + k = 0$ which is obtained for $k = -15$. The values are not acceptable because we are plotting the root locus for $> 0. Thus\ there\ is\ crossings\ to\ the\ imaginary\ axe\ for$ $k = 192.$

The auxiliary characteristic equation is: for $9s^2 + 207 = 0$ with imaginary axe crossings at $\pm j4.8$

Exercise: Determine the points of crossings of the asymptotes with the imaginary axe.



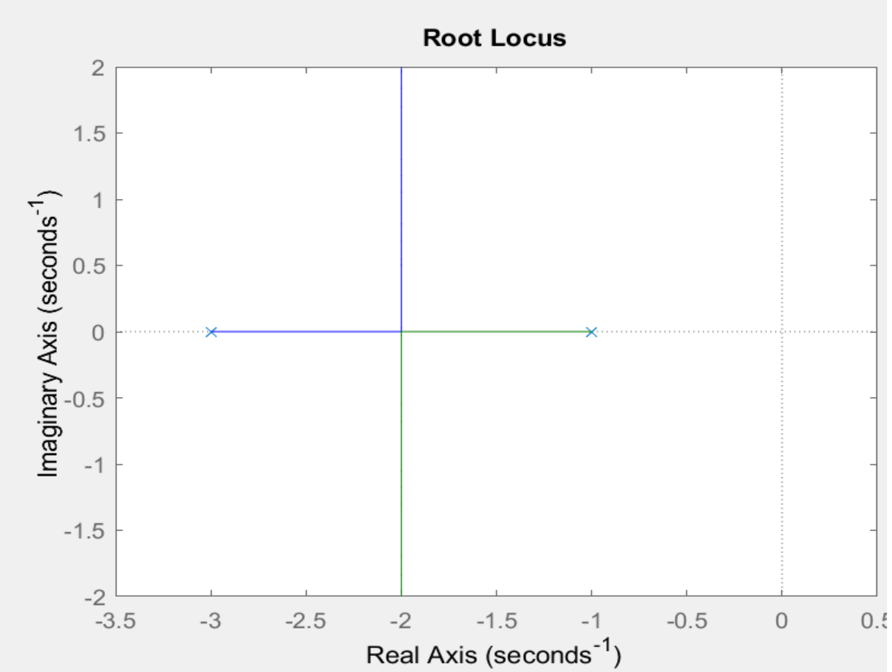
Root Locus
Imaginary Axis (seconds^{-1})
Real Axis (seconds^{-1})
15
10
5
0
-5
-10
-15
-16
-14
-12
-10
-8
-6
-4
-2
0
2
4
0.76
0.62
0.48
0.36
0.24
0.12
0.88
0.97
16
14
12
10
8
6
4
2

Design using Root Locus

Dr. Jamal Siam

Controller Design using Root Locus:

Controller design using root locus is based on introducing poles and/ or zeros to reshape the root locus and make it pass through the desired closed loop poles that achieve or almost a chieve the desired response within acceptable response tolerances.



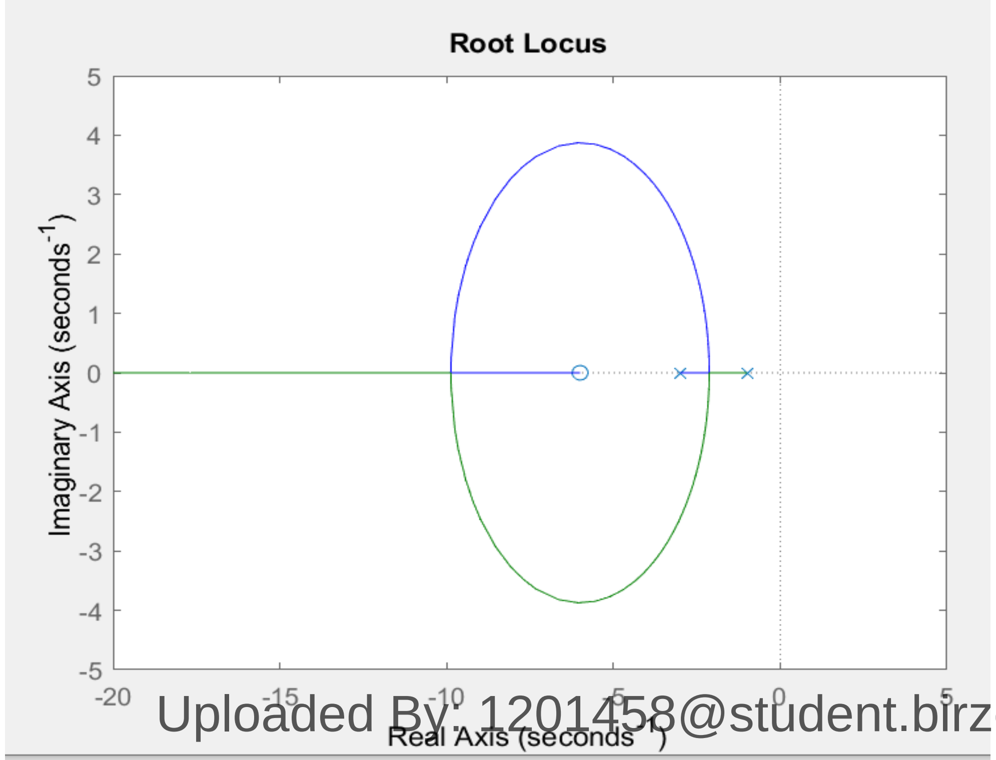
Example:

A plant+ actuator has the open-loop transfer function $G(s)H(s) = \frac{1}{(s+1)(s+3)}$ and the following closed-loop root locus with proportional positive gain control (Non-inverting P-controller).

Reshaping the root-locus:

However, this system can not achieve certain underdamped response behavior. For example, it can not achieve an underdamped closed-loop response with settling time at 2% smaller than 2 second.

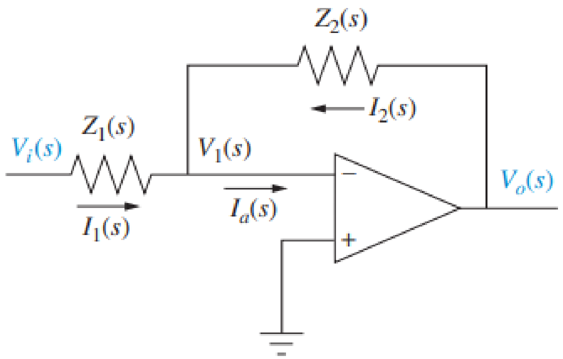
The root locus can be reshaped to achieve this possibility by introducing an open loop zero to the left of p=-3. Adding a zero at z=-6 we obtain the following root locus which obviously do the required condition.



Compensators Types

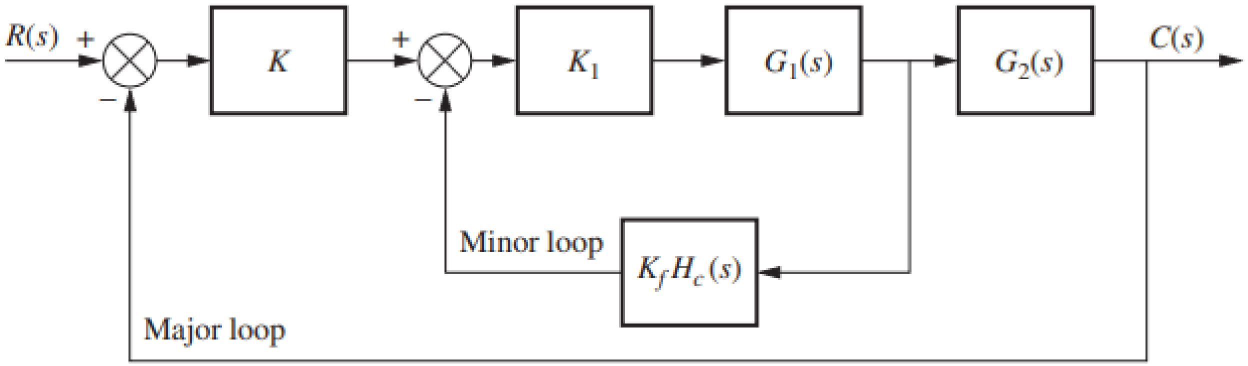
Active-Compensators: can impose certain desired response conditions. Active compensators can be implemented using any type of active elements (amplifier based circuits). The most used employs operation amplifiers as for example the dynamic gain inverting operational-amplifier.

Passive-Compensators: can adjust the response to the nearest possible desired response conditions. Depends on the general passive circuit synthesis methods.



compensation topologies:

- Cascade compensation: the compensator is inserted in the direct path of the systems is intended to affect the global behavior of the system.
- Feedback compensation: The compensator is usually inserted in feedback with a subset of system elements to improve the behavior of the specific subset of elements. It can also operate on the major loop (feedback with the system elements). However, the most general configuration is to use a feedback compensation in the minor loops coupled with a cascade compensator to set the global behavior of the feedback compensated system.

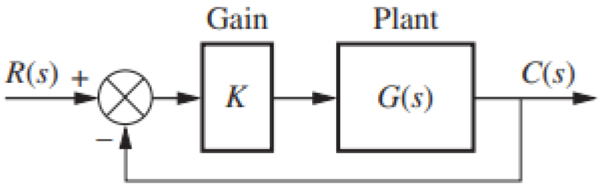


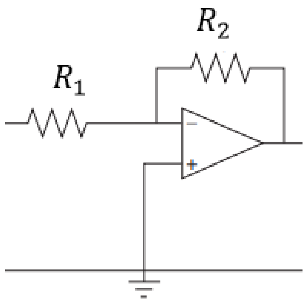
Controllers:

Proportional-Controller (P-controller) or gain controller:

- Operates on the original positive gain or negative gain locus of the system.
- The objective is to set the system closed loop poles on a selected set on the original system root-locus by moving on the root locus points through the change of the gain value.
- The transfer function is $G_p(s) = k_{comp} \rightarrow |k_{comp}| = \frac{R_2}{R1}$.
- Block diagram
- The realization assumes that the value of the static gain

  achieves the gain at the desired pole gain $|K_{tot}| = |k_{comp}| \cdot |\frac{\prod_{i=1}^{m} z_i}{\prod_{i=1}^{n} p_i}|$

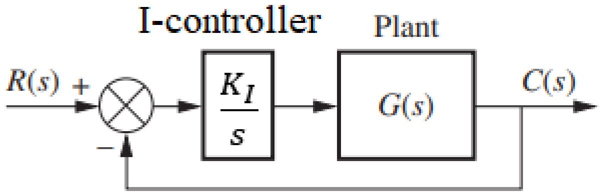


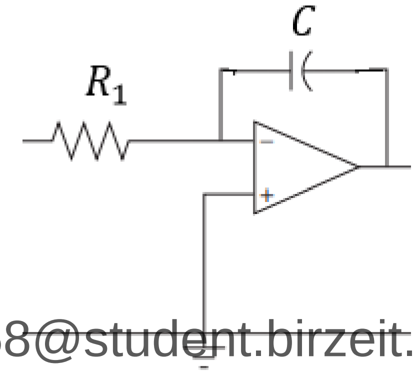


Integrative-Controller(I-controller)

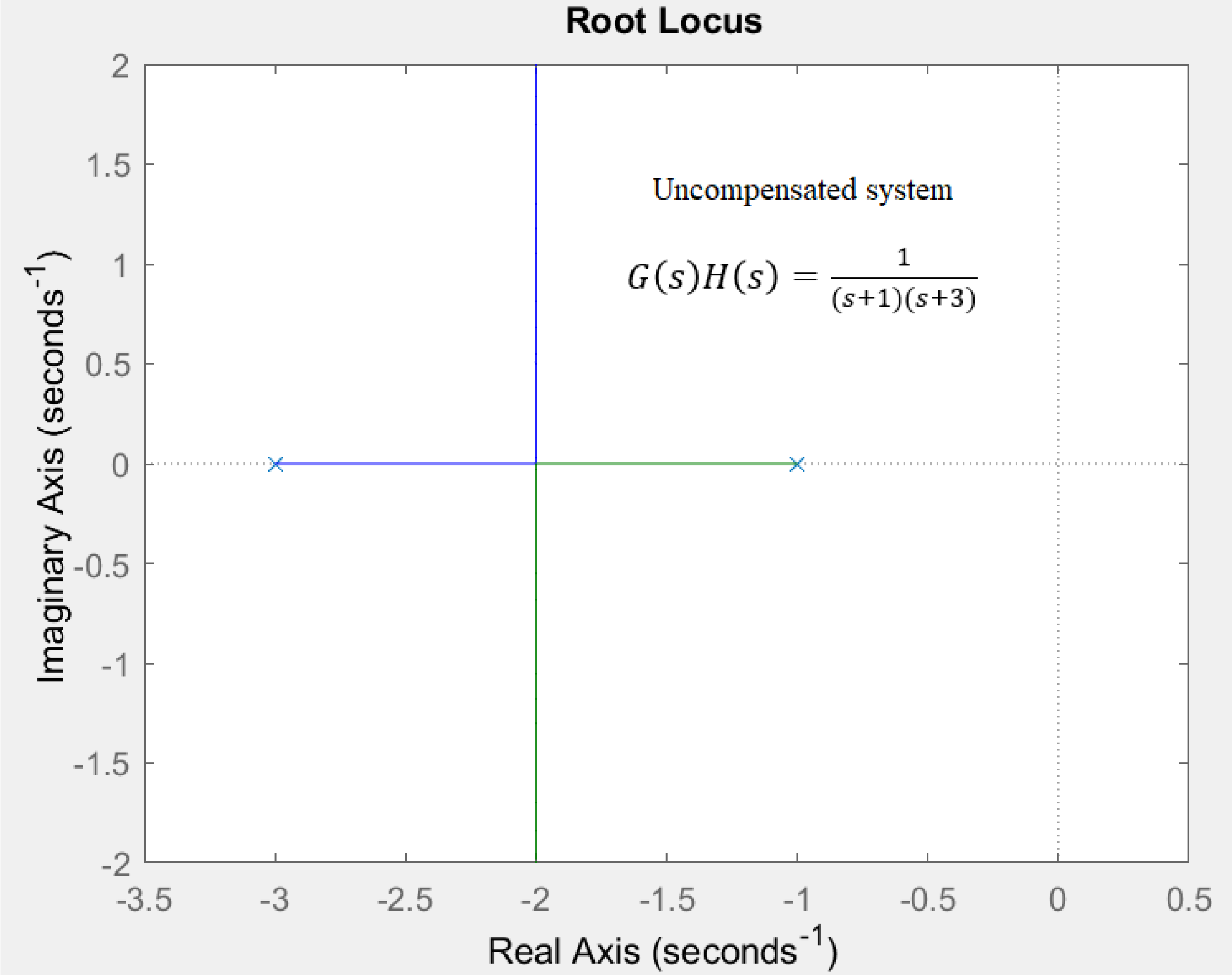
- Based on the insertion of a pole in the origin with a gain amplifier, reshapes the root locus in a single unique mode.
- The objective is setting the steady-state error to zero by increasing the system type.
- The transfer function is $G_I(s) = \frac{K_I}{s} \rightarrow |K_I| = |\frac{1}{R_1 C s}|$
- Disadvantage: the transient performance is not preserved and can change in uncontrollable mode.
- Example: Uncompensated system: $G(s) = \frac{1}{(s+1)(s+3)}$

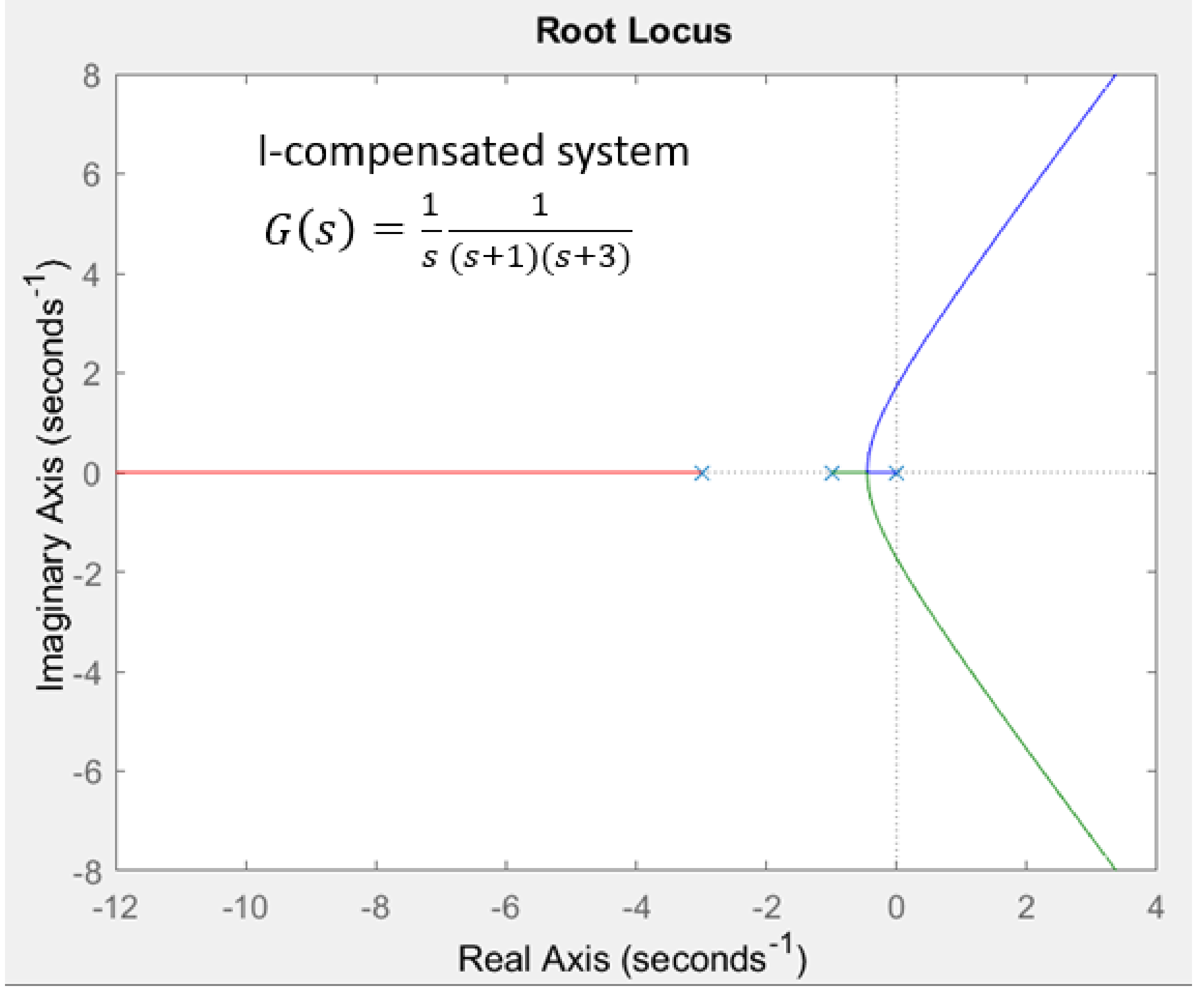
  I-compensated system: $G(s) = \frac{1}{s}\frac{1}{(s+1)(s+3)}$





- *Root locus: original and I-compensated(Next slide)*

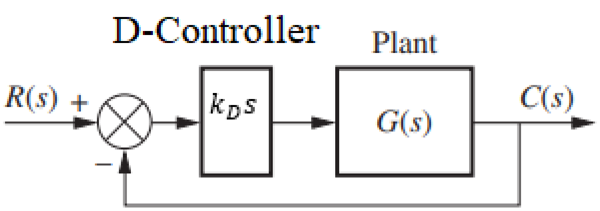

Root Locus
Uncompensated system
$G(s)H(s) = \frac{1}{(s+1)(s+3)}$
Imaginary Axis (seconds^{-1})
Real Axis (seconds^{-1})
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-3.5
-3
-2.5
-2
-1.5
-1
-0.5
0
0.5

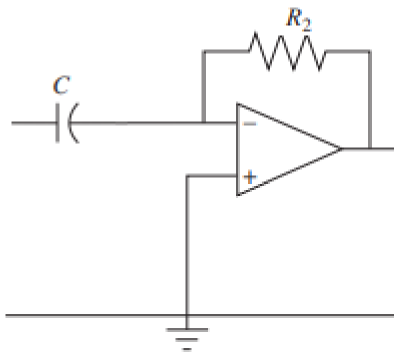

Root Locus
I-compensated system
$G(s) = \frac{1}{s}\frac{1}{(s+1)(s+3)}$
Imaginary Axis (seconds^{-1})
Real Axis (seconds^{-1})
8
6
4
2
0
-2
-4
-6
-8
-12
-10
-8
-6
-4
-2
0
2
4

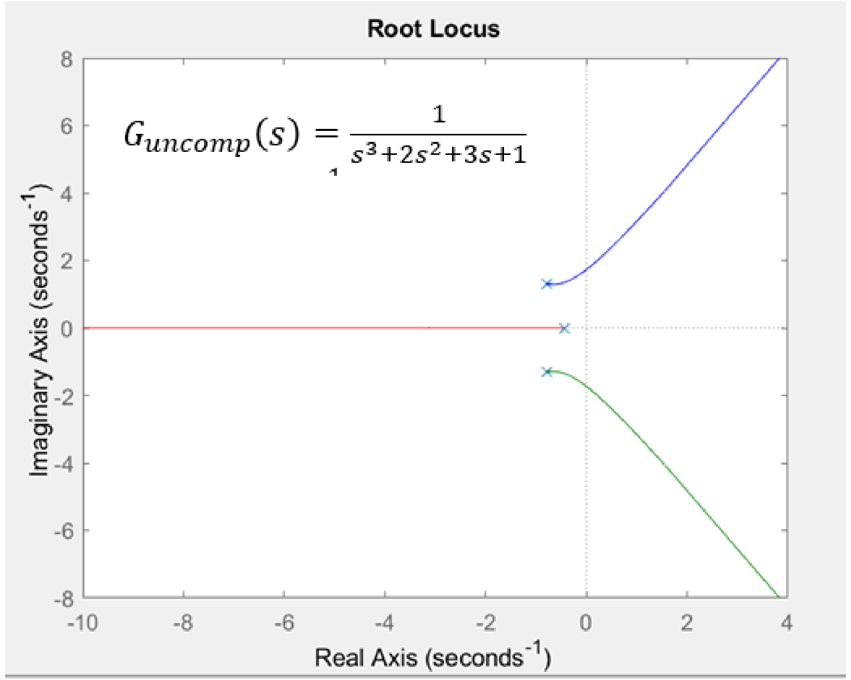
<u>Derivative Controller (D-controller):</u>

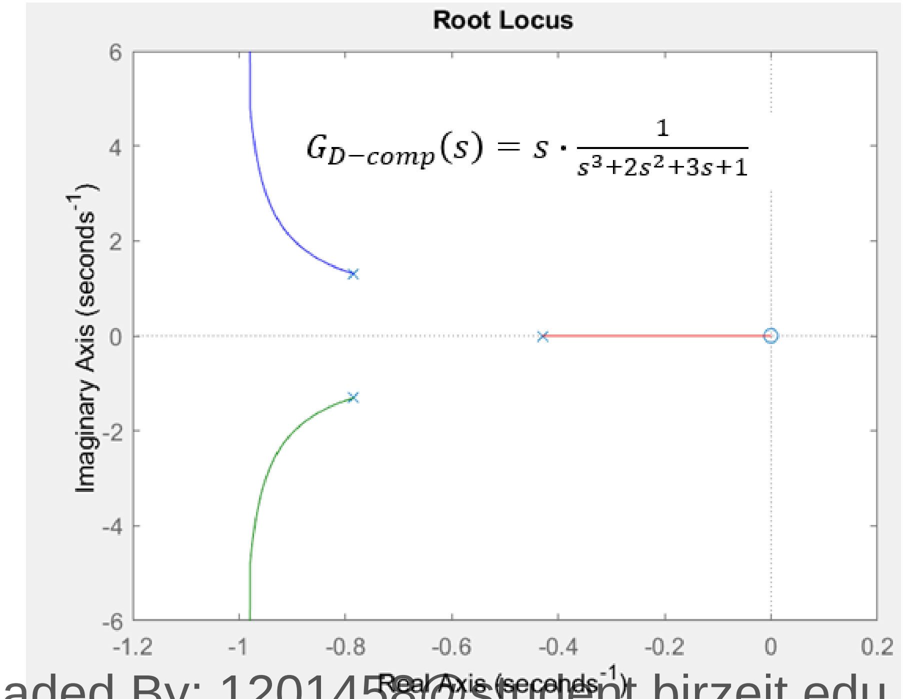
- Based on the insertion of a zero in the origin with a gain amplifier, reshapes the root locus in a single unique mode.
- The objective is changing the transient performance using the differentiation action.
- The transfer function is $G_D(s) = k_D s \rightarrow |k_D| = |R_2 C s|$
- Disadvantage: the steady state error is not preserved and can change in uncontrollable mode. It decreases the type of the system.
- Example: Uncompensated system: $G_{uncomp}(s) = \frac{1}{s^3+2s^2+3s+1}$

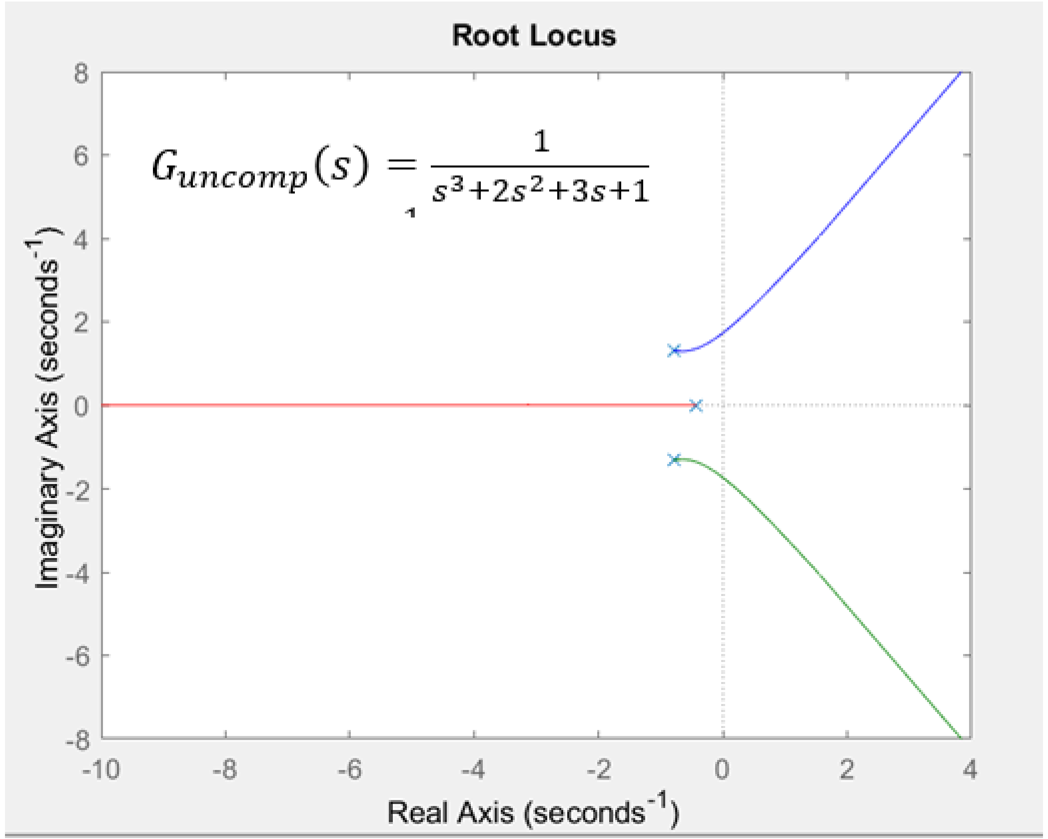
  D-compensated system: $G_{D-comp}(s) = s \cdot \frac{1}{s^3+2s^2+3s+1}$

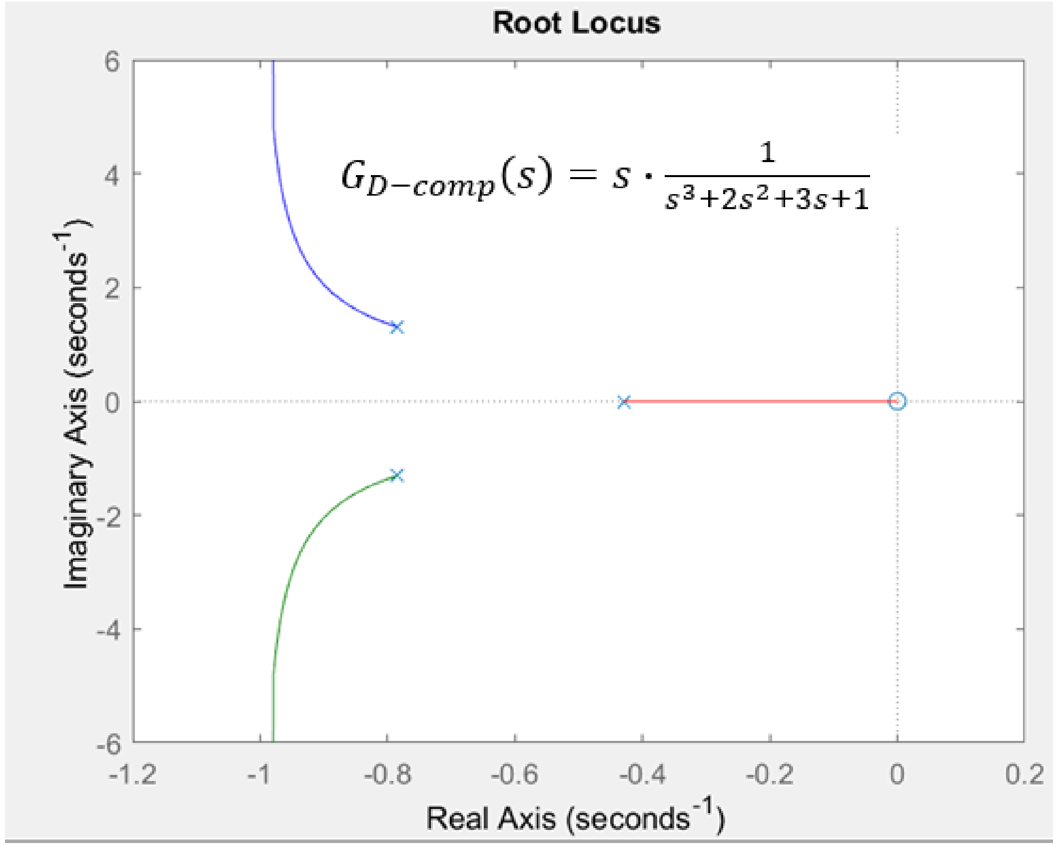








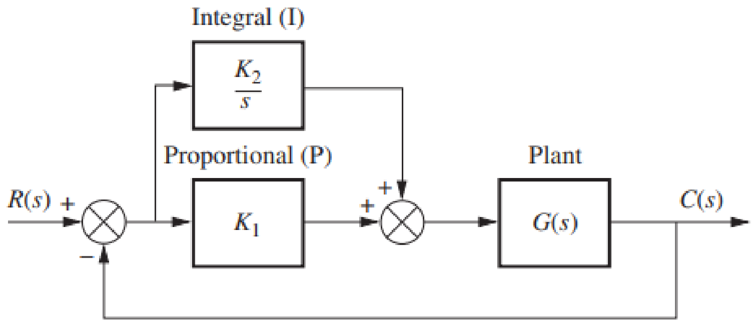

Root Locus
$G_{uncomp}(s) = \frac{1}{s^3+2s^2+3s+1}$
Imaginary Axis (seconds^{-1})
Real Axis (seconds^{-1})
8
6
4
2
0
-2
-4
-6
-8
-10
-8
-6
-4
-2
0
2
4

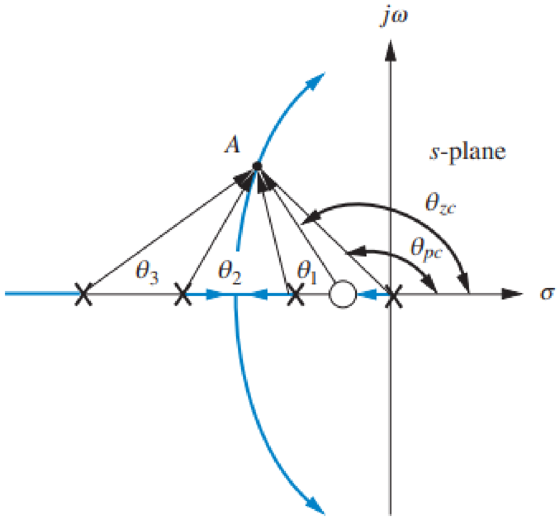

Root Locus
$G_{D-comp}(s) = s \cdot \frac{1}{s^3+2s^2+3s+1}$
Imaginary Axis (seconds^{-1})
Real Axis (seconds^{-1})
6
4
2
0
-2
-4
-6
-1.2
-1
-0.8
-0.6
-0.4
-0.2
0
0.2

Proportional-Integrative Controller (PI-controller):

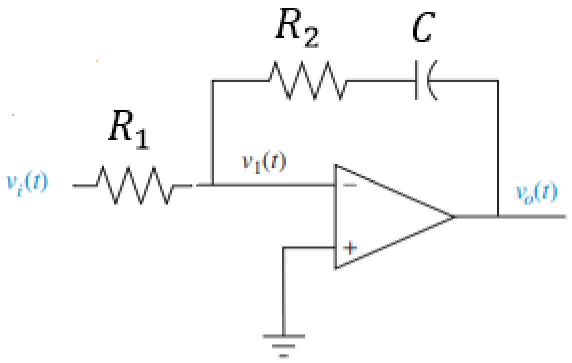
- Based on the insertion of a pole in the origin to increase the system type and a zero very closed to the pole at the origin using a dynamic gain amplifier circuit. This aims to keep the original sum of phases that defines the new root locus points almost unchanged with respect to the uncompensated system. a gain amplifier, reshapes the root locus in a single unique mode.
- The objective is setting the steady-state error to zero and preserve the original transient almost unchanged. The transfer function is $G_{PI}(s) = k_p + \frac{K_I}{s} = k_p \frac{s+\frac{k_I}{k_p}}{s} = k_1 \frac{s+\frac{k_2}{k_1}}{s} = \mathrm{k}\frac{(s+z_c)}{s}$
- Disadvantage: nearest zeros that achieve better preservation of the transient leads to slower controller.

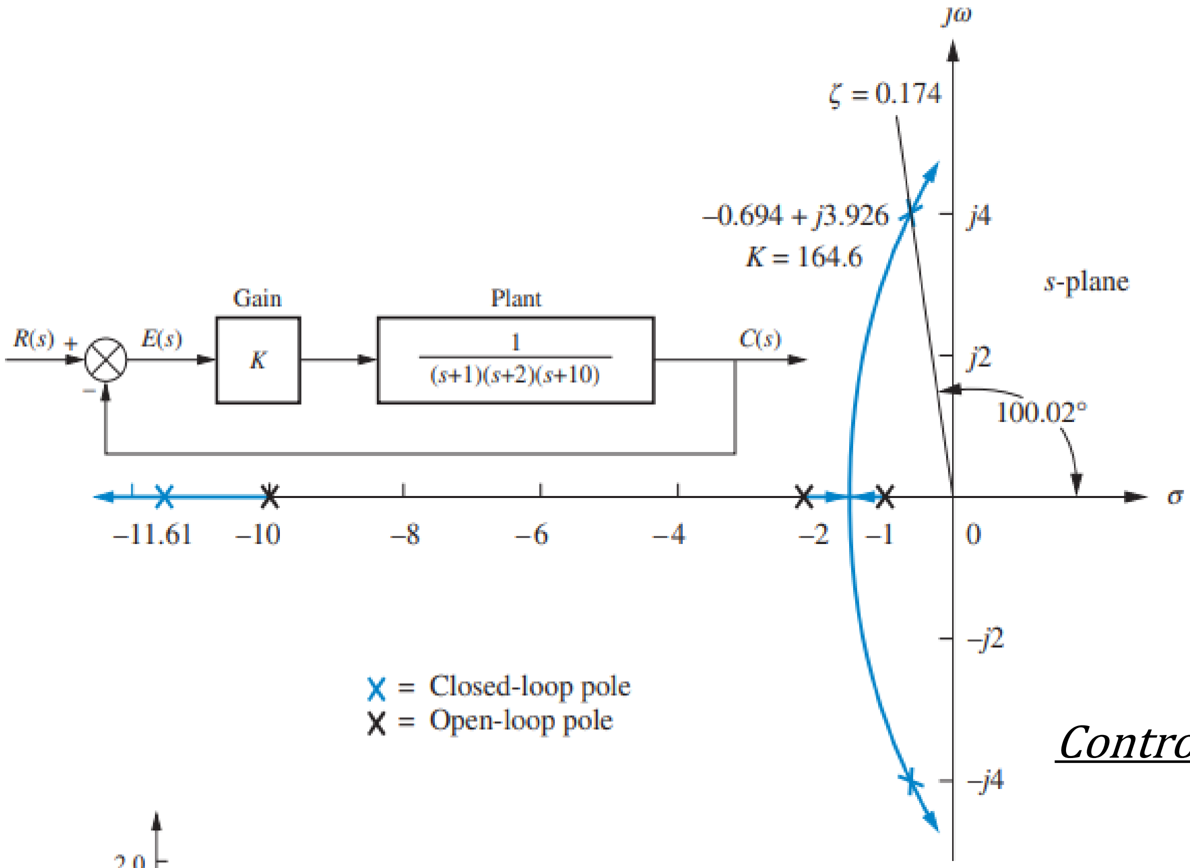
- *Root locus: original and PI-compensated + controller design (Next slide)*

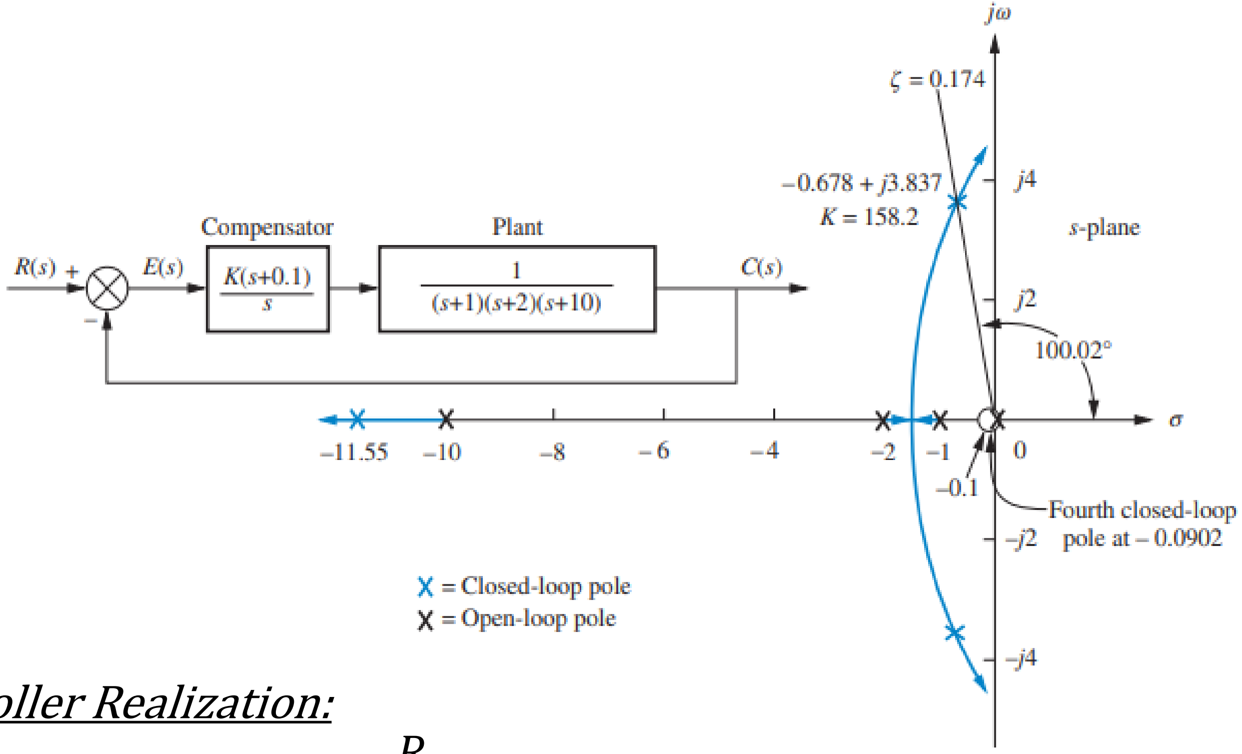


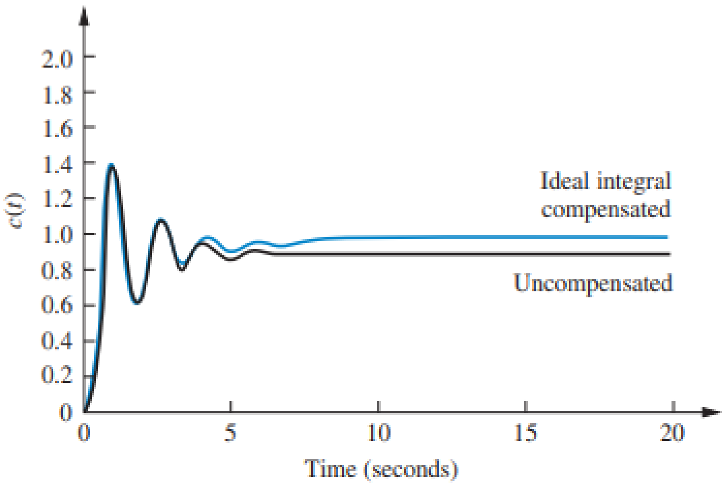


$$G_{PI_{comp}}(s) = -\frac{R_2}{R_1}\frac{(s+\frac{1}{R_2 C})}{s}$$









Controller Realization:

$$\frac{R_2}{R_1} = k = 158.2$$

$$\frac{1}{R_2 C} = z_c = 0.1$$

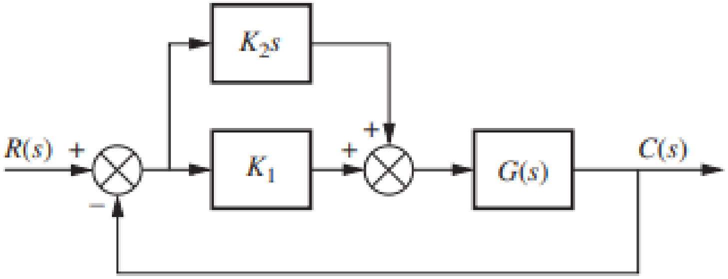
Assume $C = 10\mu F \rightarrow R_2 = 1M\Omega$

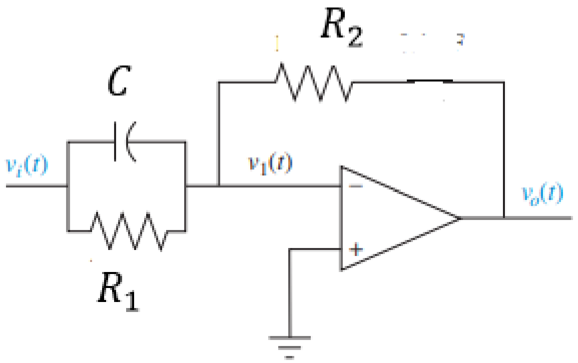
$$R_1 = \frac{1 \times 10^6}{158.2}\Omega$$

Note: you need another inverting stage with gain =-1 or you can divide the required gain between the two stages.

Proportional-Derivative Controller (PI-controller):

- Based on the insertion of a zero to reshape the original root locus and set a desired transient performance, preserving the steady sate error almost as is, using a dynamic gain amplifier circuit.
- The position of the zero is determined so that the reshaped root locus passes through the desired set of points. That is the desired poles should satisfy the phase condition with the controller zero included.
- The transfer function is $G_{PD}(s) = k_p + k_D s = k_D\left(s + \frac{k_p}{k_d}\right) = k_2\left(s + \frac{k_1}{k_2}\right) = k_D(s + z_c)$
- *Root locus: original and PI-compensated + controller design (Next slide)*

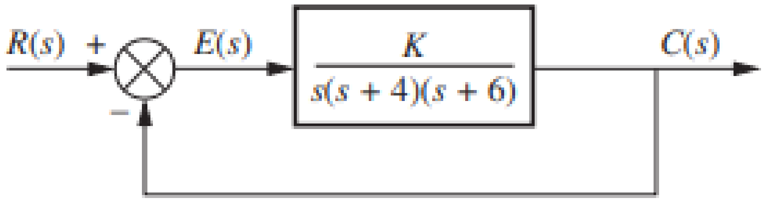




$$-\frac{R_2}{R_1}\frac{\left(s+\frac{1}{R_2C}\right)}{s}$$

PD- Compensator Design

Example: Given the system of figure, design an ideal derivative compensator to maintains the compensated system maximum overshoot less than or equal to the uncompensated system $\leq 16\%$ settling time.



Solution:

Uncompensated and compensated systems roots

- Use Matlab to determine the poles of the uncompensated system with 16% overshoot.
- Determine the value of the settling time using the second order approximation (verify the validity of the dominant poles approximation).
- Select the real part of the compensated system to be 3 times that of the uncompensated system.
- Determine the value of ζ corresponding to the selected real part at 16% overshoot using the second order approximation $\sigma_{c-des} = \zeta\omega_n \rightarrow \omega_n = \frac{\sigma_c}{\zeta} = \frac{3 \cdot 1.2}{0.503} = 7.16 \rightarrow \omega_{c-des} = \omega_n\sqrt{1-\zeta^2} = 6.17$. Thus the assumed compensated system desired poles are $P_{des} = -3.6 \pm j6.17$.
- Apply the phase condition to determine the position of the compensator zero.

$$\vartheta_{zc} - tan^{-1}\left(\frac{6.17}{4-3.6}\right) - tan^{-1}\left(\frac{6.17}{6-3.6}\right) - (180^0 - tan^{-1}\left(\frac{6.17}{3.6}\right) = 180^o$$

$$\vartheta_{zc} = 360^o + 86.29^o + 68.75^o - 59.74^o = 455.64 \rightarrow \vartheta_{zc} = 95.64^o.$$

To compute zc we compute the distance from the compensator zero and then subtract this distance from the closed pole real part because the angle is larger than 90^o.(in case the angle is less than 90^o we add the distance from the desired pole real part).

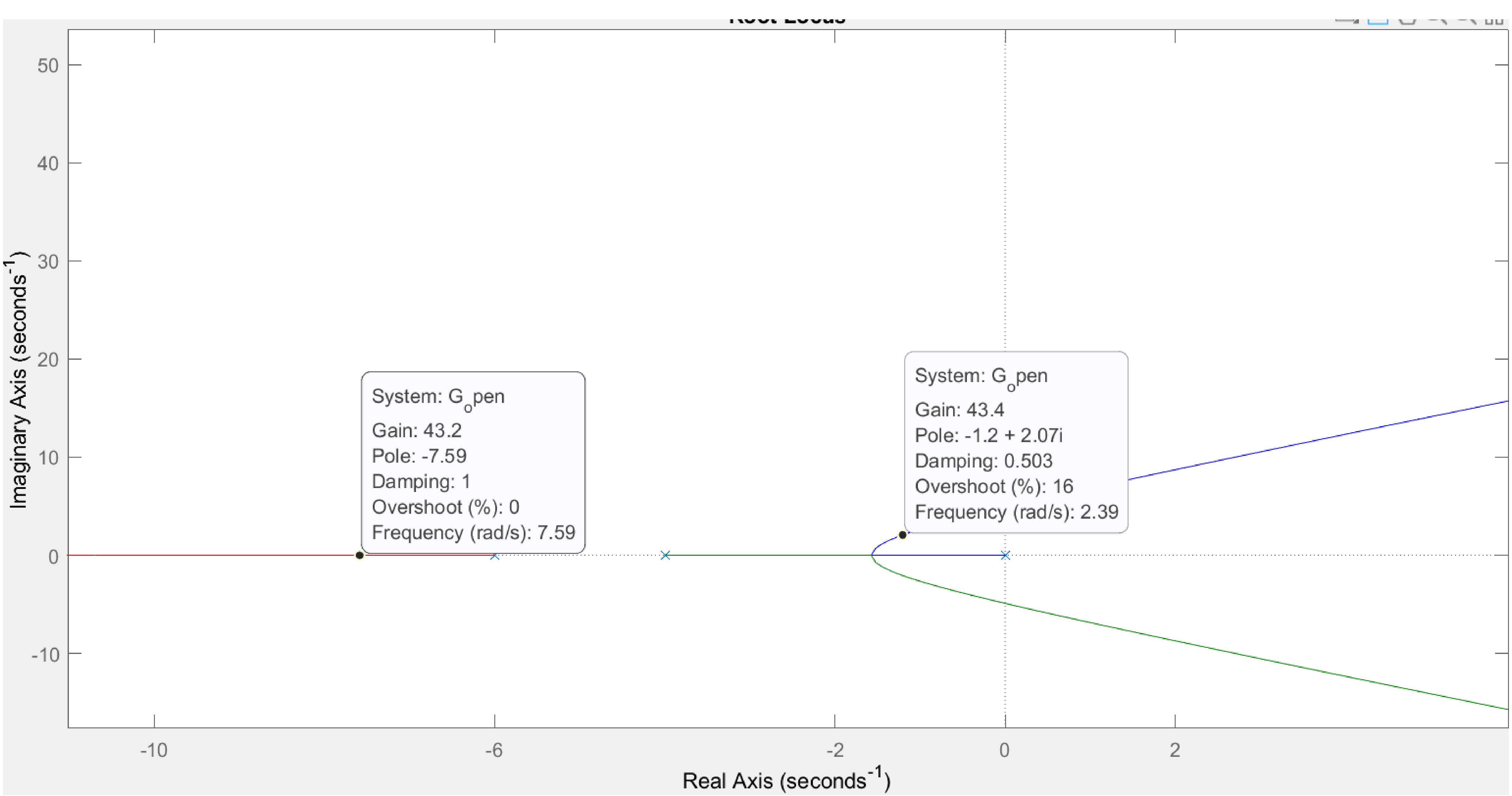
$d_{zc} = \frac{\omega_{c-des}}{\tan(180^o - 95.64^o).} = \frac{6.17}{\tan(84.36^o)} = 0.61 \rightarrow zc = -(3.6 - 0.6) = -3.$

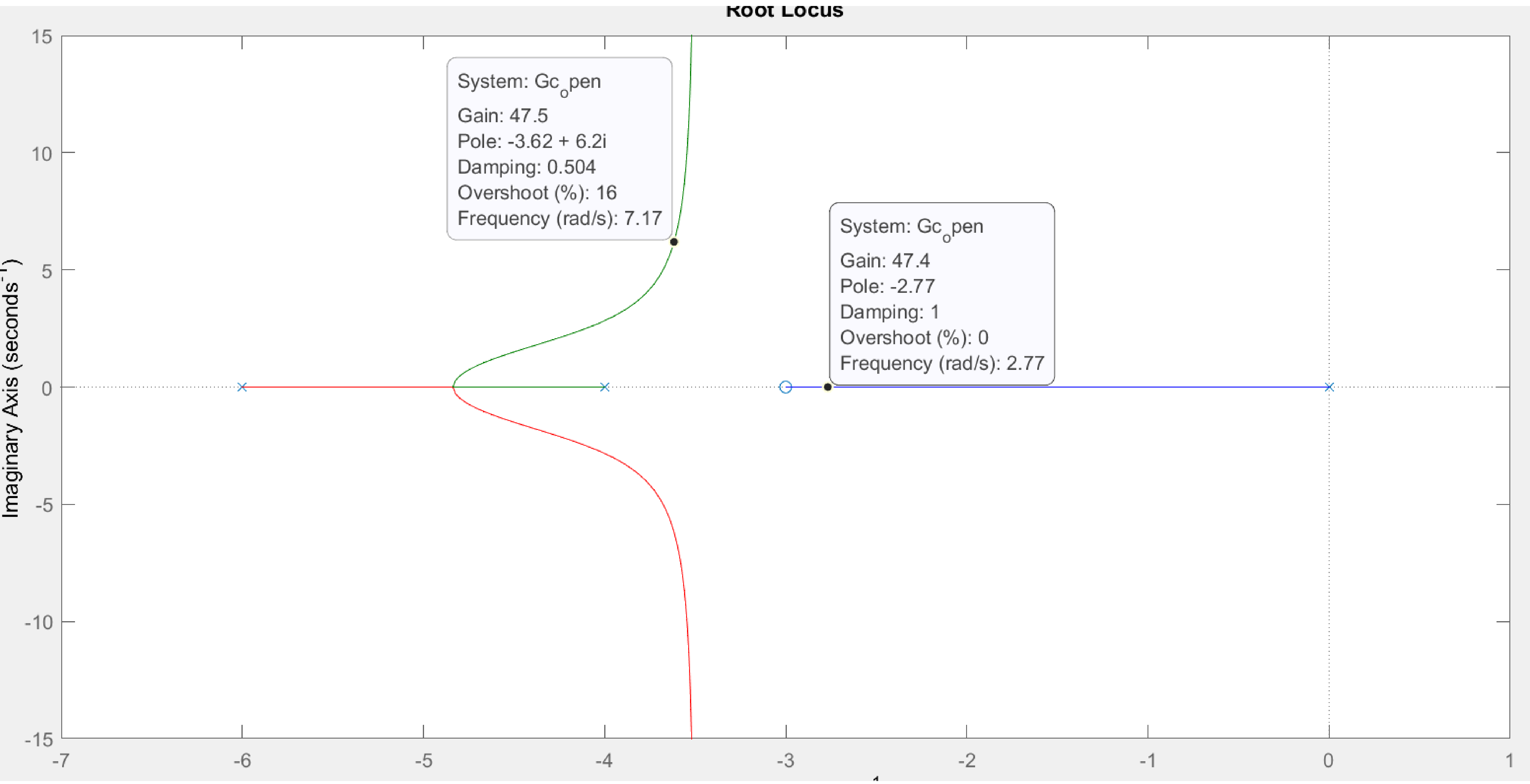
- The transfer function of the compensated system becomes $G_{c-open} = \frac{(s+3)}{s(s+4)(s+6)}$
- $T_{closed-uncomp} = \frac{43.4}{s^3+10s^2+24s+43.4}$
- $T_{closed-comp} = \frac{47.5(s+3)}{s^3+10s^2+24s+47.5(s+3)} = \frac{47.5(s+3)}{s^3+10s^2+71.5s+142.5)}$
- Check the validity of the closed loop second order dominant poles approximation.(Not satisfied)
- Make system tuning:
- $T_{cl-compTuned} = \frac{38.2(s+3.65)}{s^3+10s^2+24s+38.2.5(s+3.65)} = \frac{38.2(s+3.65)}{s^3+10s^2+62.2s+139.43)}$
- The tuned system satisfy the conditions

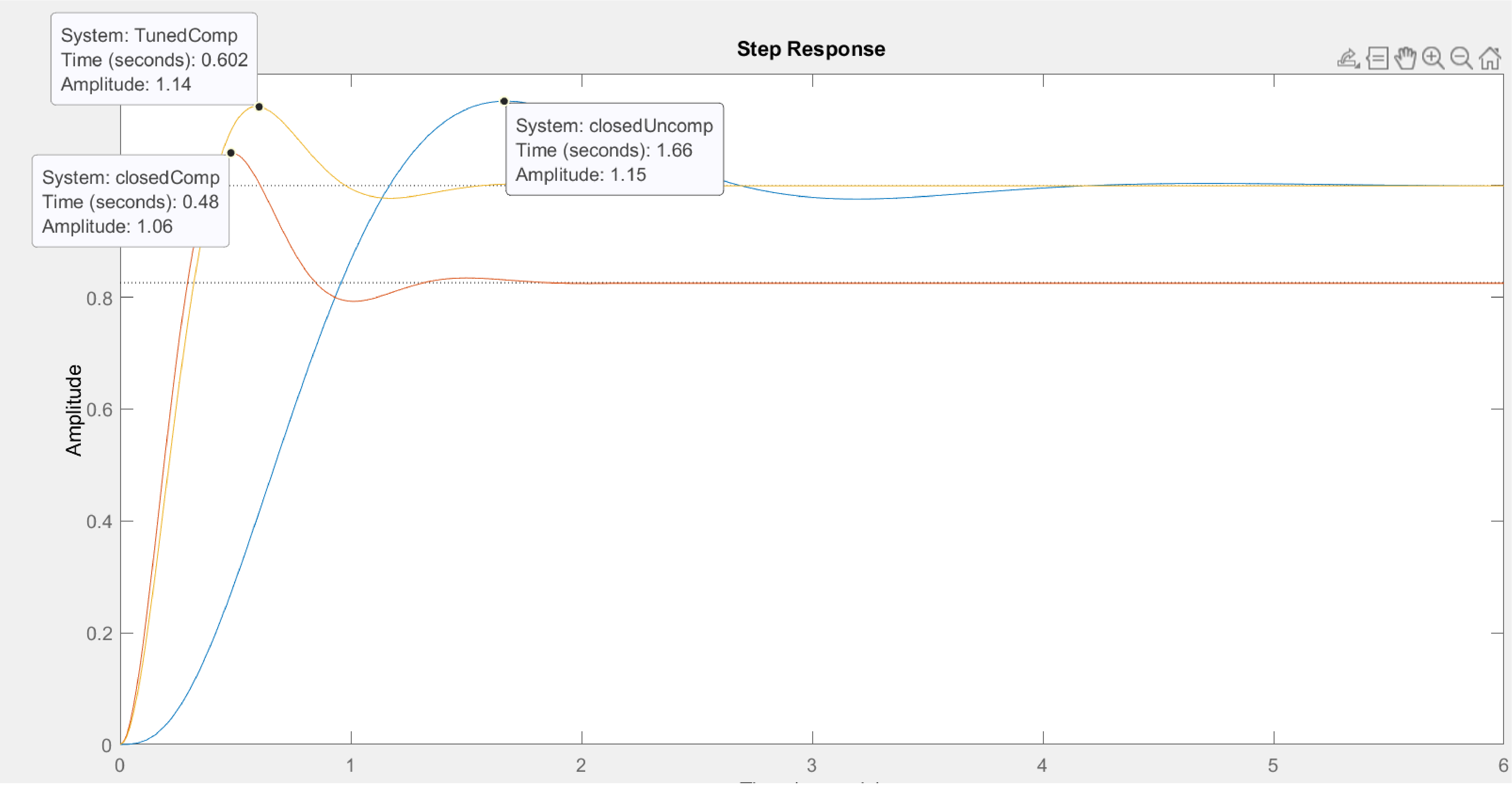
Uncompensated step-info		compensated step-info		Tuned step-info	
RiseTime:	0.7334	RiseTime:	0.1989	RiseTime:	0.2718
SettlingTime:	3.4712	SettlingTime:	1.1837	SettlingTime:	1.2574
SettlingMin:	0.9285	SettlingMin:	0.7696	SettlingMin:	0.9136
SettlingMax:	1.1509	SettlingMax:	1.0585	SettlingMax:	1.1411
Overshoot:	15.0910	Overshoot:	28.1345	Overshoot:	14.1129
Undershoot:	0	Undershoot:	0	Undershoot:	0
Peak:	1.1509	Peak:	1.0585	Peak:	1.1411
PeakTime:	1.6827	PeakTime:	0.4854	PeakTime:	0.5852

PD-Realization

$\frac{R_2}{R_1} = 38.2$, $\frac{1}{R_1 C} = 3.65$ →select $C = 50\mu F \rightarrow R_1 = 5.48 \cdot 10^3 \rightarrow R_2 = 209.34 \cdot 10^3$


Root Locus
System: G_open
Gain: 43.2
Pole: -7.59
Damping: 1
Overshoot (%): 0
Frequency (rad/s): 7.59
System: G_open
Gain: 43.4
Pole: -1.2 + 2.07i
Damping: 0.503
Overshoot (%): 16
Frequency (rad/s): 2.39
Imaginary Axis (seconds^-1)
Real Axis (seconds^-1)
50
40
30
20
10
0
-10
-10
-6
-2
0
2


Root Locus
System: Gc_open
Gain: 47.5
Pole: -3.62 + 6.2i
Damping: 0.504
Overshoot (%): 16
Frequency (rad/s): 7.17
System: Gc_open
Gain: 47.4
Pole: -2.77
Damping: 1
Overshoot (%): 0
Frequency (rad/s): 2.77
Imaginary Axis (seconds⁻¹)
15
10
5
0
-5
-10
-15
-7
-6
-5
-4
-3
-2
-1
0
1


Step Response
System: TunedComp
Time (seconds): 0.602
Amplitude: 1.14
System: closedUncomp
Time (seconds): 1.66
Amplitude: 1.15
System: closedComp
Time (seconds): 0.48
Amplitude: 1.06
Amplitude
0.8
0.6
0.4
0.2
0
0
1
2
3
4
5
6