3.6 Row Space and Column Space

Definition If A is an $m \times n$ matrix, the subspace of $\mathbb{R}^{1\times n}$ spanned by the row vectors of A is called the **row space** of A. The subspace of $\mathbb{R}^m$ spanned by the column vectors of A is called the **column space** of A.

EXAMPLE 1 Let

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}_{2\times 3}$$

The row space of A is the set of all 3-tuples of the form

$$\alpha\underbrace{(1,0,0)}_{}^{v_1} + \beta(0,1,0)^{v_2} = (\alpha,\beta,0)$$

Row space of A is $\{(\alpha,\beta,0) : \alpha,\beta \in \mathbb{R}\}$

The column space of A is the set of all vectors of the form

$$\alpha\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \beta\begin{bmatrix} 0 \\ 1 \end{bmatrix} + \gamma\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

$\{(\alpha,\beta)^T : \alpha,\beta \in \mathbb{R}\}$

Thus the row space of A is a two-dimensional subspace of $\mathbb{R}^{1\times 3}$, and the column space of A is $\mathbb{R}^2$.

Theorem 3.6.1 *Two row equivalent matrices have the same row space.*

Definition The **rank** of a matrix A, denoted rank(A), is the dimension of the row space of A.

To determine the rank of a matrix, we can reduce the matrix to row echelon form. The nonzero rows of the row echelon matrix will form a basis for the row space.

column space of A = span$\{u_1, u_2, u_3\}$

EXAMPLE 2 Let

$$A = \begin{array}{ccc} u_1 & u_2 & u_3 \\ \left[\begin{array}{c} 1 \\ 2 \\ 1 \end{array}\right. & \begin{array}{c} -2 \\ -5 \\ -4 \end{array} & \left.\begin{array}{c} 3 \\ 1 \\ -7 \end{array}\right] \end{array}$$

$u_3 = 13u_1 + 5u_2$

Reducing A to row echelon form, we obtain the matrix

$$U = \begin{array}{ccc} v_1 & v_2 & v_3 \\ \left[\begin{array}{c} 1 \\ 0 \\ 0 \end{array}\right. & \begin{array}{c} -2 \\ 1 \\ 0 \end{array} & \left.\begin{array}{c} 3 \\ 5 \\ 0 \end{array}\right] \end{array}$$

$\boxed{v_3 = 13v_1 + 5v_2}$

Clearly, $(1, -2, 3)$ and $(0, 1, 5)$ form a basis for the row space of U. Since U and A are row equivalent, they have the same row space, and hence the rank of A is 2.

In general, the rank and the dimension of the null space always add up to the number of columns of the matrix. The dimension of the null space of a matrix is called the *nullity* of the matrix.

Theorem 3.6.5 **The Rank–Nullity Theorem**

If A is an $m \times n$ matrix, then the rank of A plus the nullity of A equals n.

EXAMPLE 3 Let

$$A = \begin{array}{c} \begin{array}{cccc} u_1 & u_2 & u_3 & u_4 \end{array} \\ \begin{pmatrix} 1 & 2 & -1 & 1 \\ 2 & 4 & -3 & 0 \\ 1 & 2 & 1 & 5 \end{pmatrix} \end{array}$$

$u_2 = 2u_1$

$u_4 = 3u_1 + 2u_3$

Find a basis for the row space of A and a basis for $N(A)$. Verify that $\dim N(A) = n - r$.

Solution

$(A|0) \xrightarrow{REF} (U|0)$

The reduced row echelon form of A is given by

$$U = \begin{array}{c} \\ \\ \end{array} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \\ \left[\begin{matrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{matrix}\right] \\ x_1 & x_2 & x_3 & x_4 \end{array}$$

$v_2 = (2)v_1 + (0)v_3$

$v_4 = (3)v_1 + (2)v_3$

Thus, $\{(1, 2, 0, 3), (0, 0, 1, 2)\}$ is a basis for the row space of A, and A has rank 2. Let $x_2 = \alpha$ and $x_4 = \beta$. It follows that $N(A)$ consists of all vectors of the form

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -2\alpha - 3\beta \\ \alpha \\ -2\beta \\ \beta \end{pmatrix} = \alpha \overset{u_1}{\begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}} + \beta \overset{u_2}{\begin{pmatrix} -3 \\ 0 \\ -2 \\ 1 \end{pmatrix}}$$

The vectors $(-2, 1, 0, 0)^T$ and $(-3, 0, -2, 1)^T$ form a basis for $N(A)$. Note that

$$n - r = 4 - 2 = 2 = \dim N(A)$$

Remark:

The matrices A and U in Example 3 have different column spaces; however, their column vectors satisfy the same dependency relations. For the matrix U, the column vectors $\mathbf{u}_1$ and $\mathbf{u}_3$ are linearly independent, while

$$\mathbf{u}_2 = 2\mathbf{u}_1$$
$$\mathbf{u}_4 = 3\mathbf{u}_1 + 2\mathbf{u}_3$$

The same relations hold for the columns of A: The vectors $\mathbf{a}_1$ and $\mathbf{a}_3$ are linearly independent, while

$$\mathbf{a}_2 = 2\mathbf{a}_1$$
$$\mathbf{a}_4 = 3\mathbf{a}_1 + 2\mathbf{a}_3$$

We can use the row echelon form U of A to find a basis for the column space of A. We need only determine the columns of U that correspond to the leading 1's. These same columns of A will be linearly independent and form a basis for the column space of A.

Note

The row echelon form U tells us only which columns of A to use to form a basis. We cannot use the column vectors from U, since, in general, U and A have different column spaces.

Theorem 3.6.6 *If A is an $m \times n$ matrix, the dimension of the row space of A equals the dimension of the column space of A.*

$\boxed{A_{3\times 2} \quad , \quad \text{rank}\, A = 2}$

EXAMPLE 4 Let

$$A = \begin{bmatrix} 1 & -2 & 1 & 1 & 2 \\ -1 & 3 & 0 & 2 & -2 \\ 0 & 1 & 1 & 3 & 4 \\ 1 & 2 & 5 & 13 & 5 \end{bmatrix}$$

The row echelon form of A is given by

$$U = \begin{bmatrix} 1 & -2 & 1 & 1 & 2 \\ 0 & 1 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The leading 1's occur in the first, second, and fifth columns. Thus

$$\mathbf{a}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{a}_2 = \begin{bmatrix} -2 \\ 3 \\ 1 \\ 2 \end{bmatrix}, \quad \mathbf{a}_5 = \begin{bmatrix} 2 \\ -2 \\ 4 \\ 5 \end{bmatrix}$$

form a basis for the column space of A. ■

EXAMPLE 5 Find the dimension of the subspace of $\mathbb{R}^4$ spanned by

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} 2 \\ 5 \\ -3 \\ 2 \end{bmatrix}, \quad \mathbf{x}_3 = \begin{bmatrix} 2 \\ 4 \\ -2 \\ 0 \end{bmatrix}, \quad \mathbf{x}_4 = \begin{bmatrix} 3 \\ 8 \\ -5 \\ 4 \end{bmatrix}$$

Solution

The subspace $\text{Span}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4)$ is the same as the column space of the matrix

$= \text{span}(x_1, x_2)$

$$X = \begin{array}{c} \begin{array}{cccc} u_1 & u_2 & u_3 & u_4 \end{array} \\ \begin{bmatrix} 1 & 2 & 2 & 3 \\ 2 & 5 & 4 & 8 \\ -1 & -3 & -2 & -5 \\ 0 & 2 & 0 & 4 \end{bmatrix} \end{array}$$

$u_3 = 2u_1$

$u_4 = -u_1 + 2u_2$

$\Updownarrow$

The row echelon form of X is

$$\begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{bmatrix} 1 & 2 & 2 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

$v_3 = (2)\, v_1 + (0)\, v_2$

$v_4 = (-1)\, v_1 + (2)\, v_2$

The first two columns $\mathbf{x}_1, \mathbf{x}_2$ of X will form a basis for the column space of X. Thus, $\dim \text{Span}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4) = 2$.

Linear Systems

The concepts of row space and column space are useful in the study of linear systems. A system $A\mathbf{x} = \mathbf{b}$ can be written in the form

$$x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} \tag{1}$$

In Chapter 1 we used this representation to characterize when a linear system will be consistent. The result, Theorem 1.3.1, can now be restated in terms of the column space of the matrix.

Theorem 3.6.2 Consistency Theorem for Linear Systems

A linear system $A\mathbf{x} = \mathbf{b}$ is consistent if and only if $\mathbf{b}$ is in the column space of A.

Theorem 3.6.3 *Let A be an $m \times n$ matrix. The linear system $A\mathbf{x} = \mathbf{b}$ is consistent for every $\mathbf{b} \in \mathbb{R}^m$ if and only if the column vectors of A span $\mathbb{R}^m$. The system $A\mathbf{x} = \mathbf{b}$ has at most one solution for every $\mathbf{b} \in \mathbb{R}^m$ if and only if the column vectors of A are linearly independent.*

Corollary 3.6.4 *An $n \times n$ matrix A is nonsingular if and only if the column vectors of A form a basis for $\mathbb{R}^n$.*

$x = A^{-1} B$

SECTION 3.6 EXERCISES

1. For each of the following matrices, find a basis for the row space, a basis for the column space, and a basis for the null space.

(a) $\begin{bmatrix} 1 & 3 & 2 \\ 2 & 1 & 4 \\ 4 & 7 & 8 \end{bmatrix}$

(b) $\begin{bmatrix} -3 & 1 & 3 & 4 \\ 1 & 2 & -1 & -2 \\ -3 & 8 & 4 & 2 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 3 & -2 & 1 \\ 2 & 1 & 3 & 2 \\ 3 & 4 & 5 & 6 \end{bmatrix}$

2. In each of the following, determine the dimension of the subspace of $\mathbb{R}^3$ spanned by the given vectors.

(a) $\overset{x_1}{\begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}}, \overset{x_2}{\begin{bmatrix} 2 \\ -2 \\ 4 \end{bmatrix}}, \overset{x_3}{\begin{bmatrix} -3 \\ 3 \\ 6 \end{bmatrix}}$

$\dim(\operatorname{span}(x_1, x_2, x_3))$

Let $A = \begin{pmatrix} 1 & 2 & -3 \\ -2 & -2 & 3 \\ 2 & 4 & 6 \end{pmatrix} \xrightarrow{REF} U$

(b) $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$

(a) $A = \begin{pmatrix} 1 & 3 & 2 \\ 2 & 1 & 4 \\ 4 & 7 & 8 \end{pmatrix} \xrightarrow[R_3 - 4R_1]{R_2 - 2R_1} \begin{pmatrix} 1 & 3 & 2 \\ 0 & -5 & 0 \\ 0 & -5 & 0 \end{pmatrix}$

$\xrightarrow{R_3 - R_2} \begin{pmatrix} 1 & 3 & 2 \\ 0 & -5 & 0 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{-\frac{1}{5} R_2} \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \begin{pmatrix} 1 & 3 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{array} = U$

Basis for row space of A is $\{ (1, 3, 2), (0, 1, 0) \}$

Basis for column space of A is $\left\{ \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 7 \end{pmatrix} \right\}$

$Ax = 0 \qquad (A \mid 0) \xrightarrow{REF} (U \mid 0)$

$x_3 = \alpha$

$x_1 = 0 \quad -2\alpha \implies \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -2\alpha \\ 0 \\ \alpha \end{pmatrix} = \alpha \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$

$x_2 = 0$

$\therefore$ Basis for $N(A)$ is $\left\{ \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} \right\}$

8. Let A be an $m \times n$ matrix with $m > n$. Let $\mathbf{b} \in \mathbb{R}^m$ and suppose that $N(A) = \{\mathbf{0}\}$.

(a) What can you conclude about the column vectors of A? Are they linearly independent? Do they span $\mathbb{R}^m$? Explain.

(b) How many solutions will the system $A\mathbf{x} = \mathbf{b}$ have if $\mathbf{b}$ is not in the column space of A? How many solutions will there be if $\mathbf{b}$ is in the column space of A? Explain.

(a) $N(A) = \{0\} \Rightarrow \text{nullity}(A) = 0.$

$\Rightarrow \text{rank}(A) = n.$

$\Rightarrow \dim(\text{Col}(A)) = n.$

$\Rightarrow \{a_1, a_2, \ldots, a_n\}$ is L.I. set.

$S = \{a_1, \ldots, a_n\}$ can't span $\mathbb{R}^m$ since $|S| = n < m = \dim \mathbb{R}^m$

(b) If $b \notin col(A)$, then there is no solution

If $b \in col(A)$, then the system is consistent.

since $\{a_1, \_\_, a_n\}$ is L.I. set, then there is at most one solution.

Hence, there is exactly one solution