

• Name... Key

• Number.....

• Section.....1.....

Q1. Solve the following IVP

$$ty' + 2y = \frac{\sin t}{t}, \quad y\left(-\frac{\pi}{2}\right) = 4, \quad t < 0.$$

$$y' + \frac{2}{t}y = \frac{\sin t}{t^2}$$

$$M(t) = e^{\int \frac{2}{t} dt} = e^{2 \ln(t)} = \boxed{t^2} \quad (2)$$

$$y(t) = \frac{1}{t^2} \left[\int t^2 \cdot \frac{\sin t}{t^2} dt + C \right] \quad (1)$$

$$= \frac{1}{t^2} \int \sin t dt + \frac{C}{t^2}$$

$$= \frac{1}{t^2} (-\cos t) + \frac{C}{t^2} \quad (1)$$

$$\text{Using I.C.} \Rightarrow 4 = \frac{C}{\frac{\pi^2}{4}} \Rightarrow \boxed{C = \pi^2} \quad (1)$$

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$$\Rightarrow y(t) = \frac{-\cos t}{t^2} + \frac{\pi^2}{t^2}$$

Q₂. A tank contains 100 gal of water and 50gm of salt. Water containing a salt concentration of $\frac{1}{4}(1 + \frac{1}{2} \sin t)$ gm/gal flows into the tank at a rate of 2 gal/min, and the mixture in the tank flows out at the same rate. Find the amount of salt in the tank at any time.

Let $Q(t)$ amount of salt

$$\Rightarrow \frac{dQ}{dt} = \text{rate in} - \text{rate out}$$

$$= 2 \left(\frac{1}{4} \left(1 + \frac{1}{2} \sin t \right) \right) - 2 \frac{Q(t)}{100}$$

$$= \frac{1}{2} + \frac{1}{4} \sin t - \frac{2Q(t)}{100}$$

$$\Rightarrow \frac{dQ}{dt} + \frac{2Q(t)}{100} = \frac{1}{2} + \frac{1}{4} \sin t, \quad Q(0) = 50$$

$$M(t) = e^{\int \frac{2}{100} dt} = e^{\frac{2}{100}t}$$

$$Q(t) = \frac{1}{e^{\frac{2}{100}t}} \left[\int e^{\frac{2}{100}t} \left(\frac{1}{2} + \frac{1}{4} \sin t \right) dt + C \right]$$

$$= e^{-\frac{2}{100}t} \left[\frac{1}{2} \int e^{\frac{2}{100}t} dt + \frac{1}{4} \int e^{\frac{2}{100}t} \sin t dt + C \right]$$

$$= e^{-\frac{2}{100}t} \left[25e^{\frac{2}{100}t} + \frac{1}{4} \left(\text{Integration by parts twice} \right) + C \right]$$

(2)

Extra work (+2) Bonus