

Continuity

والتالي

cont.: continuousDef f is cont. at x_0 if

$$\textcircled{1} \quad \lim_{x \rightarrow x_0} f(x) \text{ exists and}$$

$$\textcircled{2} \quad f(x_0) \text{ exists and}$$

$$\textcircled{3} \quad \lim_{x \rightarrow x_0} f(x) = f(x_0)$$

Exp $f(x) = x^2 - 3$

cont. at $x=2$

$$f(2) = 2^2 - 3 = 4 - 3 = 1$$

$$\begin{aligned} \lim_{x \rightarrow 2} f(x) &= \lim_{x \rightarrow 2} (x^2 - 3) \\ &= 4 - 3 \\ &= 1 \end{aligned}$$

$$f(2) = \lim_{x \rightarrow 2} f(x)$$

$$\textcircled{1} = \textcircled{1}$$

$$\lim_{x \rightarrow x_0^+} f(x) = \lim_{x \rightarrow x_0^-} f(x) = \underline{\underline{L}}$$

* f cont. on $[a, b]$ if f cont. on every point $x_0 \in [a, b]$ Examples for cont. function on $(-\infty, \infty)$

$$\rightarrow f(x) = x^2 - 4x + 5 \quad \text{"all polynomials"}$$

$$f(x) = |x| \quad \checkmark$$

$$f(x) = e^x \quad \checkmark$$

$$f(x) = \cos x \quad \checkmark$$

$$f(x) = \sin x \quad \checkmark$$

Exp Rational function $R(x)$ اقران نسبي

$$R(x) = \frac{f(x)}{g(x)}$$

f, g polynomials
كثيرات مرتدة

$R(x)$ is cont. everywhere except where $g(x)=0$

Exp $R(x) = \frac{x^2 - 9}{x + 3}$ is cont. on $\mathbb{R} \setminus \{-3\}$

$R(x) = \frac{x}{x^2 - 4}$ is cont. on $\mathbb{R} \setminus \{-2, 2\}$

Exp Assume $f(x) = \begin{cases} ax + b & , x \leq 0 \\ x^2 + 3a - b & , 0 < x \leq 2 \\ 3x - 5 & , 2 < x \end{cases}$

is cont. Find a, b and sketch $f(x)$

f is cont. at $x=2 \Rightarrow f(2) = \lim_{x \rightarrow 2} f(x)$

f is cont. at x=2

$$\Rightarrow \lim_{x \rightarrow 2^+} f(x) = \lim_{x \rightarrow 2^-} f(x)$$

$$3(2) - 5 = 2^2 + 3a - b$$

$$6 - 5 = 4 + 3a - b$$

$$1 = 4 + 3a - b$$

$$3a = b - 3 \quad (2)$$

f is cont at x=0

$$\Rightarrow f(0) = \lim_{x \rightarrow 0} f(x)$$

$$\Rightarrow \lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^-} f(x)$$

$$0^2 + 3(a) - b = a(0) + b$$

$$0 + 3a - b = 0 + b$$

$$2b = 3a \quad (1)$$

$$2b = b - 3$$

$$b = -3$$

$$a = -2$$

f cont at x0

$$\Rightarrow \lim_{x \rightarrow x_0^+} f(x) = \lim_{x \rightarrow x_0^-} f(x)$$

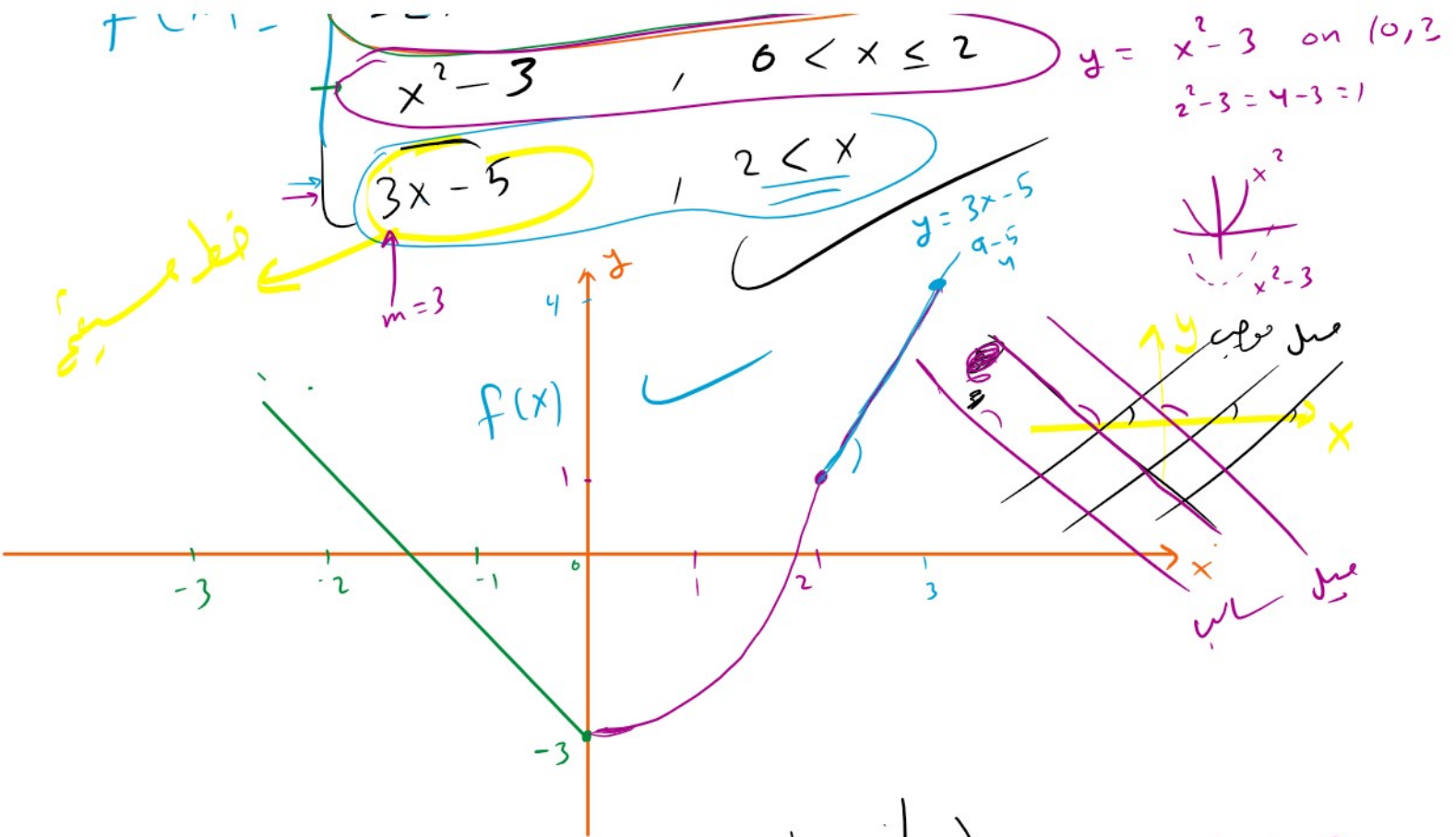
$$f(x) = \begin{cases} -2x - 3, & x \leq 0 \\ x^2 - 3, & 0 < x \leq 2 \end{cases}$$

$$y = -2x - 3$$

$m = -2 < 0$
(0, -3)

$$y = x^2 - 3 \text{ on } (0, 2]$$

$2^2 - 4 - 3 = 1$



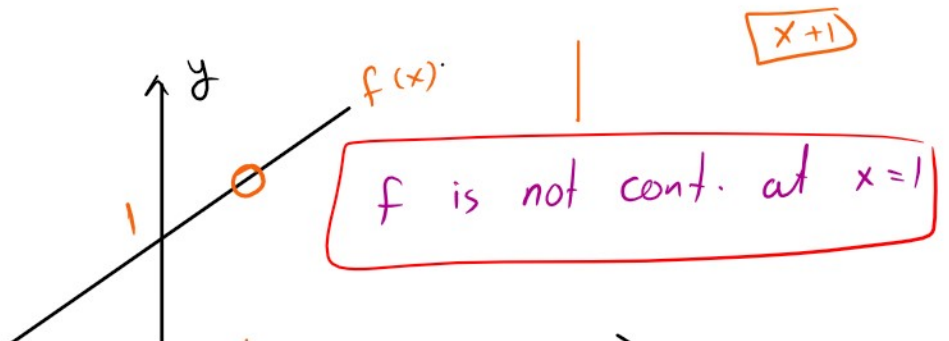
Exp (Removable Discontinuity)

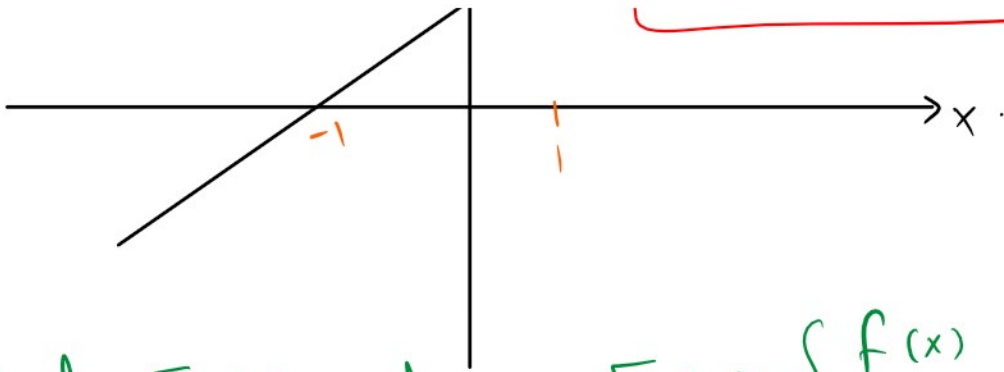
$f(1) = \frac{0}{0}$

$\Rightarrow f(x) = \frac{x^2 - 1}{x - 1} \Rightarrow D(f) = \mathbb{R} \setminus \{1\}$

(1) Find $\lim_{x \rightarrow 1} f(x) = \lim_{x \rightarrow 1} \frac{x^2 - 1}{x - 1} = \lim_{x \rightarrow 1} \frac{(x-1)(x+1)}{x-1}$ 0/0
 $= \lim_{x \rightarrow 1} (x+1) = 1 + 1 = 2$

(2) sketch $f(x)$





3) Find $F(x)$ where $F(x) = \begin{cases} f(x) & \text{if } x \neq 1 \\ \lim_{x \rightarrow 1} f(x) & \text{if } x = 1 \end{cases}$

Continuous Extension for $f(x)$

$\Rightarrow F(x)$ is cont. at $x=1$

$$= \begin{cases} \frac{x^2-1}{x-1} & \text{if } x \neq 1 \\ 2 & \text{if } x = 1 \end{cases}$$

$$\underline{F(1)} = \lim_{x \rightarrow 1} \underline{F(x)} \\ 2 = 2$$

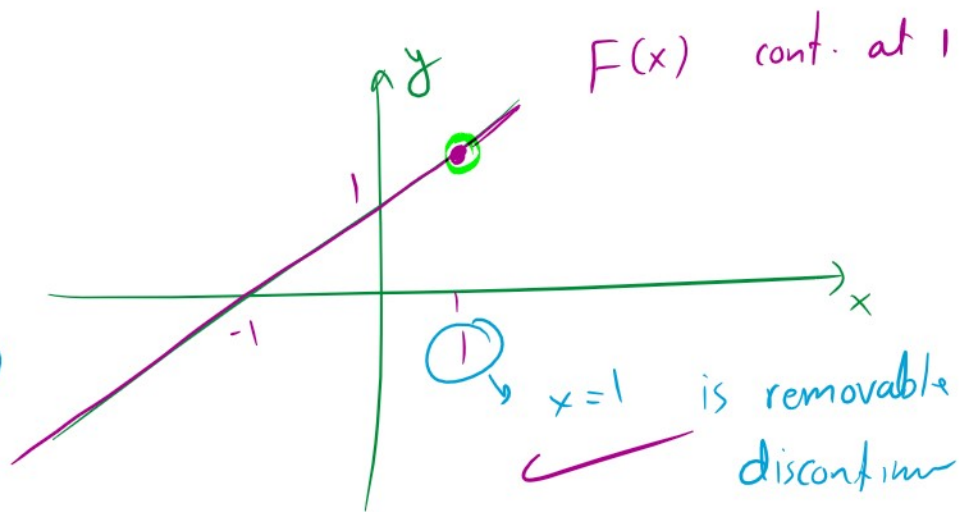
4) What is the difference between $f(x)$ and $F(x)$

$F(x)$ is cont. at $x=1$

$f(x)$ is not cont. at $x=1$

5) sketch $\underline{F(x)}$

Continuous Extension for $f(x)$ at $x=1$



✓
at $x=1$

✓ discontinuous

$$f(2) = \frac{0}{0}$$

Exp Find cont. extension for $f(x) = \frac{x^2 - 5x + 6}{2x - 4}$
at $x=2$

$$D(f) = \mathbb{R} \setminus \{2\}$$

f is not cont. at $x=2$

$$\lim_{x \rightarrow 2} f(x) = \lim_{x \rightarrow 2} \frac{x^2 - 5x + 6}{2x - 4}$$

$$= \lim_{x \rightarrow 2} \frac{(x-2)(x-3)}{(2)(x-2)}$$

$$= \lim_{x \rightarrow 2} \frac{(x-3)}{2}$$

$$= \frac{2-3}{2} = -\frac{1}{2}$$

$$F(x) = \begin{cases} f(x) & \text{if } x \neq 2 \\ \lim_{x \rightarrow 2} f(x) & \text{if } x = 2 \end{cases}$$

$$F(x) = \begin{cases} \frac{x^2 - 5x + 6}{2x - 4} & \text{if } x \neq 2 \\ -\frac{1}{2} & \text{if } x = 2 \end{cases}$$

$$F(2) = -\frac{1}{2} = \lim_{x \rightarrow 2} F(x) = \lim_{x \rightarrow 2} \frac{x^2 - 5x + 6}{2x - 4}$$

$F(x)$ cont. at $x=2$

$$x=2 \notin D(f)$$

removable
discont.

$$f(x) = \frac{D}{D}$$

$$x=c$$

removable

$$\text{if } f(c) = \frac{0}{0}$$

removable if $f(c) = \frac{0}{0}$

if $f(c) \neq \frac{0}{0}$ \Rightarrow c is not removable discont.
 (c is vertical Asymptote)
 V. Asy

Remark: If c is removable $\Rightarrow c$ is not V. Asy
 If c is V. Asy $\Rightarrow c$ is not removable

Exp Is $x=0$ removable discont. for $f(x) = \frac{1}{x}$

No $x=0$ is not removable since $f(0) = \frac{1}{0} \neq \frac{0}{0}$

$x=0$ is V. Asy.

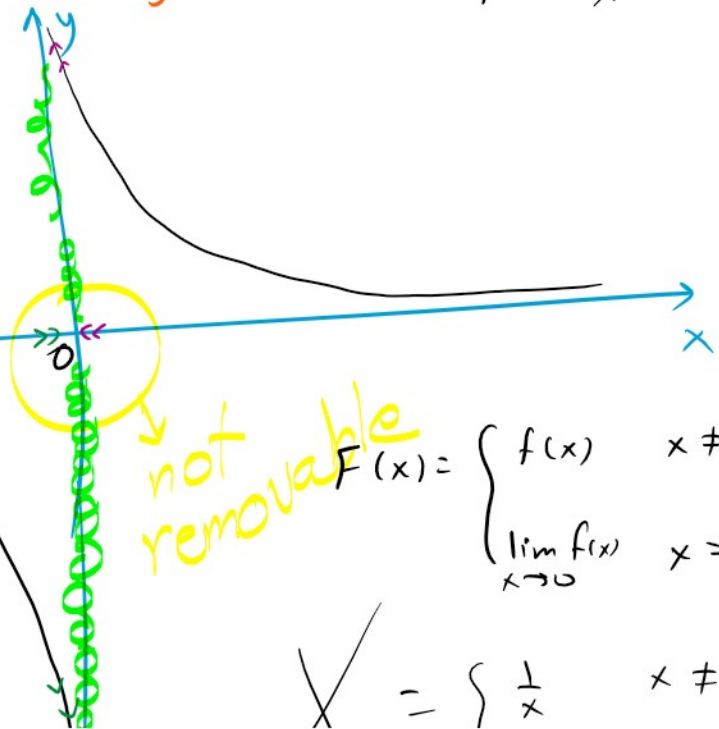
$f(x) = \frac{1}{x}$

① $\lim_{x \rightarrow 0^+} f(x) = \infty$

② $\lim_{x \rightarrow 0^-} f(x) = -\infty$

$\lim_{x \rightarrow 0} f(x)$ DNE

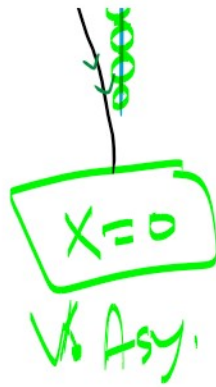
$\lim_{x \rightarrow 0} f(x)$ DNE



$$F(x) = \begin{cases} f(x) & x \neq 0 \\ \lim_{x \rightarrow 0} f(x) & x = 0 \end{cases}$$

$$f(x) = \begin{cases} \frac{1}{x} & x \neq 0 \end{cases}$$

$\lim_{x \rightarrow 0} \frac{1}{x}$ DNE



$$X = \begin{cases} \frac{1}{x} & x \neq 0 \\ \text{DNE} & x = 0 \end{cases}$$

Ex 1 $f(x) = \frac{x}{x-1}$

① $\lim_{x \rightarrow 0} f(x) = \frac{0}{0-1}$
 $= \frac{0}{-1}$
 $= 0$

f cont. at $x=0$

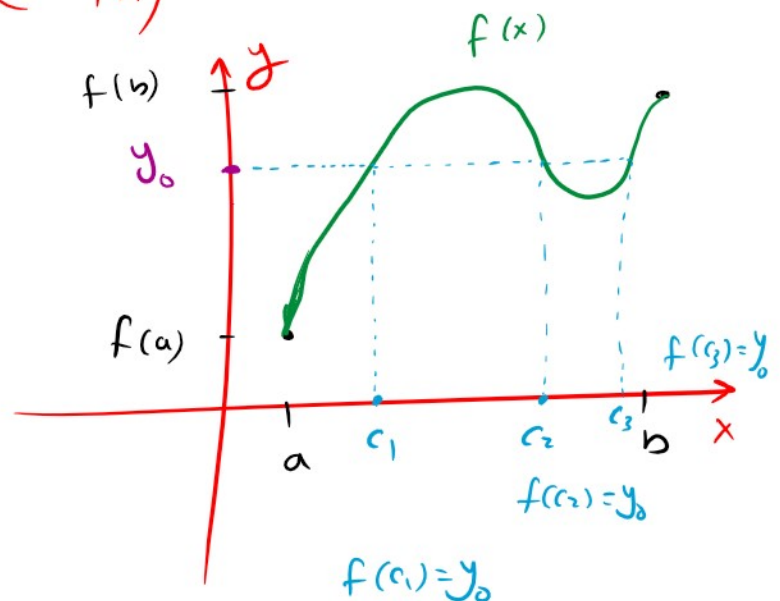
$D(f) = \mathbb{R} \setminus \{1\}$

Is $x=1$ removable? $\Rightarrow f(1) = \frac{1}{0}$

No \Rightarrow $x=1$ is v. Asy.

Th (Intermediate Value Th) - IVT

- ✓ f cont on $[a, b]$
- ✓ $y_0 \in (f(a), f(b))$
- Then \exists at least number $c \in [a, b]$ s.t $f(c) = y_0$

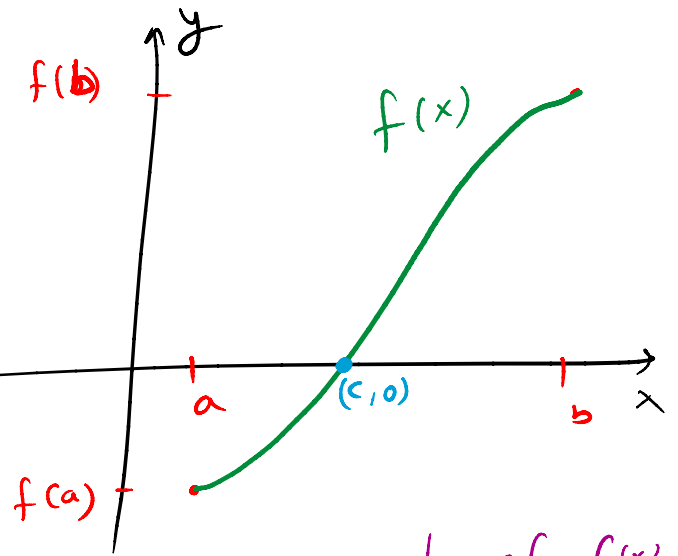


IVT ^{de sign} ^{sign}
 Th (Bolzano Th)
 $y_0 = 0$ ^{is} ^{used}

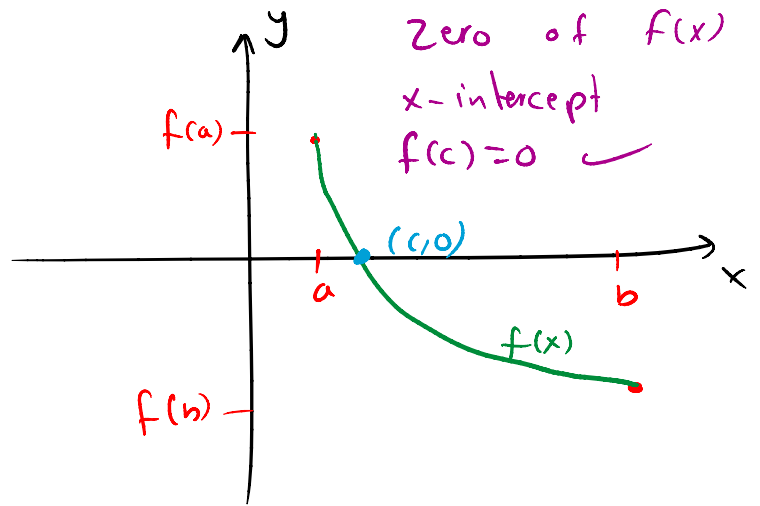
• ✓ f cont. on $[a, b]$ ✓

• ✓ $f(a) f(b) < 0$ ✓

Then \exists number $c \in [a, b]$
 s.t $f(c) = 0$ $\rightarrow y_0$



c root of $f(x)$
 Zero of $f(x)$
 x-intercept
 $f(c) = 0$ ✓



Exp Is there any root for the eq:
 $x^5 - 2x^3 - 2 = 0$??

$f(x) = 0 \Rightarrow f(x) = x^5 - 2x^3 - 2$ cont. on \mathbb{R}
Polynomial

$[a, b] = [0, 2]$

$$[a, b] = [0, 2]$$

$f(a) < 0$ $f(b) > 0$

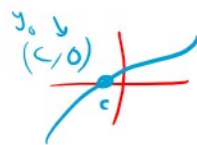
$$f(0) = 0 - 0 - 2 < 0$$

$$f(1) = 1^5 - 2 - 2 = 1 - 2 - 2 < 0$$

$$f(2) = 2^5 - 2(2)^3 - 2 = 32 - 16 - 2 > 0$$

• f cont on $[0, 2]$ } By Bolzano \Rightarrow
 • $f(0) f(2) < 0$ } \exists a number $c \in (0, 2)$
 s.t $f(c) = 0$

\Rightarrow root c



Exp $f(x) = x^2 - 9$ on $[0, 4]$

Does f intersect x-axis in $[0, 4]$

Bolzano

f cont. on $[0, 4]$

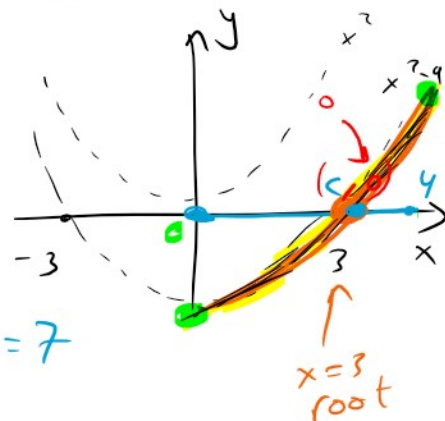
poly.

$f(0) f(4) < 0$

$$f(0) = 0^2 - 9 = -9$$

$$f(4) = 4^2 - 9 = 16 - 9 = 7$$

$(-9)(7) < 0$



By Bolzano $\Rightarrow \exists c \in (0, 4)$ s.t. $f(c) = 0$

\uparrow
 $c=3$

$f(3) = 0$

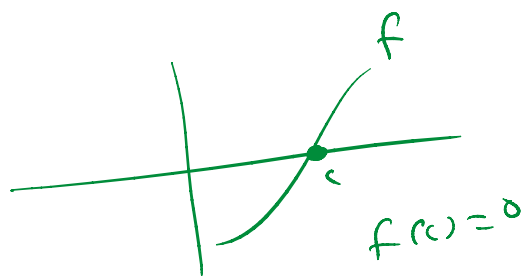
Apply IVT $f(x) = x^2 - 9$, $[0, 4]$

- f cont on $[0, 4]$
- $y_0 = 0 \in (f(0), f(4)) \Rightarrow y_0 = 0 \in (-9, 7)$



Then $\exists c \in (a, b)$ s.t. $f(c) = y_0$

$f(c) = 0$



c root