$\boxed{10.6}$ Alternating Series

The alternating series has the form

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (-1)^{n+1} U_n = U_1 - U_2 + U_3 - U_4 + \cdots \quad *$$

Question: when $*$ conv?

Th (Alternating Series Test - AST)

$*$ conv. if

1. $U_n > 0 \;\; \forall n$ and
2. $U_{n+1} \leq U_n$ for large n " $U_n \downarrow$ " and
3. $\lim_{n \to \infty} U_n = 0$

Exp (Alternating Harmonic Series)

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

$U_n = |a_n|$

① $U_n = |a_n| > 0$

② $U_n = \frac{1}{n} \downarrow$

③ $\lim_{n\to\infty} U_n = \lim_{n\to\infty} \frac{1}{n} = 0$

Hence, $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}$ conv. by AST

Exp (Alternating Geometric Series)

$$\sum_{n=1}^{\infty} (-1)^n (0.2)^n = -\frac{2}{10} + \left(\frac{2}{10}\right)^2 - \left(\frac{2}{10}\right)^3 + \cdots$$

We know it conv. to $\dfrac{-\frac{2}{10}}{1 - -\frac{2}{10}} = \dfrac{-1}{6}$

① $U_n = |a_n| > 0 \quad \forall n$

② $U_n = \left(\frac{2}{10}\right)^n \downarrow$

③ $\lim_{n\to\infty} U_n = \lim_{n\to\infty} (0.2)^n = 0$ ⟶ 10.1 ⇒ Th 5

Hence, $\sum_{n=1}^{\infty} (-1)^n (0.2)^n$ conv. by AST

Exp $\sum_{n=1}^{\infty} (-1)^n n$ this alternating series div by the n^{th} term test

since $\lim_{n\to\infty} u_n = \lim_{n\to\infty} n \neq 0$

Exp $\sum (-1)^n \frac{n}{n+3}$ " " " " " " " " " "

" $\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{n}{n+3} = 1 \neq 0$

Def (Conv. Abs.)

The infinite series $\sum a_n$ conv. Abs. if $\sum |a_n|$ conv.

Exp $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^2}$ conv. by AST since

① $u_n > 0$ ② $u_n \downarrow$ ③ $\lim_{n\to\infty} \frac{1}{n^2} = 0$

Is it converging Absolutely ?

$\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{1}{n^2}$ → Conv. p-Serie

$\Rightarrow$ so $\sum (-1)^{n+1} \frac{1}{n^2}$ conv. Abs.

Th If $\sum |a_n|$ conv. then $\sum a_n$ conv.

This means if a series conv. Abs. then it converges

Exp $\sum_{n=0}^{\infty} (-1)^{n+1} \frac{1}{2^n}$ "Alternating Geometric Series"

$$\sum |a_n| = \sum_{n=0}^{\infty} \frac{1}{2^n} = 1 + \frac{1}{2} + \frac{1}{2^2} + \cdots = \frac{1}{1-\frac{1}{2}} = \frac{1}{\frac{1}{2}} = 2$$

$\sum |a_n|$ converges to 2 $\Rightarrow \sum a_n = \sum (-1)^{n+1} \frac{1}{2^n}$ conv. "by AST"

Remark The Converse of Th above is not True:
Means: if $\sum a_n$ conv. $\not\Rightarrow \sum |a_n|$ conv.

Exp $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}$ "Alternating Harmonic Series"
This series converges by AST but not Abs. since

$$\sum |a_n| = \sum_{n=1}^{\infty} \frac{1}{n}$$ which is the divergent Harmonic Series.

This Remark says:

If a series converges, then it may not converges Abs.

<u>Exp</u> Does $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}$ Conv. Abs. ?

This is the alternating harmonic series
we have seen that it conv. by AST

But not Abs. since

$\sum |a_n| = \sum_{n=1}^{\infty} \frac{1}{n}$ which is the divergent harmonic series.

<u>Def</u> (Converge Conditionally)

The infinite series $\sum a_n$ conv. Cond.
if it conv. by AST but not Abs.

see exp. Above

<u>Exp</u> $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}$ conv. cond.

<u>Exp</u> $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ conv. by AST since $\boxed{1}$ ✓ $\boxed{2}$ ✓ $\boxed{3}$ ✓

but not Abs since $\sum |a_n| = \sum \frac{1}{\sqrt{n}}$

div. p- series

This series conv. <u>Condit.</u> ⟸

Remark $$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^p} = \begin{cases} \text{conv. Abs.} & \text{if } p > 1 \\ \text{conv. Condi} & \text{if } 0 < p \leq 1 \end{cases}$$

Th (Alternating Estimation Th)

- Assume $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (-1)^{n+1} u_n = u_1 - u_2 + u_3 + \cdots = L$

- If we approximate L by $S_n = u_1 - u_2 + u_3 - \cdots + (-1)^{n+1} u_n$

then ① the remainder $\boxed{L - S_n}$ has same sign as a_{n+1}

② the error $|L - S_n| < u_{n+1} = |a_{n+1}|$

③ $\min\{S_n, S_{n+1}\} < L < \max\{S_n, S_{n+1}\}$ $\forall n$

Exp $\sum_{n=1}^{\infty} (-1)^{n+1} \left(\frac{2}{3}\right)^n = \frac{2}{3} - \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^3 - \left(\frac{2}{3}\right)^4 + \cdots$

$$= \frac{\frac{2}{3}}{1 - \frac{-2}{3}} = \frac{4}{10} = 0.4 = L$$

If we approximate $L = 0.4$ by $S_3 = \frac{2}{3} - \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^3$

$$= \frac{14}{27} \approx 0.519$$

$L = 0.4$

$S_3 \approx 0.519$ $n = 3$

1. Remainder $L - S_n = 0.4 - 0.519 = -0.119$

$$a_{n+1} = a_{3+1} = a_4 = -\left(\frac{2}{3}\right)^4$$

2. Error $= |L - S_n| = |L - S_3| = |-0.119|$

$$E = 0.119 < U_4 = |a_4| = \left(\frac{2}{3}\right)^4 = 0.198$$

3. $S_4 = \left(\frac{1}{3}\right) - \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^3 - \left(\frac{2}{3}\right)^4 \approx 0.321$

$$\min\{S_3, S_4\} < L < \max\{S_3, S_4\}$$

$$\min\{0.519, 0.321\} < 0.4 < \max\{0.519, 0.321\}$$

$$0.321 < 0.4 < 0.519$$ ✓

Exp Approximate the sum of $\sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n)!}$ with error of magnitude less than 5×10^{-6}

- We use S_n to approx. the sum
- So we need to find $n \Rightarrow \frac{1}{(2n)!} < 5 \times 10^{-6}$

$$(2n)! > \frac{1}{5 \times 10^{-6}} = \frac{10^6}{5} = \frac{(10)(10^5)}{5} = 200{,}000$$

$$(2n)! > 200{,}000 \Rightarrow n \geq 5$$

- $S_5 = 1 - \frac{1}{2!} + \frac{1}{4!} - \frac{1}{6!} + \frac{1}{8!} \approx 0.54$
- Error $= |L - S_n| = |L - S_5| < u_6 = \frac{1}{(10)!} = 0.275 \times 10^{-6}$

10.6 Lecture Problems

14, 19, 29

Conv/Div

(14) $\sum_{n=1}^{\infty} \underbrace{(-1)^{n+1} \frac{3\sqrt{n+1}}{\sqrt{n}+1}}_{a_n}$ $\qquad U_n = |a_n|$

$$U_n = \frac{3\sqrt{n+1}}{\sqrt{n}+1}$$

$3 \to$ $$\lim_{n\to\infty} U_n = \lim_{n\to\infty} \frac{3\sqrt{n+1}}{\sqrt{n}+1} = \lim_{n\to\infty} \frac{\frac{3\sqrt{n+1}}{\sqrt{n}}}{\frac{\sqrt{n}+1}{\sqrt{n}}}$$

$$= \lim_{n\to\infty} \frac{3\sqrt{\frac{n+1}{n}}}{1+\frac{1}{\sqrt{n}}}$$

$$= \lim_{n\to\infty} \frac{3\sqrt{1+\frac{1}{n}}}{1+\frac{1}{\sqrt{n}}} = 3 \neq 0$$

so the alternating series div by n^{th} term test

(19) $\sum_{n=1}^{\infty} (-1)^{n+1} \underbrace{\frac{n}{n^3+1}}_{u_n}$ This is alternating

1. $u_n > 0$ ✓
2. $u_n \downarrow$ ✓
3. $\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{n}{n^3+1} = 0$ ✓

so this series conv. by AST

To check if it conv. Abs. $\Rightarrow$

$$\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{n}{n^3+1} < \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$\rightarrow$ Convergent p-series

$\Downarrow$

$\sum |a_n|$ conv. by DCT

$\Downarrow$

$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{n^3+1}$ conv. Abs.

$$n^3+1 > n^3$$

$$\frac{1}{n^3+1} < \frac{1}{n^3}$$

$$\frac{n}{n^3+1} < \frac{n}{n^3} = \frac{1}{n^2}$$

(29) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{\tan^{-1} n}{n^2+1}$ This is alternating

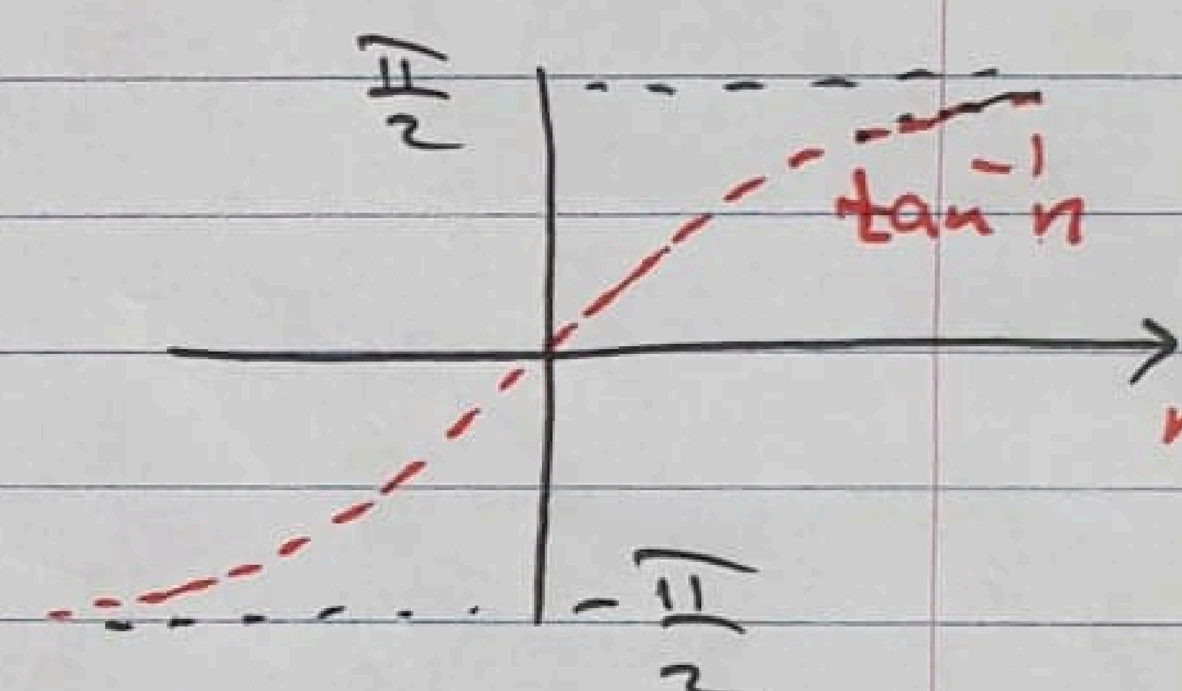
$\dfrac{\tan^{-1} n}{n^2+1} \to u_n$

- If the alternating series conv. Abs. then it conv.
- That is if $\sum |a_n|$ conv. then $\sum a_n$ conv. Abs.

$$\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{\tan^{-1} n}{n^2+1}$$

cont. , +, ↓ on $[1, \infty)$

we use IT



$$\int_1^{\infty} \frac{\tan^{-1} x}{x^2+1}\,dx = \lim_{b\to\infty} \int_1^{b} \frac{\tan^{-1} x}{x^2+1}\,dx$$

$u = \tan^{-1} x$

$du = \dfrac{dx}{x^2+1}$

$x = 1 \Rightarrow u = \tan^{-1} 1 = \frac{\pi}{4}$

$x = b \Rightarrow u = \tan^{-1} b$

$$= \lim_{b\to\infty} \int_{\frac{\pi}{4}}^{\tan^{-1} b} u\,du$$

$$= \lim_{b\to\infty} \frac{u^2}{2} \Big|_{\frac{\pi}{4}}^{\tan^{-1} b}$$

$$= \frac{1}{2} \lim_{b\to\infty} \left[\left(\tan^{-1} b\right)^2 - \left(\frac{\pi}{4}\right)^2 \right]$$

$$= \frac{1}{2} \left[\left(\frac{\pi}{2}\right)^2 - \left(\frac{\pi}{4}\right)^2 \right] = \frac{3\pi^2}{32}$$

so $\sum (-1)^n \dfrac{\tan^{-1} n}{n^2+1}$ conv. Abs.