1.2 Row Echelon Form

In Section 1.1 we learned a method for reducing an $n \times n$ linear system to strict triangular form. However, this method will fail if, at any stage of the reduction process, all the possible choices for a pivot element in a given column are 0.

Definition

A matrix is said to be in **row echelon form** if

(i) The first nonzero entry in each nonzero row is 1.

(ii) The first non-zero element of a row lies on the right of the first non-zero element of the previous row.

(iii) All zero-rows (all elements zero) are below non-zero rows.

EXAMPLE 2 The following matrices are in row echelon form:

$$\begin{bmatrix} 1 & 4 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 3 & 1 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

■

EXAMPLE 3 The following matrices are not in row echelon form:

$$\begin{bmatrix} 2 & 4 & 6 \\ 0 & 3 & 5 \\ 0 & 0 & 4 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

The first matrix does not satisfy condition (i). The second matrix fails to satisfy condition (iii), and the third matrix fails to satisfy condition (ii). ■

Definition The process of using row operations I, II, and III to transform a linear system into one whose augmented matrix is in row echelon form is called **Gaussian elimination**.

Overdetermined Systems

A linear system is said to be *overdetermined* if there are more equations than unknowns.

EXAMPLE 4 Solve each of the following overdetermined systems:

(a)
$$\begin{aligned} x_1 + x_2 &= 1 \\ x_1 - x_2 &= 3 \\ -x_1 + 2x_2 &= -2 \end{aligned}$$

(b)
$$\begin{aligned} x_1 + 2x_2 + x_3 &= 1 \\ 2x_1 - x_2 + x_3 &= 2 \\ 4x_1 + 3x_2 + 3x_3 &= 4 \\ 2x_1 - x_2 + 3x_3 &= 5 \end{aligned}$$

(c)
$$\begin{aligned} x_1 + 2x_2 + x_3 &= 1 \\ 2x_1 - x_2 + x_3 &= 2 \\ 4x_1 + 3x_2 + 3x_3 &= 4 \\ 3x_1 + x_2 + 2x_3 &= 3 \end{aligned}$$

Solution

By now the reader should be familiar enough with the elimination process that we can omit the intermediate steps in reducing each of these systems. Thus, we may write

$$\text{System (a):}\quad \left[\begin{array}{rr|r} 1 & 1 & 1 \\ 1 & -1 & 3 \\ -1 & 2 & -2 \end{array}\right] \rightarrow \left[\begin{array}{rr|r} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{array}\right]$$

It follows from the last row of the reduced matrix that the system is inconsistent. The

$$\text{System (b):}\quad \left[\begin{array}{rrr|r} 1 & 2 & 1 & 1 \\ 2 & -1 & 1 & 2 \\ 4 & 3 & 3 & 4 \\ 2 & -1 & 3 & 5 \end{array}\right] \rightarrow \left[\begin{array}{rrr|r} 1 & 2 & 1 & 1 \\ 0 & 1 & \frac{1}{5} & 0 \\ 0 & 0 & 1 & \frac{3}{2} \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Using back substitution, we see that system (b) has exactly one solution $(0.1, -0.3, 1.5)$. The solution is unique because the nonzero rows of the reduced matrix form a strictly triangular system.

$$\text{System (c):}\quad \left(\begin{array}{rrr|r} 1 & 2 & 1 & 1 \\ 2 & -1 & 1 & 2 \\ 4 & 3 & 3 & 4 \\ 3 & 1 & 2 & 3 \end{array}\right) \to \left(\begin{array}{rrr|r} 1 & 2 & 1 & 1 \\ 0 & 1 & \frac{1}{5} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Solving for x_2 and x_1 in terms of x_3, we obtain

$$\begin{aligned} x_2 &= -0.2x_3 \\ x_1 &= 1 - 2x_2 - x_3 = 1 - 0.6x_3 \end{aligned}$$

It follows that the solution set is the set of all ordered triples of the form $(1 - 0.6\alpha, -0.2\alpha, \alpha)$, where α is a real number. This system is consistent and has infinitely many solutions because of the free variable x_3. ■

Underdetermined Systems

A system of m linear equations in n unknowns is said to be *underdetermined* if there are fewer equations than unknowns ($m < n$).

EXAMPLE 5 Solve the following underdetermined systems:

$$\textbf{(a)}\quad \begin{aligned} x_1 + 2x_2 + x_3 &= 1 \\ 2x_1 + 4x_2 + 2x_3 &= 3 \end{aligned}$$

Solution

$$\text{System (a):}\quad \left(\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 2 & 4 & 2 & 3 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right)$$

Clearly, system (a) is inconsistent.

Reduced Row Echelon Form

Definition

A matrix is said to be in **reduced row echelon form** if

(i) The matrix is in row echelon form.
(ii) The first nonzero entry in each row is the only nonzero entry in its column.

The following matrices are in reduced row echelon form:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The process of using elementary row operations to transform a matrix into reduced row echelon form is called *Gauss–Jordan reduction*.

EXAMPLE 6 Use Gauss–Jordan reduction to solve the system

$$\begin{aligned} -x_1 + x_2 - x_3 + 3x_4 &= 0 \\ 3x_1 + x_2 - x_3 - x_4 &= 0 \\ 2x_1 - x_2 - 2x_3 - x_4 &= 0 \end{aligned}$$

Solution

$$\left[\begin{array}{rrrr|r} \mathbf{-1} & \mathbf{1} & \mathbf{-1} & \mathbf{3} & \mathbf{0} \\ 3 & 1 & -1 & -1 & 0 \\ 2 & -1 & -2 & -1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{rrrr|r} -1 & 1 & -1 & 3 & 0 \\ \mathbf{0} & \mathbf{4} & \mathbf{-4} & \mathbf{8} & \mathbf{0} \\ 0 & 1 & -4 & 5 & 0 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{rrrr|r} -1 & 1 & -1 & 3 & 0 \\ 0 & 4 & -4 & 8 & 0 \\ 0 & 0 & -3 & 3 & 0 \end{array}\right] \rightarrow \left[\begin{array}{rrrr|r} 1 & -1 & 1 & -3 & 0 \\ 0 & 1 & -1 & 2 & 0 \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{-1} & \mathbf{0} \end{array}\right] \quad \begin{array}{l} \text{row} \\ \text{echelon} \\ \text{form} \end{array}$$

$$\rightarrow \left[\begin{array}{rrrr|r} 1 & -1 & 0 & -2 & 0 \\ \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ 0 & 0 & 1 & -1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{rrrr|r} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{array}\right] \quad \begin{array}{l} \text{reduced} \\ \text{row echelon} \\ \text{form} \end{array}$$

If we set x_4 equal to any real number α, then $x_1 = \alpha$, $x_2 = -\alpha$, and $x_3 = \alpha$. Thus, all ordered 4-tuples of the form $(\alpha, -\alpha, \alpha, \alpha)$ are solutions of the system.

Remark:

To solve a linear system, if the method is not specified, then we solve the linear system as following:

1. Form the augmented matrix.
2. Reduce the augmented matrix to row echelon form.
3. If the system has no solution, the solution is completed.

   i.e., if we have a row consisting entirely of zeros except the most-right element which is not zero

4. If the system has a unique solution, we complete the solution using Gaussian Elimination.

   i.e., if the system has a solution, and in row echelon form, the number of non-zeros rows is equal to the number of unknowns.

5. If the system has infinite number of solutions, we proceed to reduced row echelon form and use Gauss-Jordan Elimination.

   i.e., if the system has a solution, and in row echelon form, the number of non-zeros rows is less than the number of unknowns.

EXAMPLE 5 Solve the following underdetermined systems:

$$\textbf{(b)}\quad \begin{aligned} x_1 + x_2 + x_3 + x_4 + x_5 &= 2 \\ x_1 + x_2 + x_3 + 2x_4 + 2x_5 &= 3 \\ x_1 + x_2 + x_3 + 2x_4 + 3x_5 &= 2 \end{aligned}$$

Solution

$$\text{System (b):}\quad \left(\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 2 \\ 1 & 1 & 1 & 2 & 2 & 3 \\ 1 & 1 & 1 & 2 & 3 & 2 \end{array}\right) \rightarrow \left(\begin{array}{ccccc|r} 1 & 1 & 1 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{array}\right)$$

System (b) is consistent, and since there are two free variables, the system will have infinitely many solutions. In cases such as these it is convenient to continue the elimination process and simplify the form of the reduced matrix even further. We continue eliminating until all the terms above each leading 1 are eliminated. Thus, for system (b), we will continue and eliminate the first two entries in the fifth column and then the first element in the fourth column.

$$\left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{-1} \end{array}\right] \rightarrow \left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 0 & 3 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{2} \\ 0 & 0 & 0 & 0 & 1 & -1 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccccc|c} 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{array}\right]$$

If we put the free variables over on the right-hand side, it follows that

$$\begin{aligned} x_1 &= 1 - x_2 - x_3 \\ x_4 &= 2 \\ x_5 &= -1 \end{aligned}$$

Thus, for any real numbers α and β, the 5-tuple

$$(1 - \alpha - \beta, \alpha, \beta, 2, -1)$$

is a solution of the system.

Homogeneous systems

This is a class of systems of linear equations which will be used in many topics in the later part of the course.

Definition:

A system of linear equations is said to be homogeneous if all the constant terms on right hand side are zero.

Example:

A homogeneous system of n variables and m equations is given by

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots\cdots + a_{1n}x_n &= 0 \\
a_{21}x_1 + a_{22}x_2 + \cdots\cdots + a_{2n}x_n &= 0 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots\cdots + a_{mn}x_n &= 0
\end{aligned}$$

Solving homogeneous systems

Nothing new. Similar to what we have done in examples above.

Important facts about solutions of homogeneous systems

About solutions of homogeneous systems (in general)

1) Always consistent
   - Always has at least trivial solution $x_1 = 0, x_2 = 0, \cdots, x_n = 0$ (Why?)

2) Hence has infinitely many (non-trivial) solutions only if the echelon form or reduced echelon form has a free variable

In echelon form, we cannot have a row of the form $[0 \;\; 0 \;\; \cdots\cdots \;\; 0 \;\; b]$ with b non-zero

Recall: a consistent system has unique solution if it has no free variable. Otherwise it has infinitely many solutions.

Given a homogeneous system with n variables.
If the echelon form of its augmented matrix has r nonzero rows
then the system has $n-r$ free variables.

Question

Given a homogeneous linear system with more unknowns and less equations.

Will it always have infinitely many solutions?

Exercise

The *echelon form of the augmented matrix* of a linear system is given by

$$\begin{array}{cccc} x_1 & x_2 & x_3 & \\ \end{array}$$

$$\begin{bmatrix} \boxed{1} & 3 & -6 & -5 \\ 0 & \boxed{1} & 2 & 4 \\ 0 & 0 & \boxed{1} & -2 \\ 0 & 0 & 0 & \boxed{1} \end{bmatrix}. \qquad (*)$$

What can you say about the solution? Do you need to convert it to reduced echelon form to give the answer of question?

Solution Answer: No solution of the system.

Recall that if the echelon form contains a row of the form

$$\begin{bmatrix} 0 & 0 & \cdots\cdots & 0 & b \end{bmatrix} \text{with } b \text{ non-zero}$$

then the system is inconsistent (no solution).

So while trying to convert an augmented matrix into reduced echelon form if you see a row of the form

$$\begin{bmatrix} 0 & 0 & \cdots\cdots & 0 & b \end{bmatrix} \text{with } b \text{ non-zero}$$

then you can stop and conclude immediately that the system has no solution.

Exercise Solve the linear system

$$\begin{aligned} v + 3w - 2x &= 0 \\ 2u + v - 4w + 3x &= 0 \\ 2u + 3v + 2w - x &= 0 \\ -4u - 3v + 5w - 4x &= 0 \end{aligned}$$

$$\begin{bmatrix} 0 & 1 & 3 & -2 & 0 \\ 2 & 1 & -4 & 3 & 0 \\ 2 & 3 & 2 & -1 & 0 \\ -4 & -3 & 5 & -4 & 0 \end{bmatrix}, \xrightarrow{R_{14}} \begin{bmatrix} -4 & -3 & 5 & -4 & 0 \\ 2 & 1 & -4 & 3 & 0 \\ 2 & 3 & 2 & -1 & 0 \\ 0 & 1 & 3 & -2 & 0 \end{bmatrix}, \xrightarrow{-\frac{1}{4}R_1} \begin{bmatrix} 1 & \frac{3}{4} & -\frac{5}{4} & 1 & 0 \\ 2 & 1 & -4 & 3 & 0 \\ 2 & 3 & 2 & -1 & 0 \\ 0 & 1 & 3 & -2 & 0 \end{bmatrix}$$

$$\xrightarrow{R_2 \to R_2 - 2R_1} \begin{bmatrix} 1 & \frac{3}{4} & -\frac{5}{4} & 1 & 0 \\ 0 & -\frac{1}{2} & -\frac{3}{2} & 1 & 0 \\ 2 & 3 & 2 & -1 & 0 \\ 0 & 1 & 3 & -2 & 0 \end{bmatrix}, \xrightarrow{R_3 \to R_3 - 2R_1} \begin{bmatrix} 1 & \frac{3}{4} & -\frac{5}{4} & 1 & 0 \\ 0 & -\frac{1}{2} & -\frac{3}{2} & 1 & 0 \\ 0 & \frac{3}{2} & \frac{9}{2} & -3 & 0 \\ 0 & 1 & 3 & -2 & 0 \end{bmatrix}, \xrightarrow{-2R_2} \begin{bmatrix} 1 & \frac{3}{4} & -\frac{5}{4} & 1 & 0 \\ 0 & 1 & 3 & -2 & 0 \\ 0 & \frac{3}{2} & \frac{9}{2} & -3 & 0 \\ 0 & 1 & 3 & -2 & 0 \end{bmatrix}$$

$$R_3 \to R_3 - \tfrac{3}{2}R_2 \quad \begin{bmatrix} 1 & \frac{3}{4} & -\frac{5}{4} & 1 & 0 \\ 0 & 1 & 3 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 3 & -2 & 0 \end{bmatrix}, \quad R_4 \to R_4 - R_2 \quad \begin{array}{c} \begin{array}{cccc} u & v & w & x \end{array} \\ \begin{array}{cc} & \text{free} \end{array} \\ \begin{bmatrix} 1 & \frac{3}{4} & -\frac{5}{4} & 1 & 0 \\ 0 & 1 & 3 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}, \quad R_1 \to R_1 - \tfrac{3}{4}R_2 \quad \begin{array}{c} \begin{array}{cccc} u & v & w & x \end{array} \\ \begin{array}{cc} & \text{free} \end{array} \\ \begin{bmatrix} 1 & 0 & -\frac{7}{2} & \frac{5}{2} & 0 \\ 0 & 1 & 3 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

row e.f.

$$\left.\begin{array}{l} u - \frac{7}{2}w + \frac{5}{2}x = 0 \\ v + 3w - 2x = 0 \end{array}\right\} \Rightarrow \begin{array}{l} x = r \\ w = s \\ u = \frac{7}{2}s - \frac{5}{2}r \\ v = -3s + 2r \end{array} \quad ; r, s \in \mathbb{R}$$

Exercise What condition, if any, must b_1, b_2, and b_3 satisfy in order for the following system of equations to be consistent?

$$
\begin{aligned}
x_1 + x_2 + 2x_3 &= b_1 \\
x_1 \qquad\quad + x_3 &= b_2 \\
2x_1 + x_2 + 3x_3 &= b_3
\end{aligned}
$$

$$\begin{bmatrix} 1 & 1 & 2 & b1 \\ 1 & 0 & 1 & b2 \\ 2 & 1 & 3 & b3 \end{bmatrix} \xrightarrow{R_2 \to R_2 - R_1} \begin{bmatrix} 1 & 1 & 2 & b1 \\ 0 & -1 & -1 & b2 - b1 \\ 2 & 1 & 3 & b3 \end{bmatrix} \xrightarrow{R_3 \to R_3 - 2R_1} \begin{bmatrix} 1 & 1 & 2 & b1 \\ 0 & -1 & -1 & b2 - b1 \\ 0 & -1 & -1 & b3 - 2\,b1 \end{bmatrix}$$

$$\xrightarrow{-R_2} \begin{bmatrix} 1 & 1 & 2 & b1 \\ 0 & 1 & 1 & -b2 + b1 \\ 0 & -1 & -1 & b3 - 2\,b1 \end{bmatrix} \xrightarrow{R_3 \to R_3 + R_2} \begin{bmatrix} 1 & 1 & 2 & b1 \\ 0 & 1 & 1 & -b2 + b1 \\ 0 & 0 & 0 & b3 - b1 - b2 \end{bmatrix}$$

Case 1 : $b_3 - b_1 - b_2 \neq 0$

$$\xrightarrow{\left(\frac{1}{b_3-b_1-b_2}\right)R_3}
\begin{array}{cccc}
x_1 & x_2 & x_3 & \\
\left[\begin{array}{cccc}
1 & 1 & 2 & b_1 \\
0 & 1 & 1 & -b_2+b_1 \\
0 & 0 & 0 & 1
\end{array}\right]
\end{array}$$

$\therefore$ There is no solution.

Case 2 : $b_3 - b_1 - b_2 = 0$

$$\begin{array}{cccc}
x_1 & x_2 & x_3 & \\
\left[\begin{array}{cccc}
1 & 1 & 2 & b_1 \\
0 & 1 & 1 & -b_2+b_1 \\
0 & 0 & 0 & 0
\end{array}\right]
\end{array}$$

$\therefore$ There are inf. many solutions.

Exercise Determine the values of a for which the following system has no solutions, exactly one solution, or infinitely many solutions.

$$\begin{aligned} x + 2y - 3z &= 4 \\ 3x - y + 5z &= -2 \\ 4x + y + (a^2 - 14)z &= a - 2 \end{aligned}$$

$$\begin{bmatrix} 1 & 2 & -3 & 4 \\ 3 & -1 & 5 & -2 \\ 4 & 1 & a^2-14 & a-2 \end{bmatrix} \xrightarrow{R_2 \to R_2 - 3R_1} \begin{bmatrix} 1 & 2 & -3 & 4 \\ 0 & -7 & 14 & -14 \\ 4 & 1 & a^2-14 & a-2 \end{bmatrix} \xrightarrow{R_3 \to R_3 - 4R_1} \begin{bmatrix} 1 & 2 & -3 & 4 \\ 0 & -7 & 14 & -14 \\ 0 & -7 & a^2-2 & a-18 \end{bmatrix}$$

$$\xrightarrow{-\frac{1}{7}R_2} \begin{bmatrix} 1 & 2 & -3 & 4 \\ 0 & 1 & -2 & 2 \\ 0 & -7 & a^2-2 & a-18 \end{bmatrix} \xrightarrow{R_3 \to R_3 + 7R_2} \begin{bmatrix} 1 & 2 & -3 & 4 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & a^2-16 & a-4 \end{bmatrix} \xrightarrow{R_1 \to R_1 - 2R_2} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & a^2-16 & a-4 \end{bmatrix}$$

Case 1: $a^2-16 \neq 0$

$$\xrightarrow{\left(\frac{1}{a^2-16}\right)R_3} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & 1 & \frac{a-4}{a^2-16} \end{bmatrix} \xrightarrow{R_1 \to R_1 - R_3} \begin{bmatrix} 1 & 0 & 0 & -\frac{a-4}{a^2-16} \\ 0 & 1 & -2 & 2 \\ 0 & 0 & 1 & \frac{a-4}{a^2-16} \end{bmatrix} \xrightarrow{R_2 \to R_2 + 2R_3} \begin{array}{c} \begin{matrix} x & y & z & \phantom{-\frac{1}{a+4}} \end{matrix} \\ \begin{bmatrix} 1 & 0 & 0 & -\frac{1}{a+4} \\ 0 & 1 & 0 & \frac{2(a+5)}{a+4} \\ 0 & 0 & 1 & \frac{1}{a+4} \end{bmatrix} \end{array}$$

$\therefore$ There is one solution.

Case 2: $a^2 - 16 = 0$

subcase 2.1: $(a = 4)$

$$\begin{array}{cccc} x & y & z \leftarrow \text{free} & \\ \left[\begin{array}{cccc} 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

$\therefore$ There are inf. solutions.

subcase 2.2: $(a = -4)$

$$\left[\begin{array}{cccc} 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & 0 & -8 \end{array}\right]$$

$\therefore$ there is no solution.