CH3 Differentiation

1) Sect

Def: The derivative of a function f at a denoted by $f'(a)$ is

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

provided this limit exist.

* If $f'(a)$ exists we say that f is differentiable at a
* We say f is differentiable on a closed interval $[a,b]$

iff 1) f' exists for all points in the open interval (a,b)

2) the right hand derivative of f at a exists, i.e

$$\lim_{h \to 0^+} \frac{f(a+h) - f(a)}{h} \text{ exists}$$

and we denote it by $f'_+(a)$

3) The left hand derivative of f at b exists, i.e

$$\lim_{h \to 0^-} \frac{f(b+h) - f(b)}{h} \text{ exists}$$

and we denote it by $f'_-(b)$

Remark(1): A function f is differentiable at $x = c$ iff the left hand derivative and the right hand derivative both exists at $x = c$ and are equall

Remark 2: If f is differentiable at $x=c$ then f is continous at $x=c$ But the converse is not true.

Example: The function $f(x)=|x|$ is continous at $x=0$, But is not diff. at $x=0$, because

$$f'_+(0)=\lim_{h\to 0^+}\frac{f(0+h)-f(0)}{h}$$

$$=\lim_{h\to 0^+}\frac{|h|-0}{h}=\lim_{h\to 0^+}\frac{h}{h}=1$$

$$f'_-(0)=\lim_{h\to 0^-}\frac{|f(0+h)|-f(0)}{h}$$

$$=\lim_{h\to 0^-}\frac{|h|-0}{h}=\lim_{h\to 0^-}\frac{-h}{h}=-1$$

$\Rightarrow f$ is not diff at $x=0$

sect (3)

Differentiation Rules: If $f(x)$ and $g(x)$ are diff. at x then

1) $(f(x)\pm g(x))'=f'(x)\pm g'(x)$ Sum Rule
2) $(f(x)\,g(x))'=f'(x)\,g(x)+f(x)\,g'(x)$ product Rule
3) $\left(\frac{f(x)}{g(x)}\right)'=\frac{f'(x)g(x)-f(x)g'(x)}{(g(x))^2}$ Quotient
4) $(f\circ g)'(x)=(f(g(x)))'$ chain Rule

$$=f'(g(x))\cdot g'(x)$$

Sec 3.4 Derivatives of the Trigonometric functions

1) $(\sin x)' = \cos x$

2) $(\cos x)' = \sin x$

3) $(\tan x)' = \sec^2 x$

4) $(\sec x)' = \sec x \tan x$

5) $(\csc x)' = -\csc x \cot x$

6) $(\cot x)' = -\csc^2 x$

Ex's

1) $\left(\frac{x+1}{x^2+1} \right)' = \frac{(1)(x^2+1) - (x+1)(2x)}{(x^2+1)^2} = \frac{1-x^2-2x}{(x^2+1)^2}$

2) $(\tan(\sqrt{x}))' = \sec^2(\sqrt{x}) \cdot \frac{1}{2\sqrt{x}}$

3) $(\sec x \tan x)' = \sec x \cdot \sec^2 x + \sec x \tan x \cdot \tan x$

$= \sec^3 x + \sec x \tan^2 x$

$\boxed{\text{Ex 4}}$ Find the equation of the tangent line to the curve $f(x) = \sec x \tan x$ at $x = \frac{\pi}{4}$

Soln the slope of the tangent line at $x = \frac{\pi}{4}$

is $f'(\frac{\pi}{4}) = \sec^3(\frac{\pi}{4}) + \sec(\frac{\pi}{4}) \cdot \tan^2(\frac{\pi}{4})$

$= (\sqrt{2})^3 + \sqrt{2} = 2\sqrt{2} + \sqrt{2} = 3\sqrt{2}$

and $f(\frac{\pi}{4}) = \sqrt{2}$ so

$y - \sqrt{2} = 3\sqrt{2}(x - \frac{\pi}{4})$

Ex 4 Find the equation of the tangent line of the curve $f(x) = x^3$ at $x = 2$

Soln the slope of the tangent line at $x=2$ is $f'(2) = 12$ (since $f'(x) = 3x^2$) and $f(2) = 8$ so

$$y - 8 = 12(x-2)$$

Implicit Differentiation: whenever we derive y we multiply by y',

Ex 1 $x^2 + y^2 = 1$

$$\Rightarrow \quad 2x + 2y\,y' = 1 \Rightarrow y' = \frac{-2x}{2y} = \frac{-x}{y}$$

Ex 2 $xy = \cot(xy)$

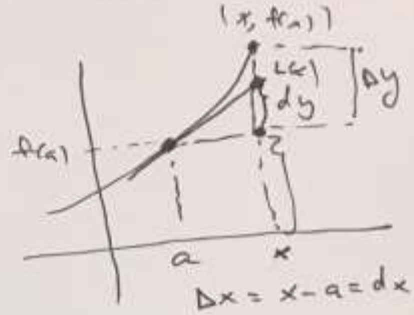
$$1 \cdot y + xy' = -\csc^2(xy)\,(y + xy')$$

$$xy' + \csc^2(xy)\,(xy') = -y - y\csc^2(xy)$$

$$y'(x(1 + \csc^2(xy))) = -y(1 + \csc^2 xy)$$

$$y' = -\frac{y}{x}$$

Linearization and differentials



$L(x) \simeq f(x)$

$L(x) = f(a) + dy$

$\quad = f(a) + f'(a)\,dx$

$L(x) = f(a) + f'(a)(x-a)$ $\qquad \dfrac{dy}{dx} = f'(a)$

$\qquad dy = f'(a)\,dx$

$L(x)$ is called the Linearization of $f(x)$ at $x=a$ and $f(x) \simeq L(x)$ is called the standard linear approximation of f at a.

<u>Ex1</u> The Linearization of $f(x) = \sqrt{1+x}$ at $x=0$

$\qquad f'(x) = \dfrac{1}{2\sqrt{1+x}}$

is $L(x) = 1 + \frac{1}{2}x$

so to approximate $\sqrt{1.2}$, $x = 0.2$,

$\sqrt{1.2} \simeq 1 + \frac{1}{2}(0.2) = 1.1$

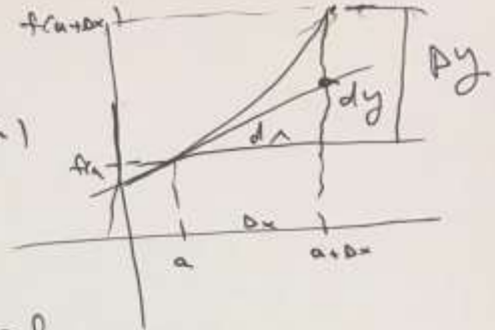
& $\sqrt{1.05} \simeq 1 + \frac{1}{2}(0.05) = 1.025$

<u>Ex</u> The Linearization of $f(x) = \sec x$ at $x = \frac{\pi}{4}$

$L(x) = f(\frac{\pi}{4}) + f'(\frac{\pi}{4})(x - \frac{\pi}{4})$ $\qquad f(\frac{\pi}{4}) = \sqrt{2}$

$\quad = \sqrt{2} + \sqrt{2}\left(x - \frac{\pi}{4}\right)$ $\qquad f'(x) = \sec x \tan x$

$\qquad f'(\frac{\pi}{4}) = \sqrt{2}$



<u>Notice that</u>

$dy \approx \Delta y = f(a+\Delta x) - f(a)$

$\frac{dy}{dx} = f'(a)$

$dy = f'(a)\,dx \approx \Delta y = \Delta f.$

dy is called the differential of f at a

For example if $y = \tan^2 x$

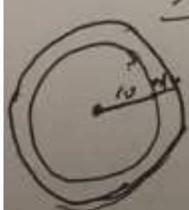
then $dy = 2\tan x \sec^2 x\,dx$

<u>Ex 2</u> The radius r of a circle increases from 10 to 10.1 <u>m</u>. Use dA to estimate the increase in circle area A.

Estimate the area of the enlarged circle and compare it to the true area by direct calculation

<u>Soln</u> since the radius changed from 10 to 10.1

so $dr = 0.1$ m.



since $A = \pi r^2 \Rightarrow dA = 2\pi r\,dr$
$= 2\pi(10)(0.1) = 2\pi m^2$

So the estimate of the area of the enlarged circle

is $A(10.1) \approx A(10) + dA = 100\pi + 2\pi = 102\pi$

But the exact area of the enlarged circle

is $\pi(10.1)^2 = 102.01\pi$

Error $= 0.01\pi$