6.3

Wednesday, November 10, 2021 12:44 PM

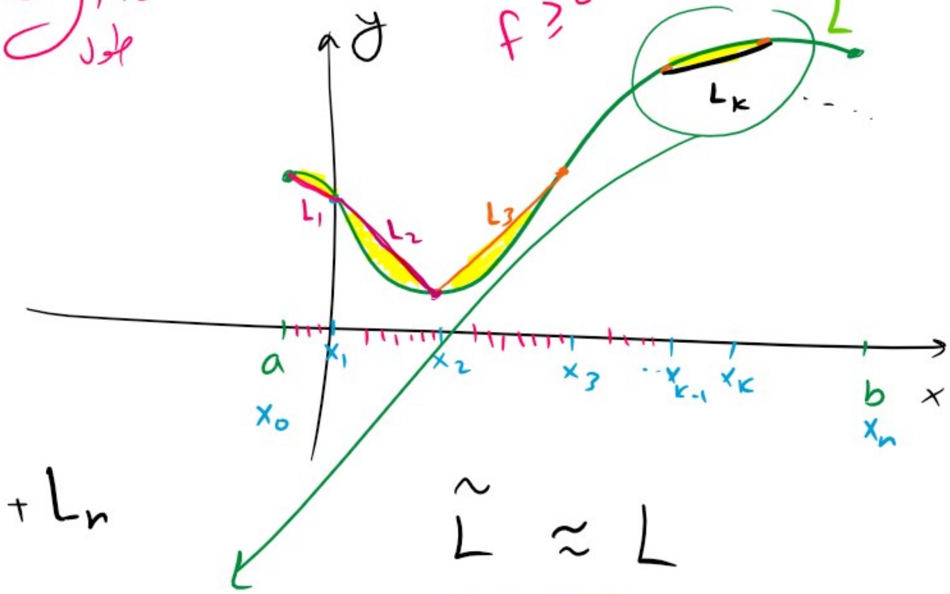
Arc length

طول منحنى

L : True length

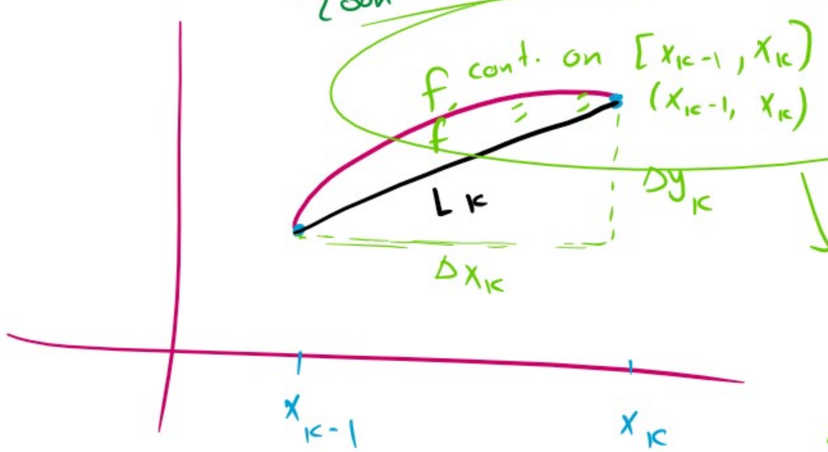
$$\tilde{L} = \sum_{k=1}^{n} K_k$$

$$= L_1 + L_2 + \dots + L_n$$



$$\tilde{L} \approx L$$

Zoom



$$L_k = \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}$$

by MVT

$\exists\, c_k \in (x_{k-1}, x_k)$

s.t $f'(c_k) = \dfrac{\Delta y_k}{\Delta x_k}$

$$L_k = \sqrt{(\Delta x_k)^2 \left[1 + \left(\frac{\Delta y_k}{\Delta x_k}\right)^2\right]}$$

$$\tilde{L} = \sum_{k=1}^{n} L_k = \sum_{k=1}^{n} \sqrt{\left(1 + \left(\frac{\Delta y_k}{\Delta x_k}\right)^2\right)} \, \Delta x_k$$

$$= \sum^{n} \sqrt{1 + (f'(c_k))^2} \, \Delta x_k$$

$$= \sum_{k=1} \sqrt{1 + (f'(c_k))^2}\ \Delta x_k$$

$$\lim_{n\to\infty} \tilde{L} = \lim_{n\to\infty} \sum_{k=1}^{n} \sqrt{1 + (f'(c_k))^2}\ \Delta x_k$$

$$\boxed{L = \int_a^b \sqrt{1 + (f'(x))^2}\ dx} \quad *$$

$$\frac{df}{dx} = \frac{dy}{dx}$$

Def: f cont $[a,b]$ ✓
f' cont. $[a,b]$ ✓

length for the curve $y = f(x)$ is $*$

If f or f' not cont. on $[a,b]$ $\Rightarrow$

$$L = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\ dy$$

Ex: Find length of the

Exp Find length of the curve

$$f(x) = \frac{x^3}{12} + \frac{1}{x} \quad , \quad 1 \le x \le 4$$

$x=0$??

f cont. on $[1,4]$ ✓

$f' = \frac{x^2}{4} - \frac{1}{x^2}$ cont. $[1,4]$

$$L = \int_a^b \sqrt{1 + (f'(x))^2}\, dx$$

$$= \int_1^4 \sqrt{1 + \left(\frac{x^2}{4} - \frac{1}{x^2}\right)^2}\, dx$$

$$= \int_1^4 \sqrt{1 + \frac{x^4}{16} - \cancel{2}\left(\frac{\cancel{x^2}}{\cancel{4}_{2}}\right)\left(\frac{1}{\cancel{x^2}}\right) + \left(\frac{-1}{x^2}\right)^2}\, dx$$

$$= \int_1^4 \sqrt{1 + \underbrace{\frac{x^4}{16} - \frac{1}{2} + \frac{1}{x^4}}}\, dx$$

$$= \int_1^4 \sqrt{\underbrace{\frac{x^4}{16} + \frac{1}{2} + \frac{1}{x^4}}}\, dx$$

$$= \int_1^4 \cancel{\sqrt{\left(\frac{x^2}{4}+\frac{1}{x^2}\right)^2}}\, dx$$

$$= \int_1^4 \left(\frac{x^2}{4}+\frac{1}{x^2}\right) dx = \int_1^4 \left(\frac{x^3}{2} + x^{-2}\right) dx$$

$$= \frac{x^3}{3} + \frac{x^{-1}}{-1}\Bigg|_1^4 = \dots = 6$$

عوض

2 Find arc length of $y = \left(\frac{x}{2}\right)^{\frac{2}{3}}$ from $x = 0$ to $x = 2$

$$\cancel{L = \int_a^b \sqrt{1 + \left(f'(x)\right)^2}\, dx} \qquad \left(\frac{dy}{dx}\right)^2$$

$$= \int_c^d \sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy$$

$y = \sqrt[3]{\left(\frac{x}{2}\right)^2}$ cont. $[0, 2]$

$y' = \frac{2}{3}\left(\frac{x}{2}\right)^{-\frac{1}{3}}\left(\frac{1}{2}\right)$

$= \frac{1}{3}\,\frac{1}{\sqrt[3]{\frac{x}{2}}}$ discont at $\boxed{x=0}$

$y = \left(\frac{x}{2}\right)^{\frac{2}{3}}$

$y^{\frac{3}{2}} = \frac{x}{2}$

$x = 2y^{\frac{3}{2}}$ $[0, 1]$

$x = 0 \Rightarrow y = 0$

$x=0 \Rightarrow y=0$

$x=2 \Rightarrow y=1$

$= \int_0^1 \sqrt{1+9y}\, dy$

$u = 1+9y$

$du = 9\, dy \Rightarrow dy = \frac{du}{9}$

$= \int_1^{10} \sqrt{u}\ \frac{du}{9}$

when $y=0 \Rightarrow u=1$

$y=1 \Rightarrow u=10$

$= \frac{1}{9}\int_1^{10} u^{\frac{1}{2}}\, du = \frac{1}{9}\left.\frac{u^{\frac{3}{2}}}{\frac{3}{2}}\right|_1^{10} = \cdots \approx 2.27$

$= \frac{2}{27}\left(10\sqrt{10}-1\right)$

$\boxed{x = 2\, y^{\frac{3}{2}}}$ $[0,1]$

x cont. on $[0,1]$

$\frac{dx}{dy} = x' = \cancel{2}\,\frac{3}{\cancel{2}}\, y^{\frac{1}{2}} = 3\sqrt{y}$ cont. on $[0,1]$

$\left(\frac{dx}{dy}\right)^2 = 9y$

Exp Find length of $y = \int_0^x \sqrt{\cos 2t}\, dt$

from $x=0$ to $x=\frac{\pi}{4}$

$L = \int_a^b \sqrt{1+\left(f'(x)\right)^2}\, dx$

$y' = \sqrt{\cos 2x}$

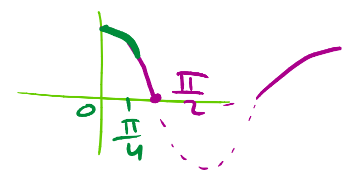
$(y')^2 = \cos 2x$

π

$$= \int_0^{\frac{\pi}{4}} \sqrt{1+\cos 2x}\; dx \qquad \cos 2x = 2\cos^2 x - 1$$

$$= \int_0^{\frac{\pi}{4}} \sqrt{\cancel{1} + 2\cos^2 x \cancel{-1}}\; dx$$

$$= \int_0^{\frac{\pi}{4}} \sqrt{2\cos^2 x}\; dx = \sqrt{2}\int_0^{\frac{\pi}{4}} |\cos x|\; dx$$



$$= \sqrt{2}\int_0^{\frac{\pi}{4}} \cos x\; dx$$

$$= \sqrt{2}\; \sin x \Big|_0^{\frac{\pi}{4}} = \sqrt{2}\left[\sin\frac{\pi}{4} - \sin 0\right]$$

$$= \sqrt{2}\left(\frac{1}{\sqrt{2}} - 0\right)$$

$$= 1$$

Q4 (4) Find length of this curve

$$x = \frac{y^{\frac{3}{2}}}{3} - y^{\frac{1}{2}}, \quad y=1 \text{ to } y=9$$

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d

$x = \frac{\sqrt{y^3}}{3} - \sqrt{y}$ cont. on [1,9]

$x = \frac{\sqrt{y^3}}{3} - \sqrt{y}$ cont. on $[1,9]$

$x' = \cancel{\frac{3}{2}} \cdot \frac{1}{3} y^{\frac{1}{2}} - \frac{1}{2\sqrt{y}}$

$= \frac{1}{2}\sqrt{y} - \frac{1}{2\sqrt{y}}$ cont. on $[1,9]$

$= \frac{1}{2}\left(\sqrt{y} - \frac{1}{\sqrt{y}}\right)$

$$L = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

$$= \int_1^9 \sqrt{1 + \left(\frac{1}{2}\sqrt{y} - \frac{1}{2}\frac{1}{\sqrt{y}}\right)^2}\, dy$$

$$= \int_1^9 \sqrt{1 + \frac{1}{4}\left(\sqrt{y} - \frac{1}{\sqrt{y}}\right)^2}\, dy$$

الأول تربيع — 2 (الأول)(الثاني) — الثاني تربيع

$$= \int_1^9 \sqrt{1 + \frac{1}{4}\left(y - 2 + \frac{1}{y}\right)}\, dy$$

$$= \int_1^9 \sqrt{1 + \frac{y}{4} - \frac{1}{2} + \frac{1}{4y}}\, dy$$

$$= \int_1^9 \sqrt{\frac{y}{4} + \frac{1}{2} + \frac{1}{4y}}\, dy$$

$$= \int_1^9 \sqrt{\frac{1}{4}\left(y + 2 + \frac{1}{y}\right)}\, dy$$

$$= \int_1^9 \sqrt{\frac{1}{4}\left(\sqrt{y} + \frac{1}{\sqrt{y}}\right)^2} \, dy$$

$$= \frac{1}{2} \int_1^9 \left(\sqrt{y} + \frac{1}{\sqrt{y}}\right) dy$$

$$= \frac{1}{2} \int_1^9 \left(y^{\frac{1}{2}} + y^{-\frac{1}{2}}\right) dy$$

$$= \frac{1}{2} \left(\frac{2}{3} y^{\frac{3}{2}} + 2 y^{\frac{1}{2}}\right) \Big|_1^9 = \cdots = \frac{32}{3}$$

19 Q Find the curve passes through (1,1) whose length $L = \int_1^4 \sqrt{1 + \frac{1}{4x}} \, dx$

How many curves are ~~they~~?

$$L = \int_a^b \sqrt{1 + (f'(x))^2} \, dx$$

we need to find $\underline{f(x)}$

$$(f'(x))^2 = \frac{1}{\ldots}$$

$$\left(f'(x)\right)^2 = \frac{1}{4x}$$

$$f'(x) = \sqrt{\frac{1}{4x}} = \frac{1}{2\sqrt{x}}$$

عدد لا نهائي

$$f(x) = \int f'(x)\,dx = \int \frac{1}{2\sqrt{x}} = \sqrt{x} + C$$

$$\left.\begin{aligned} f(1) &= \sqrt{1} + C \\ 1 &= 1 + C \\ \boxed{0} &\boxed{= C} \end{aligned}\right\} \Rightarrow \boxed{f(x) = \sqrt{x}}$$

unique since it is the only curve passes (1,1)