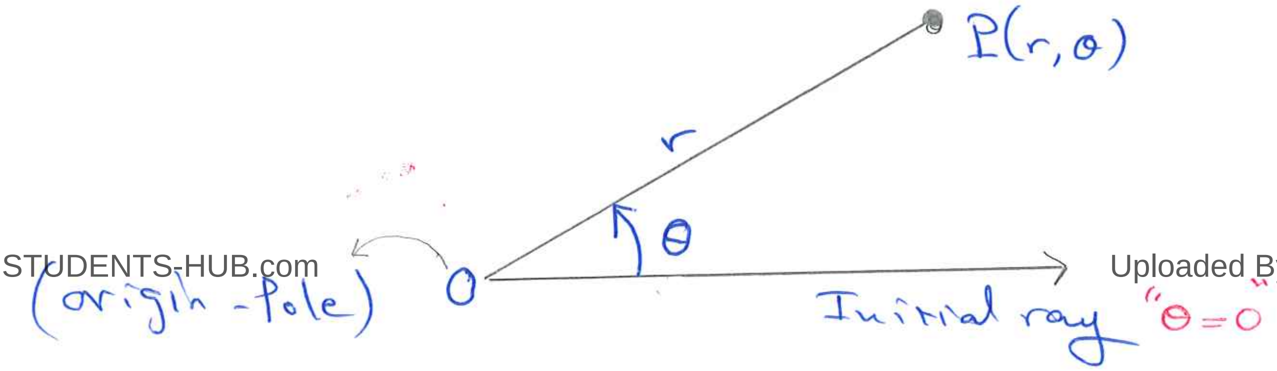
11.3 Polar Coordinates

We Use Polar Coordinates to help in Calculating some Integrals. To define Polar Coordinates, we fix an origin O, we call it "pole", and an initial ray from O, then each point P can be located by assigning to it polar Coordinate pair (r, θ) ; $P(r, \theta)$; where:

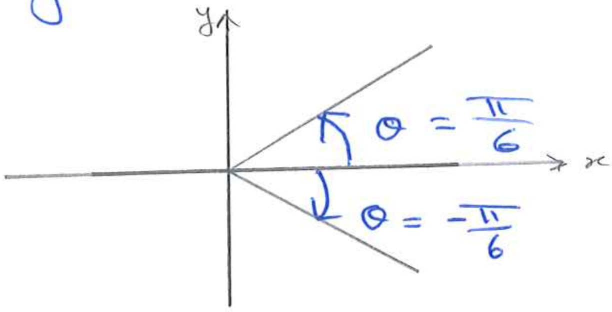
r : directed distance from O to P. "Can be negative"

θ : directed angle from the Initial ray to OP

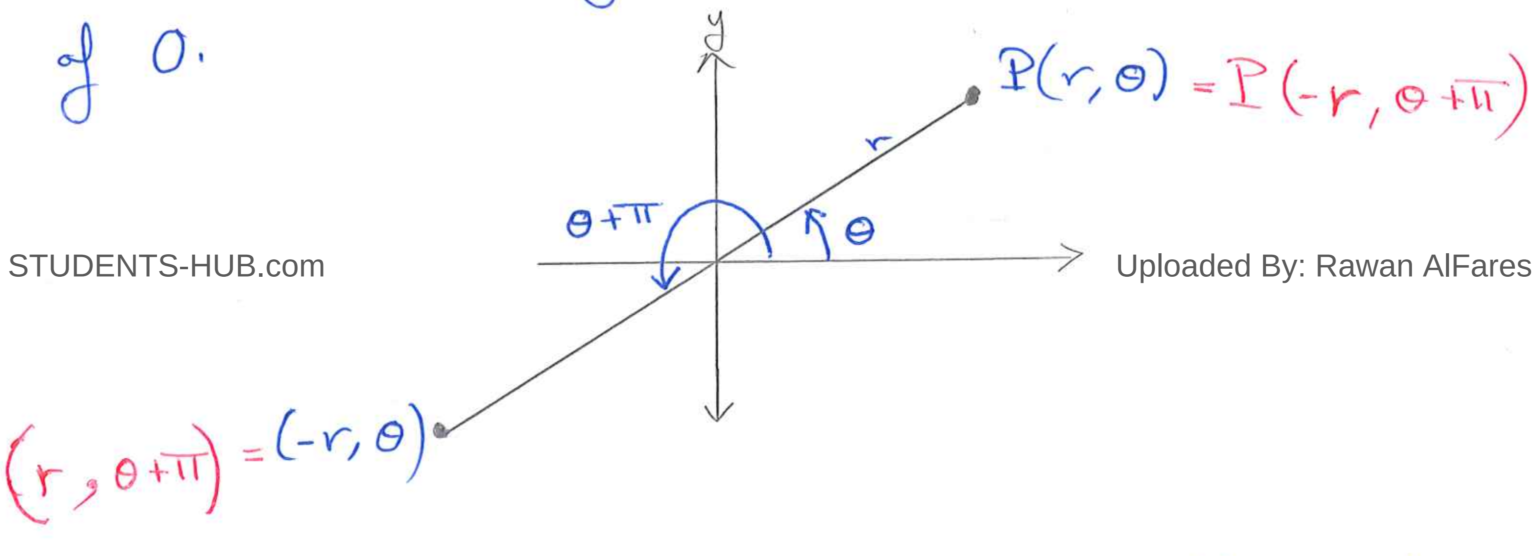


Remarks:

1) The angle θ will be positive if the directed angle from the Initial ray to OP is measured Counter clockwise, and will be negative if the directed angle is measured clockwise.



2) The points (r, θ) and $(-r, \theta)$ Lies on the same line through O, but on opposite sides of O.



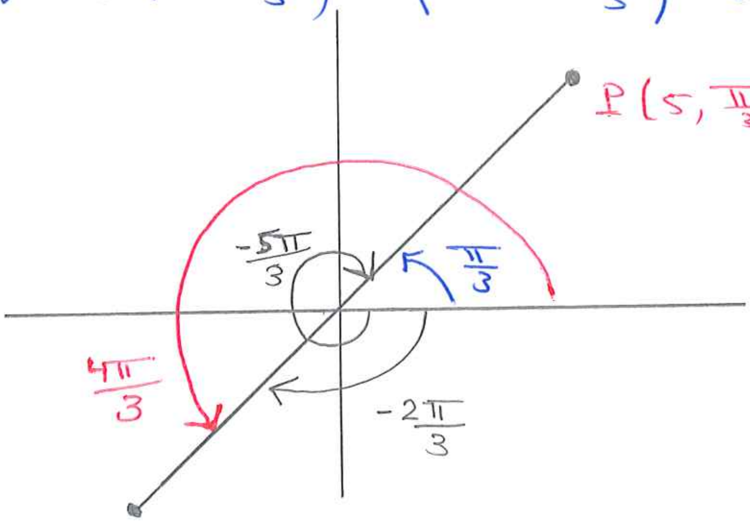
(i-e) if r is negative, we can write it positive by adding or subtracting π to the angle θ.

3) If $r>0$, then the point (r,θ) lies on the same quadrant as θ.

If $r<0$, then the point (r,θ) will be end up in the quadrant exactly opposite θ.

4) The representation of a point P in polar Coordinate is Not Unique. as in Cartesian Coordinates. In fact it has infinitly many pairs of Polar Coordinates. For instance:

$$\left(5,\frac{\pi}{3}\right)=\left(5,-\frac{5\pi}{3}\right)=\left(-5,\frac{4\pi}{3}\right)=\left(-5,-\frac{2\pi}{3}\right)$$

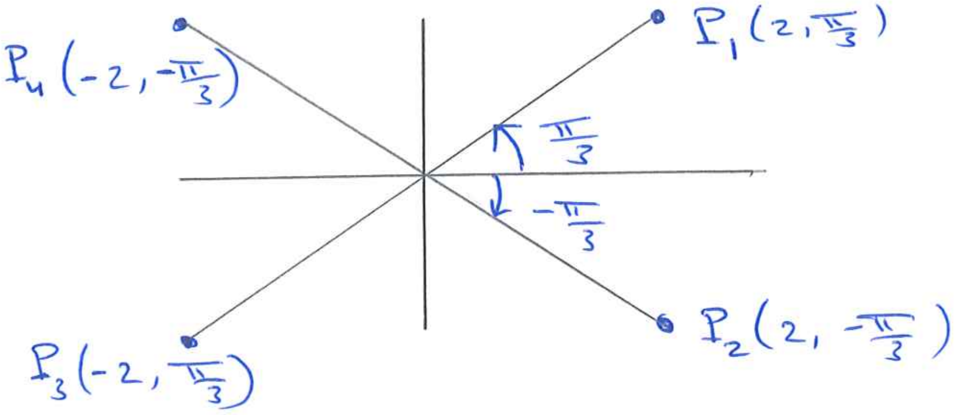


$$\Rightarrow P(r,\theta)=P(r,\theta+2\pi m)=P(-r,(\theta\pm\pi)+2\pi m)$$

$$=P(-r,\theta\pm(2m+1)\pi),\quad m=0,\pm1,\pm2,\ldots$$

Example: Plot $P_1\left(2, \frac{\pi}{3}\right)$, $P_2\left(2, -\frac{\pi}{3}\right)$

$P_3\left(-2, \frac{\pi}{3}\right)$, $P_4\left(-2, -\frac{\pi}{3}\right)$



Example: Find all Polar Coordinates of $P\left(2, \frac{\pi}{6}\right)$

Case 1: If $r = 2$, then $\theta = \frac{\pi}{6} + 2m\pi$

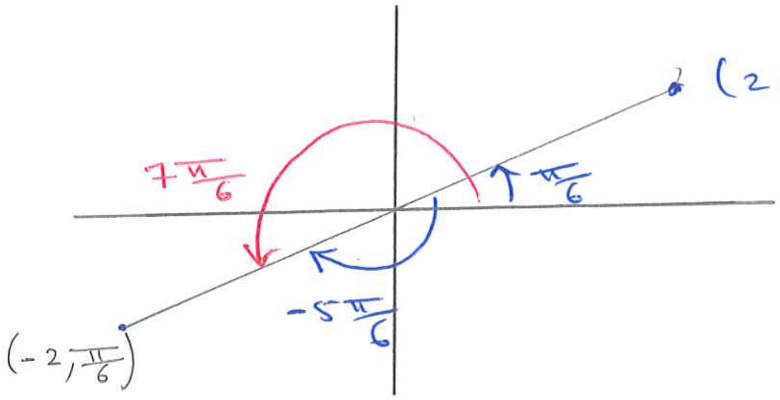
where $m = 0, \pm 1, \pm 2, \ldots$.

Case 2: If $r = -2$, then $\theta = \left(\frac{\pi}{6} + \pi\right) + 2m\pi$
$= \frac{\pi}{6} + (2m+1)\pi$

where $m = 0, \pm 1, \pm 2, \ldots$.

$\Rightarrow P\left(2, \frac{\pi}{6}\right) = P\left(2, \frac{\pi}{6} + 2m\pi\right) = P\left(-2, \frac{7\pi}{6} + 2m\pi\right)$

where $m = 0, \pm 1, \pm 2, \ldots$



$\left(2, \frac{\pi}{6}\right) = \left(-2, -\frac{5\pi}{6}\right)$

$= \left(-2, \frac{7\pi}{6}\right)$

$= \ldots$

Polar Equations and Graphs.

- $\boxed{r = a}$ $\Rightarrow$ Circle of radius $|a|$ centered at O.

where θ varies over any interval of length 2π.

- $\boxed{\theta = \theta_0}$ $\Rightarrow$ Line through O making an angle θ_0 with the Initial ray.

In this case r varies between $-\infty$ and ∞.

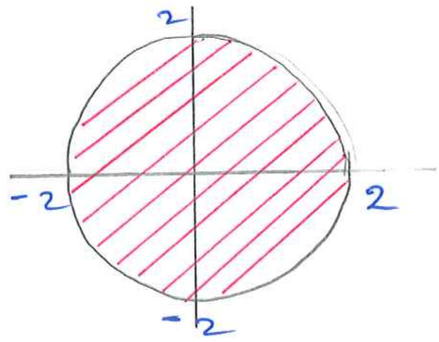
Example: $\boxed{1}$ $r = 9$ and $r = -9$ are equations for the Circle of radius 3 centered at O.

$\boxed{2}$ $\theta = \frac{\pi}{6}$, $\theta = \frac{7\pi}{6}$ and $\theta = -\frac{5\pi}{6}$ are equations for the line.

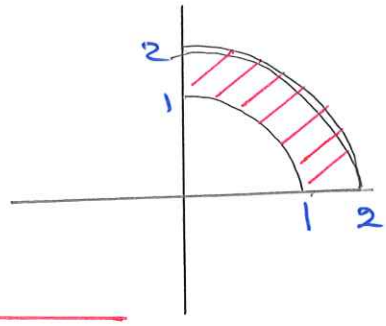
Example: Graph the sets of points whose polar Coordinates satisfy the following equations and identities.

(a) $0 \leq r \leq 2$

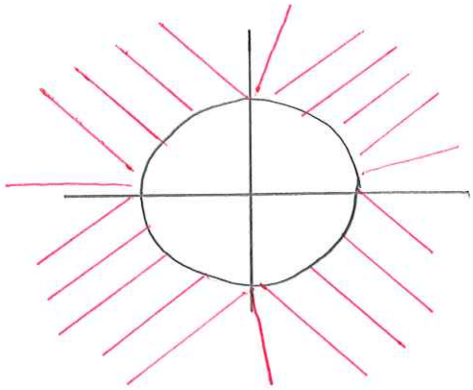
(No restriction on θ)



(b) $1 \leq r \leq 2$ and $0 \leq \theta \leq \frac{\pi}{2}$

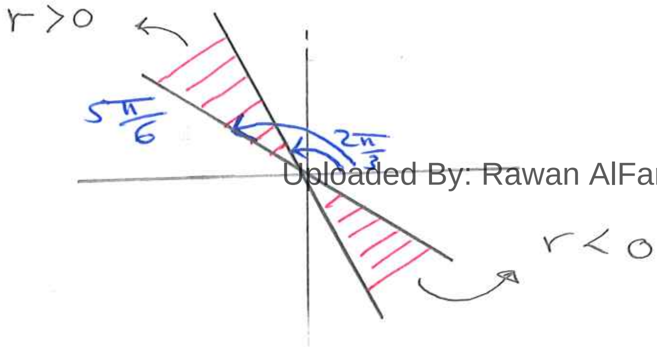


(c) $r \geq 1$

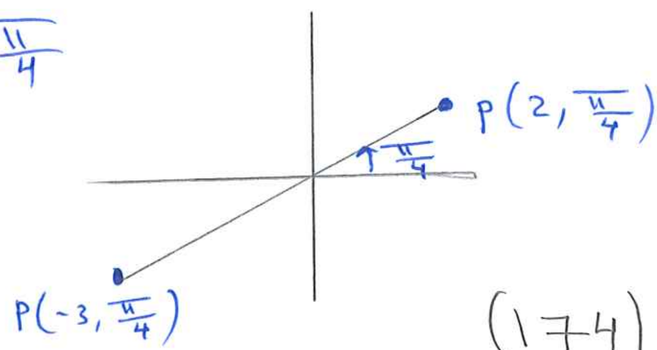


(d) $\frac{2\pi}{3} \leq \theta \leq \frac{5\pi}{6}$

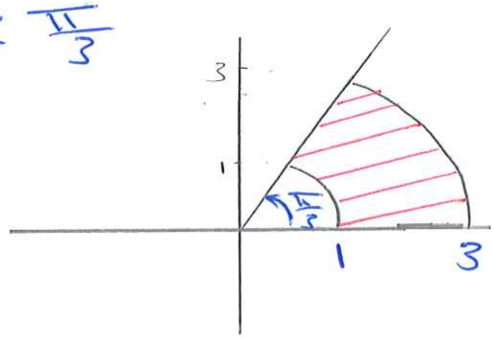
(No restriction on r)



(e) $-3 \leq r \leq 2$ and $\theta = \frac{\pi}{4}$

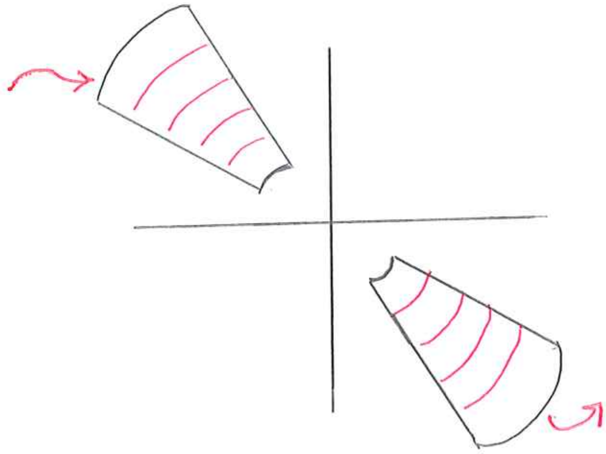


(f) $1 \leq r \leq 3$ and $0 \leq \theta \leq \frac{\pi}{3}$

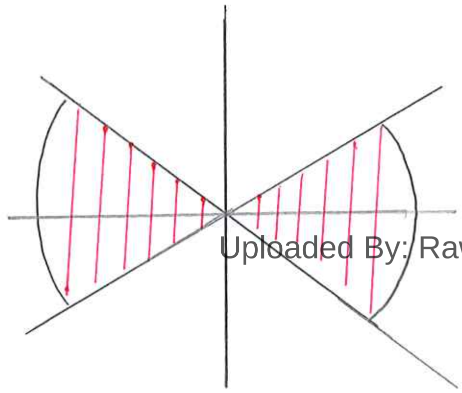


(g) $\frac{2\pi}{3} \leq \theta < \frac{5\pi}{6}$ and $1 \leq |r| \leq 2$.

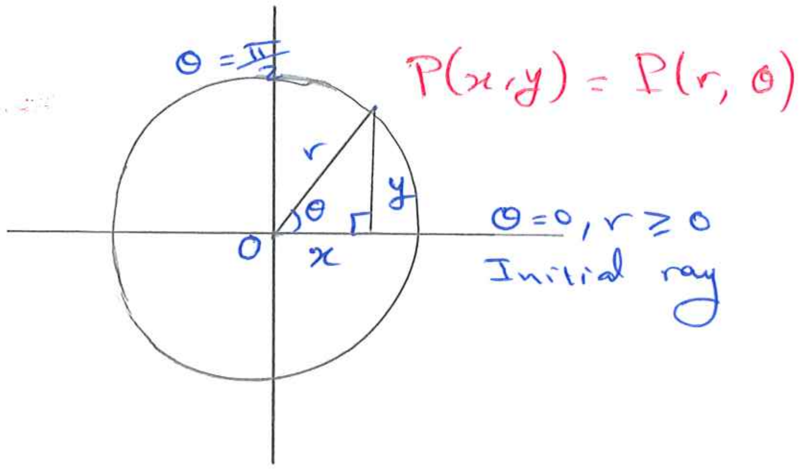
First, note that $1 \leq |r| \leq 2 \Leftrightarrow 1 \leq r \leq 2$ or $-2 \leq r \leq -1$



(h) $-\frac{\pi}{4} \leq \theta \leq \frac{\pi}{4}$ and $-1 \leq r \leq 1$



Equations Relating Polar and Cartesian Coordinates.



- $x = r\cos\theta$
- $y = r\sin\theta$
- $r^2 = x^2 + y^2$
- $\tan\theta = \frac{y}{x}$ here $\cancel{\Leftrightarrow}$ $\theta = \tan^{-1}\left(\frac{y}{x}\right)$

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Notice that $\tan^{-1}x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, but θ could be any angle.

Example : Replace the Polar Coordinates with Cartesian.

$\boxed{1}$ $2 = r\cos\theta \Rightarrow x = 2$, vertical line passes $(2,0)$

$\boxed{2}$ $r = -3\sec\theta \Rightarrow r = \frac{-3}{\cos\theta} \Rightarrow -3 = r\cos\theta$

$\Rightarrow x = -3$, vertical line passes $(-3,0)$

$\boxed{3}$ $r^2 = 1 \Rightarrow x^2 + y^2 = 1$, circle centered at $(0,0)$ with radius $r = 1$.

$\boxed{4}$ $r = \dfrac{5}{\sin\theta - 2\cos\theta}$

$\Rightarrow \quad 5 = r\sin\theta - 2r\cos\theta$

$\Rightarrow \quad 5 = y - 2x \quad \Rightarrow \quad y = 2x+5$, Line.

$m = 2$

y-int. $= 5$

$\boxed{5}$ $r = \csc\theta \, e^{r\cos\theta} = \dfrac{e^{r\cos\theta}}{\sin\theta}$

$\Rightarrow \quad r\sin\theta = e^{r\cos\theta} \quad \Rightarrow \quad y = e^{x}$, Exponential.

$\boxed{6}$ $\left(-\sqrt{2}, \frac{\pi}{4}\right) = (r, \theta)$, then the Cartesian Coordinates

$x = r\cos\theta = -\sqrt{2}\cos\frac{\pi}{4} = -\sqrt{2}\cdot\frac{1}{\sqrt{2}} = -1$

$y = r\sin\theta = -\sqrt{2}\sin\frac{\pi}{4} = -\sqrt{2}\cdot\frac{1}{\sqrt{2}} = -1$

$\Rightarrow \quad (x, y) = (-1, -1)$

$\boxed{7}$ $r^2 = 4r\cos\theta$

$x^2+y^2 = 4x \iff x^2 - 4x + y^2 = 0$

$\iff (x^2 - 4x + 4) + y^2 = 4$

$\iff (x-2)^2 + y^2 = 4$

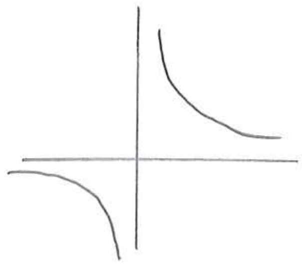
which is Circle with center $(2,0)$ & radius $= 2$.

$\boxed{8}$ (Q32) $r^2 \sin(2\theta) = 2 \iff r^2(2\sin\theta\cos\theta) = 2$

$\iff (r\sin\theta)(r\cos\theta) = 1$

$\iff y \cdot x = 1$

$\iff y = \frac{1}{x}$ (Hyperbola).



$\boxed{9}$ (Q40) $r = 4\tan\theta\sec\theta$

$\iff r = 4\frac{\sin\theta}{\cos\theta} \cdot \frac{1}{\cos\theta}$

$\iff r\cos^2\theta = 4\sin\theta$

$\iff r^2\cos^2\theta = 4r\sin\theta$

$\iff x^2 = 4y$, which is parabola $y = \frac{x^2}{4}$

$\boxed{10}$ (Q51) $r\sin\left(\theta + \frac{\pi}{6}\right) = 2$

$\iff r\left[\sin\theta\cos\frac{\pi}{6} + \cos\theta\sin\frac{\pi}{6}\right] = 2$

$\iff \frac{\sqrt{3}}{2} r\sin\theta + \frac{1}{2} r\cos\theta = 2$

$\iff \frac{\sqrt{3}}{2} y + \frac{1}{2} x = 2$

$\iff y = -\frac{1}{\sqrt{3}} x + \frac{4}{\sqrt{3}}$, which is a Line

with $m = -\frac{1}{\sqrt{3}}$, y-intercept $= \frac{4}{\sqrt{3}}$.

Example: Replace the Cartesian equations with Polar Coordinates.

① (Q55) $x = y \iff r\cos\theta = r\sin\theta \iff \cos\theta = \sin\theta$

$\iff \boxed{\theta = \frac{\pi}{4}}$

② (Q57) $x^2 + y^2 = 4 \iff r^2 = 4 \iff \boxed{r = \pm 2}$

③ (Q64) $(x-5)^2 + y^2 = 25$

$x^2 - 10x + 25 + y^2 = 25$

$x^2 + y^2 = 10x$

$r^2 = 10 r\cos\theta \iff r(r - 10\cos\theta) = 0$

$\Rightarrow \boxed{r = 10\cos\theta}$ or $\boxed{r = 0}$ Included here for $\theta = \frac{\pi}{2}$

④ ([illegible]) The point $(x,y) = (-2,-2)$ if $-\pi \le \theta < \pi$, $r \geq 0$.

$r^2 = x^2 + y^2 = (-2)^2 + (-2)^2 = 8$

$\iff r = \sqrt{8} = 2\sqrt{2}$.

$\tan\theta = \frac{y}{x} = \frac{-2}{-2} = 1 \Rightarrow \theta = -\frac{3\pi}{4}$ الربع الثالث

$\Rightarrow (r,\theta) = \left(2\sqrt{2}, -\frac{3\pi}{4}\right)$.

⑤ (Q60) $xy = 2 \Rightarrow y = \frac{2}{x}$

$(r\cos\theta)(r\sin\theta) = 2$

$\Leftrightarrow \quad r^2 = \frac{1}{\cos\theta}\frac{1}{\sin\theta} \cdot 2 = 2\sec\theta\csc\theta.$

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⑥ (Q61) $y^2 = 4x$

$r^2\sin^2\theta = 4r\cos\theta$

$\Leftrightarrow \quad r(r\sin^2\theta - 4\cos\theta) = 0$

$\Leftrightarrow \quad r = 0 \quad$ or $\quad r\sin^2\theta = 4\cos\theta.$

$\Rightarrow \quad \boxed{r\sin^2\theta = 4\cos\theta}$ (Includes $r=0$ for $\theta = \frac{\pi}{2}$)

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⑦ $(x,y) = (-1,1)$

$r^2 = x^2+y^2 = 2 = \sqrt{2}$

$\tan\theta = \frac{y}{x} = \frac{1}{-1} = -1 \Rightarrow \theta = \tan^{-1}(-1) + \pi = \frac{3\pi}{4}$

Notice that $\tan^{-1}(-1) = -\frac{\pi}{4}$ which occurs in the fourth quadrant., but the θ required must be in the 2nd quadrant.

(النقطة (1,-1) في الربع الثاني)

Example: Convert to the other Coordinates.

(1) $r = a \iff r^2 = a^2 \iff x^2 + y^2 = a^2$

(2) $\theta = \theta_0 \iff \tan\theta = \tan\theta_0 = m$ (number)

$\iff \frac{y}{x} = m \iff y = mx.$

(3) $2x - 5x^3 = 1 + xy.$

$2(r\cos\theta) - 5(r\cos\theta)^3 = 1 + (r\cos\theta)(r\sin\theta).$

$2r\cos\theta - 5r^3\cos^3\theta = 1 + r^2\sin\theta\cos\theta.$

(4) $r = -8\cos\theta \Rightarrow r^2 = -8r\cos\theta$

$\Rightarrow x^2 + y^2 = -8x.$

Example: Is the Curve $r = 2\sin 2\theta$ passes through the point $P(2, \frac{3\pi}{4})$?

Let $F(r,\theta) = r - 2\sin 2\theta = 0$

$F(2, \frac{3\pi}{4}) = 2 - 2\sin\frac{3\pi}{2} = 4 \neq 0$

But $P(2, \frac{3\pi}{4}) = P(-2, \frac{3\pi}{4} - \pi) = P(-2, -\frac{\pi}{4})$

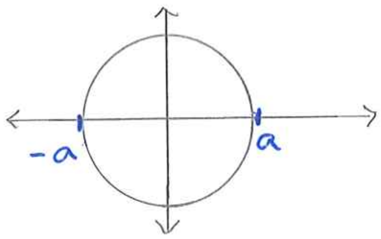
$F(-2, -\frac{\pi}{4}) = -2 - 2\sin(-\frac{\pi}{2}) = -2 + 2 = 0$

$\Rightarrow P(-2, -\frac{\pi}{4}) = P(2, \frac{3\pi}{4})$ lies on the Curve.

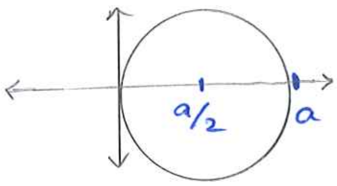
11.4 Graphing in Polar Coordinates.

Recall:

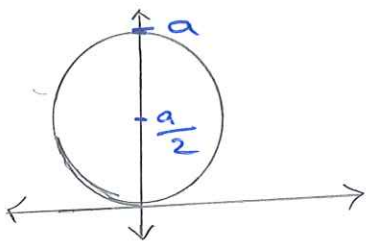
(1) $r = a \quad , \; a \neq 0$



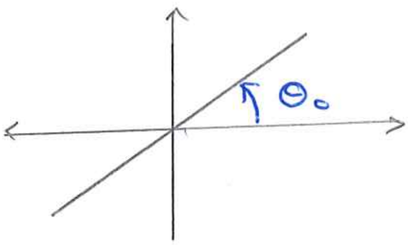
(2) $r = a\cos\theta$



(3) $r = a\sin\theta$



(4) $\theta = \theta_0$.



Symmetry Tests for Polar Graphs:

$\boxed{1}$ symmetry about x-axis:

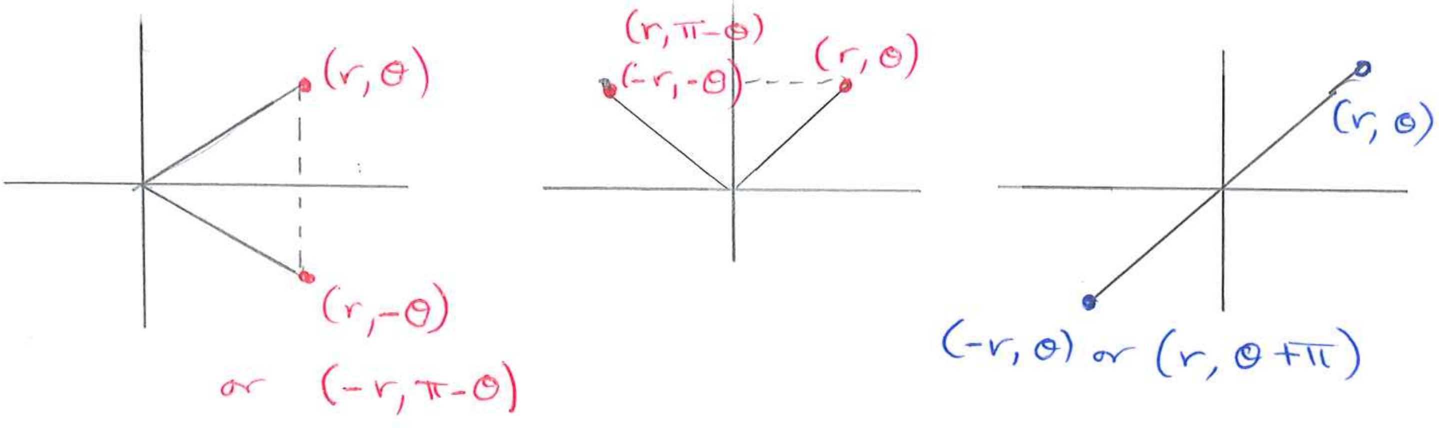
$$(r,\theta) \longleftrightarrow (r,-\theta) \text{ or } (-r, \pi-\theta).$$

$\boxed{2}$ symmetry about y-axis:

$$(r,\theta) \longleftrightarrow (r,\pi-\theta) \text{ or } (-r,-\theta).$$

3 Symmetry about the Origin :

$$(r,\theta) \longleftrightarrow (-r,\theta) \text{ or } (r,\theta+\pi).$$



About x-axis. About y-axis. About the Origin

Remark : Each Graph has one of the followings

1) NO symmetries.

2) Only one symmetry.

3) All the three symmetries.

Remark: Any Curve of the form:

symmetry y-axis ← $r = a + a\sin\theta$ or $r = a + a\cos\theta$ → symmetry x-axis

is called a "Cardioid".

Example: sketch the graph of the following

$\boxed{(1)}$ $r = 1 - \cos\theta$, (Cardioid).

First, we need to check symmetries:

(1) About the x-axis: $(r, -\theta)$ or $(-r, \pi - \theta)$??

$(r, \theta) \in$ graph $\Rightarrow$ $r = 1 - \cos\theta = 1 - \cos(-\theta)$

$\Rightarrow$ $(r, -\theta) \in$ graph $\Rightarrow$ The Curve is symmetric about x-axis.

(2) About the y-axis: $(r, \pi - \theta)$ or $(-r, -\theta)$??

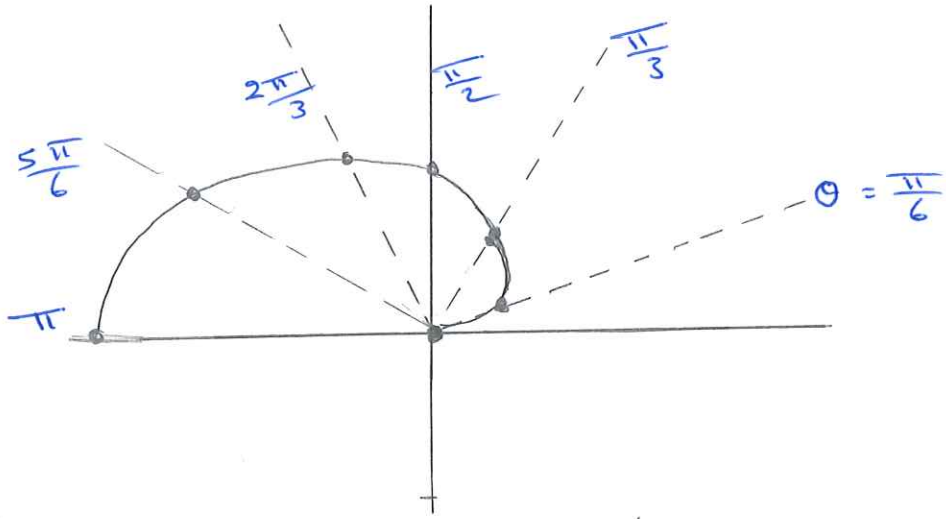
$$\left.\begin{array}{l} 1 - \cos(-\theta) = 1 - \cos\theta \neq -r \\ 1 - \underbrace{\cos(\pi - \theta)}_{-\cos\theta} = 1 + \cos\theta \neq r \end{array}\right\} \Rightarrow$$ The Curve is Not symmetric about y-axis.

(3) About the Origin: $(-r, \theta)$ or $(r, \theta + \pi)$??

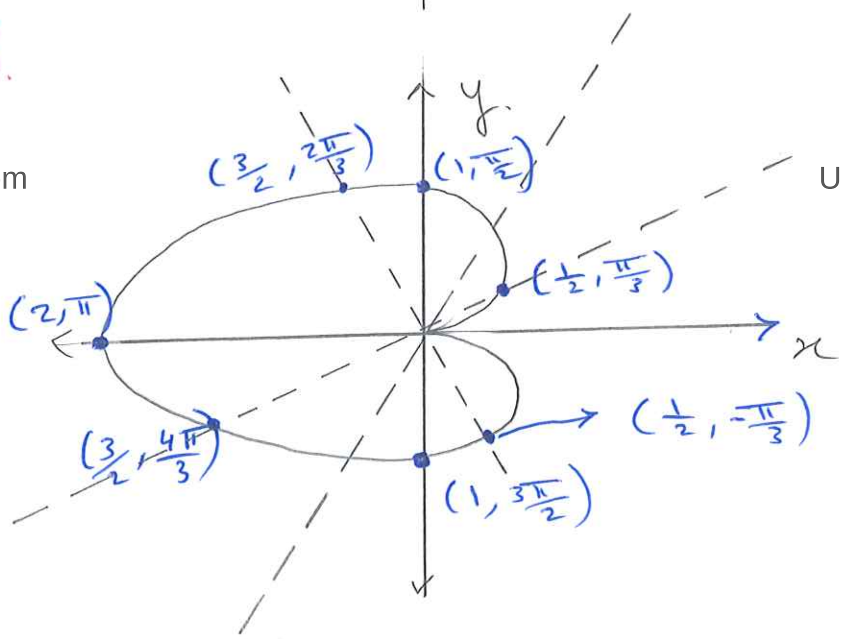
$$\left.\begin{array}{l} 1 - \cos\theta \neq -r \\ 1 - \cos(\theta + \pi) = 1 + \cos\theta \neq r \end{array}\right\} \Rightarrow$$ The Curve is Not symmetric about the origin

Now, we need to Construct a table for r and θ taking in Consideration if r is periodic in θ and the symmetry about x-axis. (i.e 0 to π) is enough here.

θ	$r = 1 - \cos\theta$	(r, θ)
0	0	$(0, 0)$
$\frac{\pi}{3}$	$\frac{1}{2}$	$(\frac{\pi}{3}, \frac{1}{2})$
$\frac{\pi}{2}$	1	$(1, \frac{\pi}{2})$
$\frac{2\pi}{3}$	$\frac{3}{2}$	$(\frac{3}{2}, \frac{2\pi}{3})$
π	2	$(2, \pi)$
$\frac{3\pi}{2}$	1	$(1, \frac{3\pi}{2})$

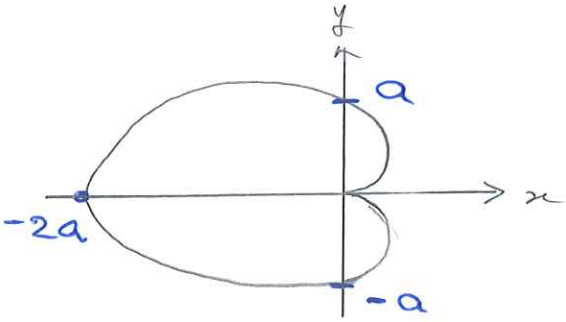


Cardioid.

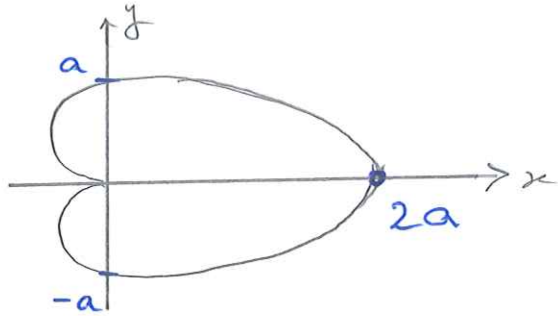


Graphs:

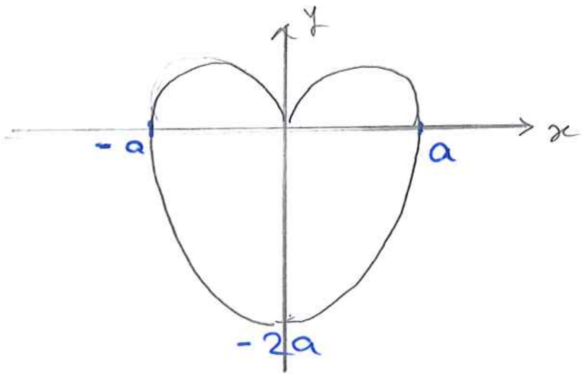
1. $r = a(1 - \cos\theta)$



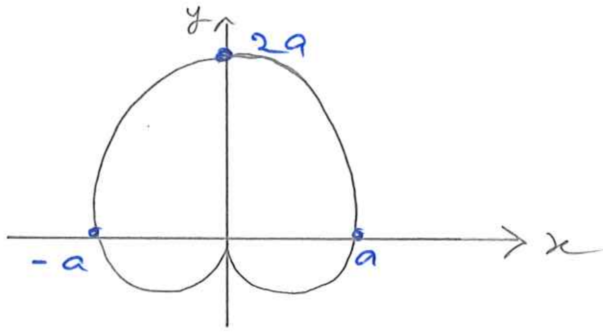
2. $r = a(1 + \cos\theta)$



3. $r = a(1 - \sin\theta)$



4. $r = a(1 + \sin\theta)$



(Q6)

$\boxed{(2)}$ sketch $r = 1+2\sin\theta$.

First, we need to check symmetries.

(1) About the x-axis. $(r,-\theta)$ or $(-r,\pi-\theta)$??

$(r,\theta) \in$ graph $\Rightarrow$ $r = 1+2\sin\theta$.

But : $1+2\sin(-\theta) = 1-2\sin\theta \neq r$.

& $1+2\sin(\pi-\theta) = 1+\sin\theta \neq -r$

$\Rightarrow$ No symmetry about x-axis.

(2) About y-axis : $(r,\pi-\theta)$ or $(-r,-\theta)$

$1+2\sin(\pi-\theta) = 1+2\sin\theta = r \Rightarrow$ The Curve is symmetric about y-axis

(3) NO symmetry about the origin $\Rightarrow$ (check).

since the symmetry is about the y-axis, and the function $\sin\theta$ is periodic of period 2π

it's enough to draw the graph in the first and fourth quadrants. (i-e), $\left(-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}\right)$.

θ	$r = 1 + 2\sin\theta$	(r, θ)
$-\frac{\pi}{2}$	-1	$(-1, -\frac{\pi}{2})$
$-\frac{\pi}{3}$	≈ -0.73	$(-0.73, -\frac{\pi}{3})$
$-\frac{\pi}{6}$	0	$(0, -\frac{\pi}{6})$
0	1	$(1, 0)$
$\frac{\pi}{6}$	2	$(2, \frac{\pi}{6})$
$\frac{\pi}{3}$	≈ 2.73	$(2.73, \frac{\pi}{3})$
$\frac{\pi}{2}$	3	$(3, \frac{\pi}{2})$

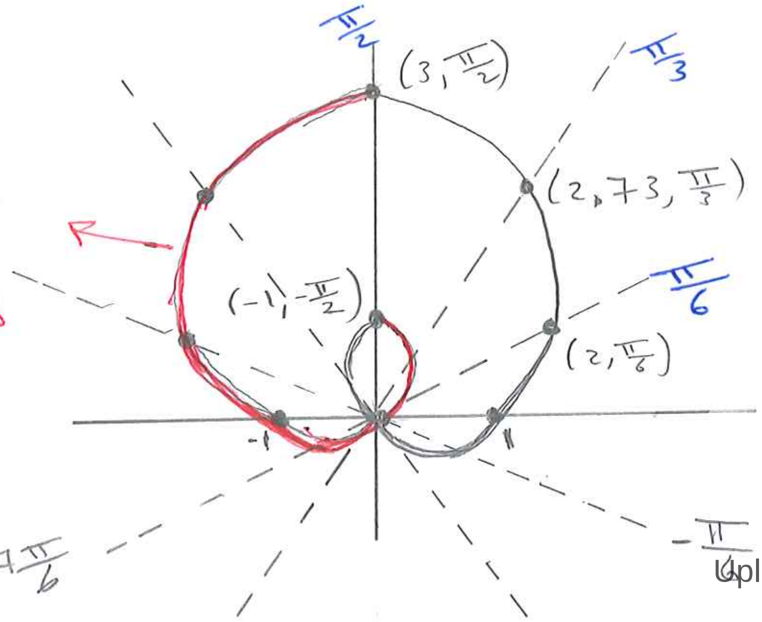
$r = 1 + 2\sin\theta = 0$

$\Rightarrow \sin\theta = -\frac{1}{2}$

$\Rightarrow \theta = \frac{7\pi}{6}$

or

$\theta = \frac{11\pi}{6} = -\frac{\pi}{6}$



Remark: The curve is called Limaçon with inner loop., with general form:

$$r = a + b\sin\theta$$

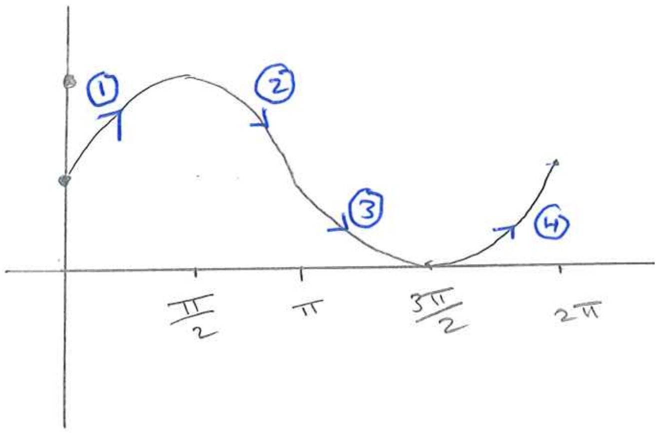
محور التماثل هو y-axis

Remark: If $r = a + b\cos\theta$, then the Curve is also Lemaçon with inner loop but its symmetric about x-axis.

If $b > 0$, then its to the right side.

If $b < 0$, then its to the left side

Example: sketch $r = 1 + \sin\theta$.



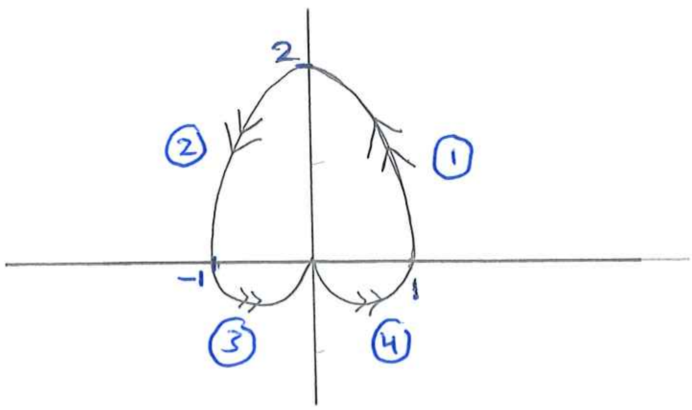
$\Rightarrow$ $r = 1 + \sin\theta$ In the Cartesian Coordinates.

Notice that: (1) As θ increases from 0 to $\frac{\pi}{2}$, r increases from 1 to 2.

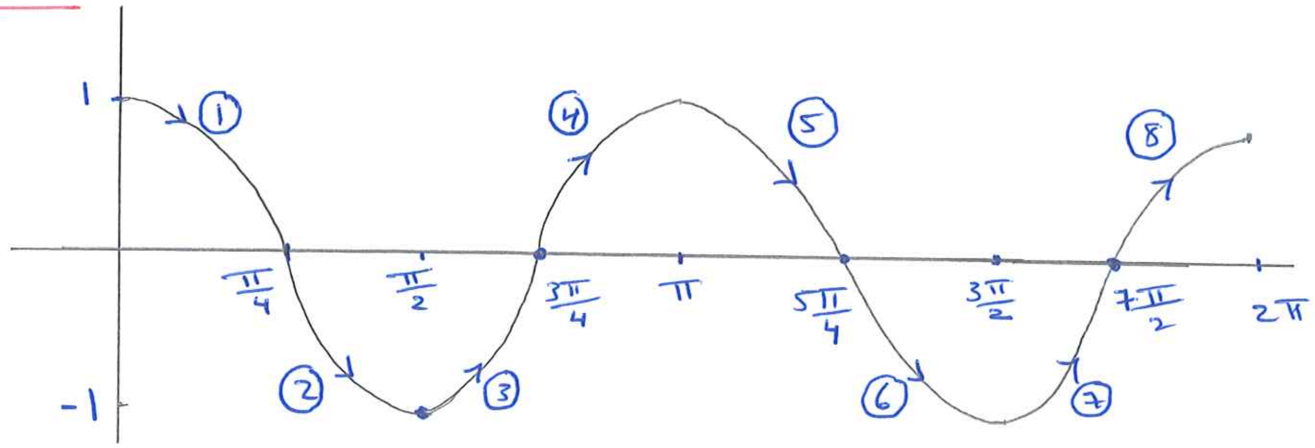
(2) As θ increases from $\frac{\pi}{2}$ to π, r decreases from 2 to 1.

(3) As θ increases from π to $\frac{3\pi}{2}$, r decreases from 1 to 0

(4) As θ increases from $\frac{3\pi}{2}$ to 2π, r increases from 0 to 1



Example: Sketch the Curve $r = \cos 2\theta$.

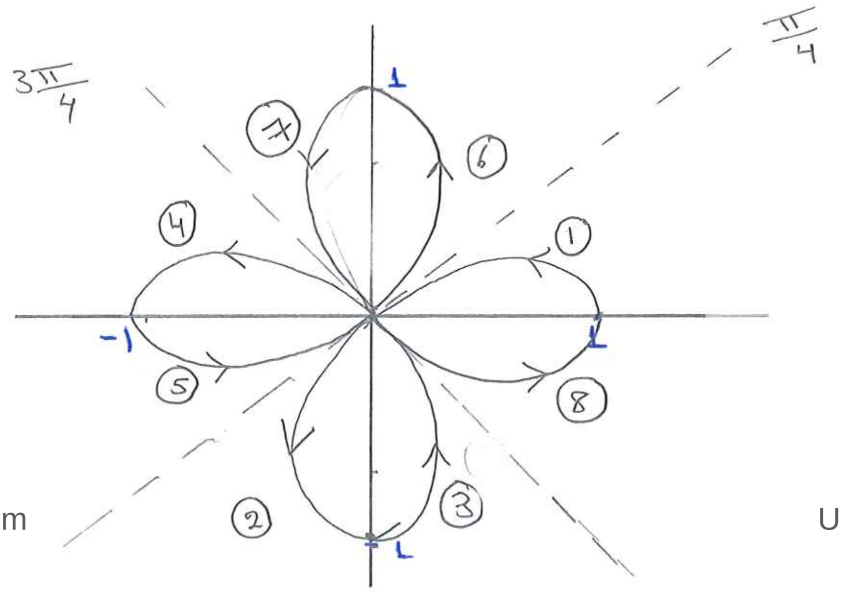


Notice that: ① As θ increases from 0 to $\frac{\pi}{4}$

r decreases from 1 to 0.

② As θ increases from $\frac{\pi}{4}$ to $\frac{\pi}{2}$, r decreases from

0 to -1, .. etc ..



Remarks:

① This Curve is called Four Leaved Rose.

② This graph is a good graph since its

symmetric about x-axis, y-axis & the Origin.

1) x-axis : If $(r,\theta) \in$ graph, then $(r,-\theta) \in$ graph also, $[\, r = \cos 2\theta = \cos(-2\theta) \,]$.

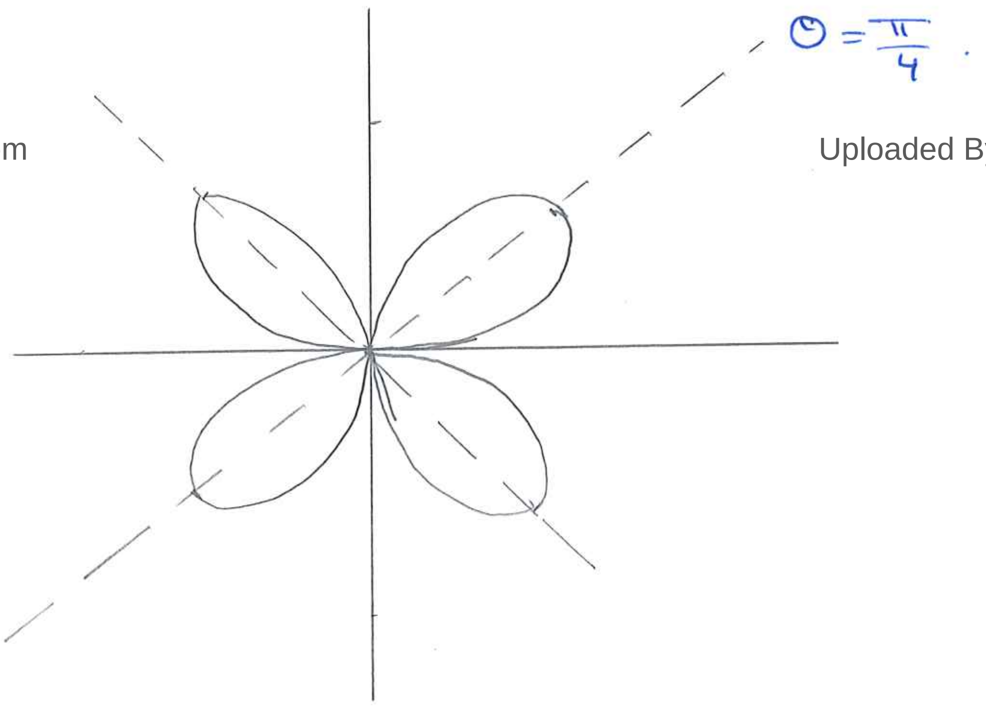
2) y-axis : $(r,\theta) \in$ graph, then $(r, \pi-\theta) \in$ graph also, $[\ \cos(2(\pi-\theta)) = \cos 2\pi \cos 2\theta + \sin 2\pi \sin 2\theta$

$= \cos 2\pi = r\ ]$.

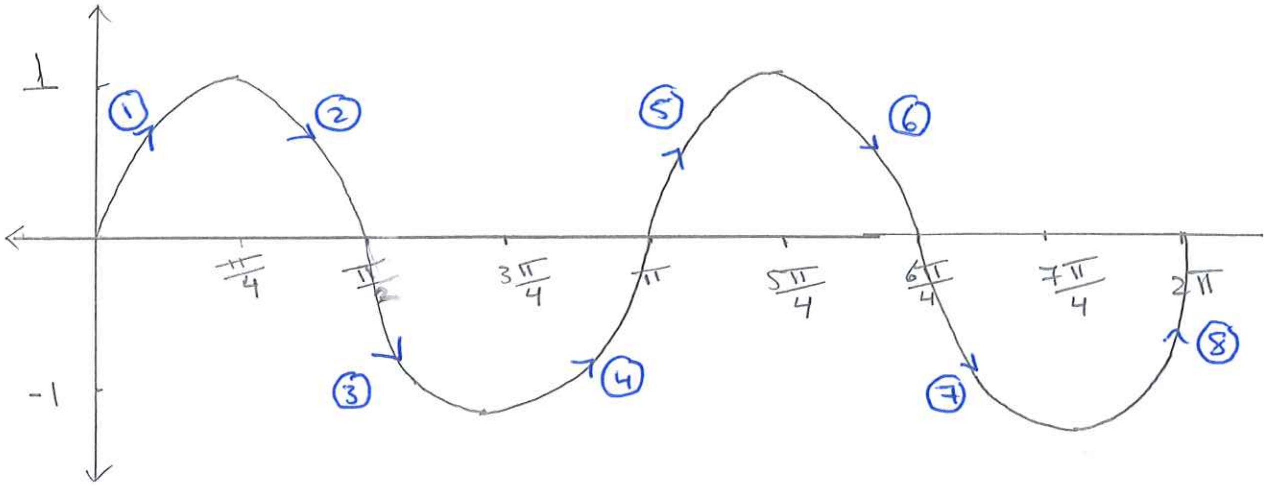
3) Origin : $(r,\theta) \in$ graph, then $(r, \theta+\pi) \in$ graph also, $[\ \cos(2(\theta+\pi)) = \cos 2\theta \cos 2\pi - \sin 2\theta \sin 2\pi$

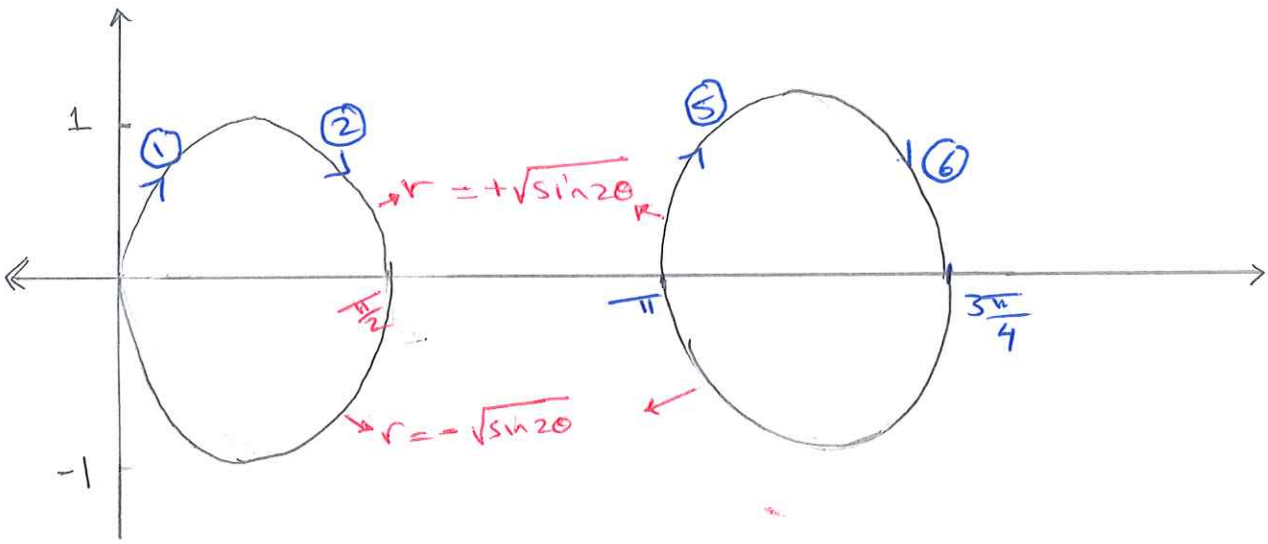
$= \cos 2\theta = r\ ]$.

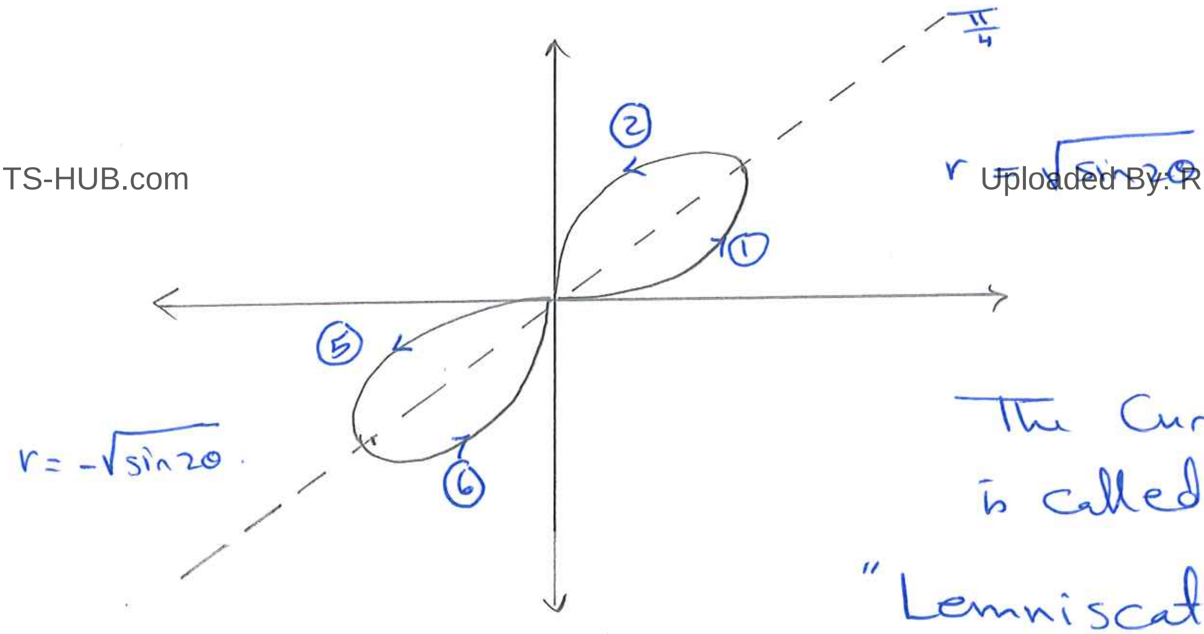
Example: $r = \sin 2\theta$



Example: sketch $r^2 = \sin 2\theta$

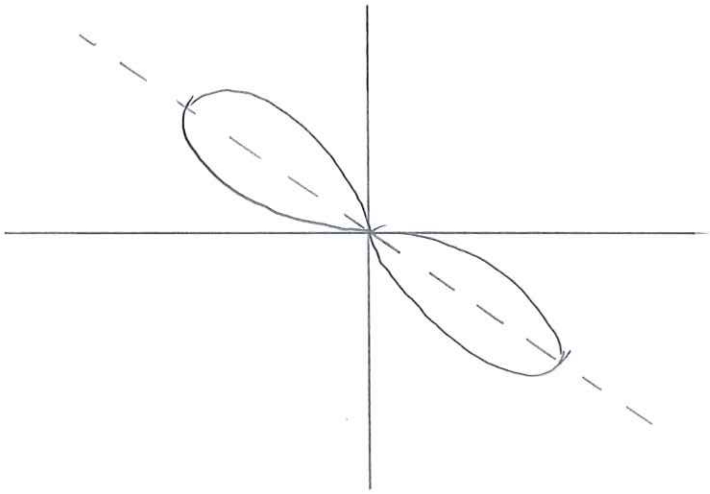




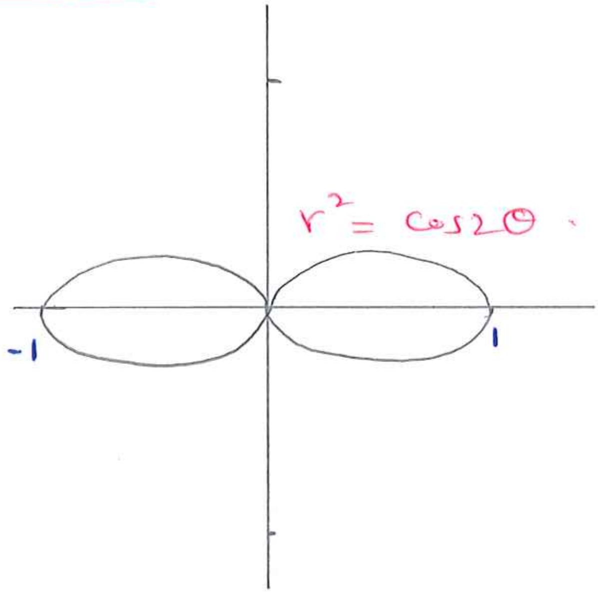


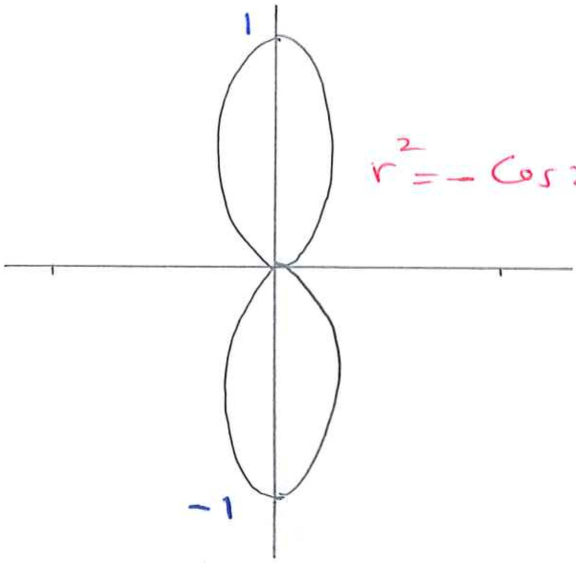
The Curve is called "Lemniscate"

Example : sketch $r^2 = -\sin 2\theta$.



Example: sketch $r^2 = \cos 2\theta$ & $r^2 = -\cos 2\theta$.





Slope of the Curve $r = f(\theta)$

If we have a curve that is given in Polar Coordinates and we want to find the slope at (r, θ), then we need to find $\frac{dy}{dx}$.

$\left(\frac{dy}{dx} \neq f'(\theta) \right)$.

Recall that : $x = r\cos\theta = f(\theta)\cos\theta.$

$$y = r\sin\theta = f(\theta)\sin\theta.$$

$$\Rightarrow \text{Slope} = \left.\frac{dy}{dx}\right|_{(r,\theta)} = \left.\frac{dy/d\theta}{dx/d\theta}\right|_{(r,\theta)}$$

$$= \frac{f(\theta)\cos\theta + f'(\theta)\sin\theta}{-f(\theta)\sin\theta + f'(\theta)\cos\theta}, \quad \frac{dx}{d\theta} \neq 0$$

Remark : If $r = f(\theta)$ passes through the Origin $(0, \theta_0)$, then

$$\text{slope}\Big|_{(0,\theta_0)} = \frac{f'(\theta_0)\sin\theta_0}{f'(\theta_0)\cos\theta_0} = \tan\theta_0$$

Example: Find the Slope of $r = \cos 2\theta$ at $\theta = 0$ & $\theta = \frac{\pi}{2}$.

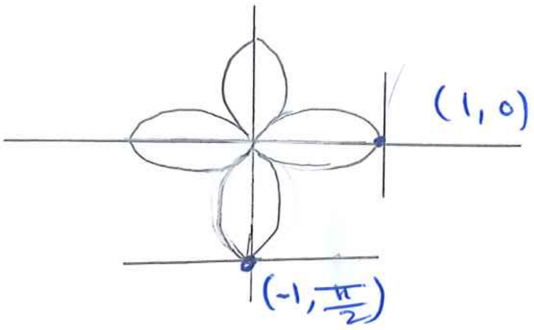
sol: when $\theta = 0$, then $r = 1$.

$$\text{slope}\Big|_{(1,0)} = \left.\frac{r\cos\theta + r'\sin\theta}{-r\sin\theta + r'\cos\theta}\right|_{(1,0)} = \frac{1(\cos 0) + -2\sin 0(\sin 0)}{-1(\sin 0) - 2\sin 0(\cos 0)}$$

$$= \frac{1}{0}, \text{ Undefined}$$

When $\theta = \frac{\pi}{2}$, $r = -1$

$$\text{slope}\Big|_{(-1,\frac{\pi}{2})} = \frac{-1\cos(\frac{\pi}{2}) - 2\sin(\pi)\sin(\frac{\pi}{2})}{1\sin(\frac{\pi}{2}) - 2\sin(\pi)\cos(\frac{\pi}{2})} = \boxed{0}$$



Example: Find the slope of $r = -2 + 3\cos\theta$ at $(0, \frac{\pi}{3})$.

$$\text{slope}\Big|_{(0,\frac{\pi}{3})} = \tan\left(\frac{\pi}{3}\right) = \sqrt{3}.$$

Example: For the Cardioid $r = 1 + \sin\theta$.

1) Find the slope of the tangent line at $\theta = \frac{\pi}{3}$.

2) Find the points on the Cardioid where the tangent line is Horizontal or Vertical.

sol: 1) $$\frac{dy}{dx} = \frac{r\cos\theta + r'\sin\theta}{-r\sin\theta + r'\cos\theta}.$$

$$= \frac{(1+\sin\theta)\cos\theta + \cos\theta\sin\theta}{-(1+\sin\theta)\sin\theta + \cos\theta \cdot \cos\theta}$$

$$\Rightarrow \left.\frac{dy}{dx}\right|_{(\theta=\frac{\pi}{3})} = \frac{(1+\sin\frac{\pi}{3})\cos(\frac{\pi}{3}) + \cos(\frac{\pi}{3})\sin(\frac{\pi}{3})}{-(1+\sin\frac{\pi}{3})\sin(\frac{\pi}{3}) + (\cos\frac{\pi}{3})^2}$$

$$= \frac{(1+\frac{\sqrt{3}}{2})(\frac{1}{2}) + (\frac{1}{2})(\frac{\sqrt{3}}{2})}{-(1+\frac{\sqrt{3}}{2})(\frac{\sqrt{3}}{2}) + (\frac{1}{2})^2}$$

$$= \frac{(1+\sqrt{3})/2}{-(1+\sqrt{3})/2} = \boxed{-1}$$

2) Recall that the Horizontal Tangent Lines Occur when $\frac{dy}{dx} = 0 \iff \frac{dy}{d\theta} = 0$

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& the vertical tangent where the deniomenator is 0, Now, $y = r\sin\theta$, then

$$\therefore \frac{dy}{d\theta} = r\cos\theta + r'\sin\theta = (1+\sin\theta)\cos\theta + \sin\theta\cos\theta$$

$$= \cos\theta + 2\sin\theta\cos\theta = \cos\theta(1+2\sin\theta) = 0$$

$$\Rightarrow \cos\theta = 0 \Rightarrow \boxed{\theta = \frac{\pi}{2}, \frac{3\pi}{2}}.$$

$$\text{or } (1+2\sin\theta) = 0 \Rightarrow \sin\theta = -\frac{1}{2} \Rightarrow \boxed{\theta = \frac{7\pi}{6}, \frac{11\pi}{6}}$$

Now, $x = r\cos\theta$

$$\Rightarrow \frac{dx}{d\theta} = -r\sin\theta + r'\cos\theta = -(1+\sin\theta)\sin\theta + \cos^2\theta$$

$(1-\sin^2\theta)$

$$= (1-\sin^2\theta) - \sin\theta(1+\sin\theta)$$

$$= (1-\sin\theta)(1+\sin\theta) - \sin\theta(1+\sin\theta)$$

$$= (1+\sin\theta)(1-2\sin\theta) = 0$$

$$\Rightarrow (1+\sin\theta) = 0 \Rightarrow \sin\theta = -1 \Rightarrow \boxed{\theta = \frac{3\pi}{2}}$$

$$\text{or } (1-2\sin\theta) = 0 \Rightarrow \sin\theta = \frac{1}{2} \Rightarrow \boxed{\theta = \frac{\pi}{6}, \frac{5\pi}{6}}$$

Notice that $\left.\dfrac{dy/d\theta}{dx/d\theta}\right|_{\theta=\frac{3\pi}{2}} = \dfrac{0}{0}$.

So, we need to check if there is a Horizontal or vertical Asymptote at $\theta = \frac{3\pi}{2}$.

$$\lim_{\theta \to \frac{3\pi}{2}^-} \frac{dy/d\theta}{dx/d\theta} = \infty \quad \& \quad \lim_{\theta \to \frac{3\pi}{2}^+} \frac{dy/d\theta}{dx/d\theta} = -\infty$$

Therefore, at $\theta = \frac{3\pi}{2}$, there is a Vertical Asymptote.

Therefore, The Cardioid has a Horizontal Tangent Lines at $\left(2, \frac{\pi}{2}\right)$, $\left(\frac{1}{2}, \frac{7\pi}{6}\right)$, $\left(\frac{1}{2}, \frac{11\pi}{6}\right)$

and a Vertical Tangent Lines at

$\left(\frac{3}{2}, \frac{\pi}{6}\right)$, $\left(\frac{3}{2}, \frac{5\pi}{6}\right)$, $\left(0, \frac{3\pi}{2}\right)$

Lecture Problems:

(Q13) sketch $r^2 = 4\cos 2\theta$.

We need to check symmetries:

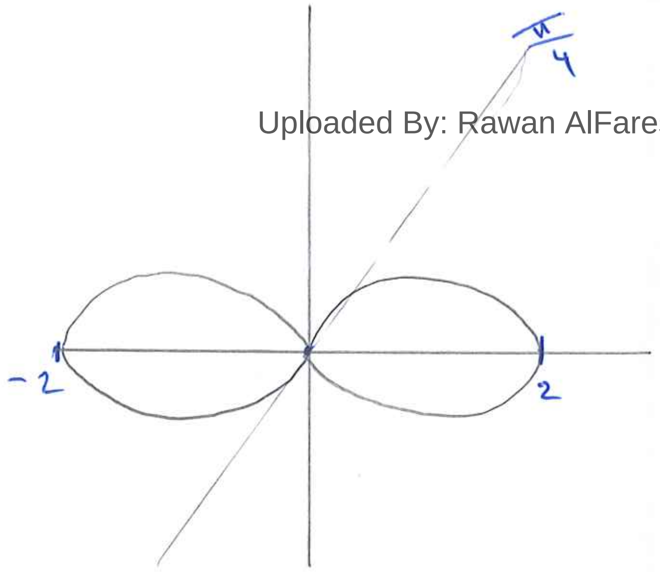
1) x-axis : If $(r, \theta) \in$ graph, then $(r, -\theta) \in$ graph.
(since : $4\cos(2(-\theta)) = 4\cos 2\theta = r$).

2) y-axis : If $(r, \theta) \in$ graph, then $(-r, -\theta) \in$ graph.
(since : $(-r)^2 = 4\cos(-2\theta) \Rightarrow r^2 = 4\cos 2\theta$.

3) Origin : Yes , (Check !!)

Notice that $r^2 = 4\cos 2\theta$ is always nonnegative.
therefore 2θ must be in the first & fourth quadrant. So take 2θ in the first quadrants then apply the symmetrices.

θ	$r = \pm\sqrt{4\cos 2\theta}$	(r, θ)
0	± 2	$(\pm 2, 0)$
$\frac{\pi}{12}$	± 1.86	$(\pm 1.86, \frac{\pi}{12})$
$\frac{\pi}{6}$	± 1.414	$(\pm 1.414, \frac{\pi}{6})$
$\frac{\pi}{4}$	0	$(0, \frac{\pi}{4})$



(Q14) $r^2 = 4\sin 2\theta$

check the symmmetries:

1) x-axis : If $(r,\theta) \in$ graph, then $(r,-\theta) \notin$ graph

& $(-r, \pi-\theta) \notin$ graph.

(NO)

$4\sin(-2\theta) = -4\sin 2\theta \neq r^2$.

& $4\sin(2(\pi-\theta)) = 4\sin(-2\theta) = -4\sin 2\theta \neq r^2$.

2) y-axis : If $(r,\theta) \in$ graph, then $(r,\pi-\theta) \notin$ graph (check !!) & $(-r,-\theta) \notin$ graph. (NO)

3) Origin: $(r,\theta) \in$ graph $\Rightarrow$ $(-r,\theta) \in$ graph. (Yes)

θ	$r = \pm\sqrt{4\sin 2\theta}$	(r,θ)
0	0	$(0,0)$
$\frac{\pi}{6}$	± 1.86	$(\pm 1.86, \frac{\pi}{6})$
$\frac{\pi}{4}$	± 2	$(\pm 2, \frac{\pi}{4})$
$\frac{\pi}{3}$	± 1.86	$(\pm 1.86, \frac{\pi}{3})$
$\frac{\pi}{2}$	0	$(0, \frac{\pi}{2})$