8.1 Integration by Parts:

$$\int f(x)\, g(x)\, dx$$

we Let $u = f(x)$ $\qquad dv = g(x)\,dx$ (The easiest to Integrate)

$\Rightarrow du = f'(x)\,dx \longrightarrow v = \int g(x)\,dx = h(x)$

then $\int f(x)\,g(x)\,dx = f(x)h(x) - \int h(x)\, f'(x)\,dx.$

Example: $\int x \cos x\, dx.$

Let $u = x$ $\qquad dv = \cos x\, dx$

$\Rightarrow du = dx$ $\qquad v = \sin x$

the $\int x \cos x\, dx = x \sin x - \int \sin x\, dx = x\sin x + \cos x + C_1$

Example: $\int \ln x\, dx.$

$\left.\begin{array}{ll} u = \ln x & dv = dx \\ du = \frac{1}{x}\,dx & v = x. \end{array}\right\} \Rightarrow \int \ln x = x \ln x - \int \frac{1}{x} \cdot x\, dx$

$\Rightarrow \int \ln x\, dx = x \ln x - x + C$

Example: $\int x^2 e^x \, dx$

$u = x^2 \qquad dv = e^x \, dx$

$du = 2x \, dx \qquad v = e^x.$

$\Rightarrow \int x^2 e^x \, dx = x^2 e^x - \int 2x e^x \, dx.$

Again Let $u = x \qquad dv = e^x \, dx$

$du = dx \qquad v = e^x.$

$\Rightarrow \int x^2 e^x \, dx = x^2 e^x - x e^x + \int e^x \, dx$

$= x^2 e^x - 2x e^x + 2e^x + C$

Example: $\int e^x \cos x \, dx.$

Let $u = e^x$, $dv = \cos x \, dx$

$du = e^x \, dx \qquad v = \sin x.$

$\Rightarrow \int e^x \cos x \, dx = e^x \sin x - \int e^x \sin x \, dx$

Again Let $u = e^x \qquad dv = \sin x \, dx$

$du = e^x \, dx \qquad v = -\cos x$

$\Rightarrow \int e^x \cos x \, dx = e^x \sin x + e^x \cos x - \int e^x \cos x \, dx + C$

$\Rightarrow 2\int e^x \cos x \, dx = e^x \sin x + e^x \cos x + C$

$\Rightarrow \int e^x \cos x \, dx = \frac{1}{2} e^x \sin x + \frac{1}{2} e^x \cos x + C.$

Tabular Integration:

We use this method to find $\int x^n g(x)\, dx$ when n is large & $g(x)$ is easy to Integrate.

derivatives		Integrations
x^n		$g(x)$
$n\, x^{n-1}$	$+$	$\int g(x)\, dx = h(x)$
$n(n-1)\, x^{n-2}$	$-$	$\int h(x)\, dx = r(x)$
$n(n-1)(n-2)\, x^{n-3}$	$+$	$\int r(x)\, dx = s(x)$
$\vdots$		
$\boxed{0}$		

$$\Rightarrow \int x^n g(x)\, dx = x^n h(x) - n x^{n-1} (r(x)) + n(n-1) x^{n-2} s(x) - \cdots$$

Example: $\int x^2 e^x \, dx. = x^2 e^x - 2x e^x + 2e^x + C$

x^2		e^x
$2x$	$+$	e^x
2	$-$	e^x
0	$+$	e^x

Example: $\int x^3 \sin x \, dx = -x^3 \cos x + 3x^2 \sin x + 6x \cos x - 6 \sin x + C$

x^3	$+$	$\sin x$
$3x^2$	$-$	$-\cos x$
$6x$	$+$	$-\sin x$
6	$-$	$\cos x$
0		$\sin x$

Example: Find the area of the region bounded by the Curve $y = xe^{-x}$ and the x-axis, $x=0$ to $x=4$

$$A = \int_0^4 x e^{-x} \, dx.$$

Let $u = x$ $\qquad dv = e^{-x} dx$

$du = dx$ $\qquad v = -e^{-x}.$

$$\Rightarrow A = -xe^{-x} \Big|_0^4 + \int_0^4 e^{-x} \, dx$$

$$= -4e^{-4} + -e^{-x} \Big|_0^4 = -4e^{-4} - e^{-4} + 1 = 1 - 5e^{-4}$$

(16) p^4 $\xrightarrow{+}$ $-e^{-p}$ (e^{-p})
$4p^3$ $\xrightarrow{-}$ e^{-p}
$12p^2$ $\xrightarrow{+}$ $-e^{-p}$
$24p$ $\xrightarrow{-}$ e^{-p}
24 $\xrightarrow{+}$ $-e^{-p}$
0

$= -p^4e^{-p} - 4p^3e^{-p} - 12p^2e^{-p} - 24pe^{-p} - 24e^{-p} + C$

(22) $u = \cos y$ & $dv = e^{-y}\,dy$
$du = -\sin y$ & $v = -e^{-y}$
$\Rightarrow -e^{-y} - \int e^{-y} \sin y.$
[Again Integration by parts]
$\Rightarrow \frac{1}{2}(e^{-y}\sin y - e^{-y}\cos y) + C$

We combine the products of the functions connected by the arrows according to the operation signs above the arrows to obtain

$$\int x^2e^x\,dx = x^2e^x - 2xe^x + 2e^x + C.$$

Compare this with the result in Example 3.

EXAMPLE 8 Evaluate

$$\int x^3 \sin x\,dx.$$

Solution With $f(x) = x^3$ and $g(x) = \sin x$, we list:

$f(x)$ and its derivatives		$g(x)$ and its integrals
		$\sin x$
x^3	$(+)$	$-\cos x$
$3x^2$	$(-)$	$-\sin x$
$6x$	$(+)$	$\cos x$
6	$(-)$	$\sin x$
0		

Again we combine the products of the functions connected by the arrows according to the operation signs above the arrows to obtain

$$\int x^3 \sin x\,dx = -x^3\cos x + 3x^2\sin x + 6x\cos x - 6\sin x + C.$$

The Additional Exercises at the end of this chapter show how tabular integration can be used when neither function f nor g can be differentiated repeatedly to become zero.

Exercises 8.1

Integration by Parts

Evaluate the integrals in Exercises 1–24 using integration by parts.

1. $\int x \sin \frac{x}{2}\,dx$
2. ~~2.~~ $\int \theta \cos \pi\theta\,d\theta$ $u = \theta$, $dv = \cos\pi\theta\,d\theta$ $\frac{\theta}{\pi}\sin\pi\theta + \frac{1}{\pi^2}\cos\pi\theta + C$
3. $\int t^2 \cos t\,dt$
4. $\int x^2 \sin x\,dx$
5. $\int_1^2 x \ln x\,dx$
6. ~~6.~~ $\int_1^e x^3 \ln x\,dx$ $\rightarrow u = \ln x$, $dv = x^3\,dx$
7. $\int xe^x\,dx$
8. $\int xe^{3x}\,dx$
9. $\int x^2 e^{-x}\,dx$
10. $\int (x^2 - 2x + 1)\,e^{2x}\,dx$
11. ~~11.~~ $\int \tan^{-1} y\,dy$
12. $\int \sin^{-1} y\,dy$
13. $\int x \sec^2 x\,dx$
14. $\int 4x \sec^2 2x\,dx$
15. $\int x^3e^x\,dx$
16. ~~16.~~ $\int p^4e^{-p}\,dp$
17. $\int (x^2 - 5x)e^x\,dx$
18. $\int (r^2 + r + 1)e^r\,dr$
19. $\int x^5e^x\,dx$
20. $\int t^2e^{4t}\,dt$
21. $\int e^\theta \sin\theta\,d\theta$
22. ~~22.~~ $\int e^{-y}\cos y\,dy$
23. $\int e^{2x}\cos 3x\,dx$
24. $\int e^{-2x}\sin 2x\,dx$

Using Substitution

Evaluate the integrals in Exercises 25–30 by using a substitution prior to integration by parts.

25. $\int e^{\sqrt{3s+9}}\,ds$
26. $\int_0^1 x\sqrt{1-x}\,dx$

(11) $u = \tan^{-1} y \Rightarrow du = \frac{1}{1+y^2}\,dy$, & $dv = dy$, $v = y$

$= y\tan^{-1} y - \int \left(\frac{y}{1+y^2}\right) du = y\tan^{-1} y - \frac{1}{2}\ln(1+y^2) + C$

(28) $u = \ln(x+x^2)$, $dv = dx$
$du = \frac{1+2x}{x+x^2}$, $v = x$

$\Rightarrow x\ln(x+x^2) - \int \frac{x+2x^2}{x+x^2}\,dx = x\ln(x+x^2) - \int\left(\frac{2x+1}{x+1}\right)dx$
$= x\ln(x+x^2) - \int \frac{2(x+1)-1}{(x+1)}\,dx =$

$x\ln(x+x^2) - 2x + \ln|x+1| + c$

27. $\int_0^{\pi/3} x\tan^2 x\,dx$
28. $\int \ln(x+x^2)\,dx$
29. $\int \sin(\ln x)\,dx$
30. $\int z(\ln z)^2\,dz$

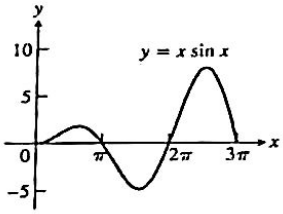
Evaluating Integrals

Evaluate the integrals in Exercises 31–50. Some integrals do not require integration by parts.

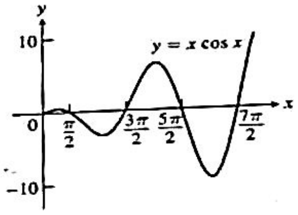
31. $\int x\sec x^2\,dx$
32. $\int \frac{\cos\sqrt{x}}{\sqrt{x}}\,dx$
33. $\int x(\ln x)^2\,dx$
34. $\int \frac{1}{x(\ln x)^2}\,dx$
35. $\int \frac{\ln x}{x^2}\,dx$
36. $\int \frac{(\ln x)^3}{x}\,dx$
37. $\int x^3 e^{x^4}\,dx$
38. $\int x^5 e^{x^3}\,dx$
39. $\int x^3\sqrt{x^2+1}\,dx$
40. $\int x^2\sin x^3\,dx$
41. $\int \sin 3x\cos 2x\,dx$
42. $\int \sin 2x\cos 4x\,dx$
43. $\int e^x\sin e^x\,dx$
44. $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$
45. $\int \cos\sqrt{x}\,dx$
46. $\int \sqrt{x}\,e^{\sqrt{x}}\,dx$
47. $\int_0^{\pi/2} \theta^2\sin 2\theta\,d\theta$
48. $\int_0^{\pi/2} x^3\cos 2x\,dx$
49. $\int_{2/\sqrt{3}}^{2} t\sec^{-1} t\,dt$
50. $\int_0^{1/\sqrt{2}} 2x\sin^{-1}(x^2)\,dx$

Theory and Examples

51. **Finding area** Find the area of the region enclosed by the curve $y = x\sin x$ and the x-axis (see the accompanying figure) for
   a. $0 \le x \le \pi$.
   b. $\pi \le x \le 2\pi$.
   c. $2\pi \le x \le 3\pi$.
   d. What pattern do you see here? What is the area between the curve and the x-axis for $n\pi \le x \le (n+1)\pi$, n an arbitrary nonnegative integer? Give reasons for your answer.



52. **Finding area** Find the area of the region enclosed by the curve $y = x\cos x$ and the x-axis (see the accompanying figure) for
   a. $\pi/2 \le x \le 3\pi/2$.
   b. $3\pi/2 \le x \le 5\pi/2$.
   c. $5\pi/2 \le x \le 7\pi/2$.
   d. What pattern do you see? What is the area between the curve and the x-axis for
   $$\left(\frac{2n-1}{2}\right)\pi \le x \le \left(\frac{2n+1}{2}\right)\pi,$$
   n an arbitrary positive integer? Give reasons for your answer.



53. **Finding volume** Find the volume of the solid generated by revolving the region in the first quadrant bounded by the coordinate axes, the curve $y = e^x$, and the line $x = \ln 2$ about the line $x = \ln 2$.
54. **Finding volume** Find the volume of the solid generated by revolving the region in the first quadrant bounded by the coordinate axes, the curve $y = e^{-x}$, and the line $x = 1$
   a. about the y-axis.
   b. about the line $x = 1$.
55. **Finding volume** Find the volume of the solid generated by revolving the region in the first quadrant bounded by the coordinate axes and the curve $y = \cos x$, $0 \le x \le \pi/2$, about
   a. the y-axis.
   b. the line $x = \pi/2$.
56. **Finding volume** Find the volume of the solid generated by revolving the region bounded by the x-axis and the curve $y = x\sin x$, $0 \le x \le \pi$, about
   a. the y-axis.
   b. the line $x = \pi$.
   (See Exercise 51 for a graph.)
57. Consider the region bounded by the graphs of $y = \ln x$, $y = 0$, and $x = e$.
   a. Find the area of the region.
   b. Find the volume of the solid formed by revolving this region about the x-axis.
   c. Find the volume of the solid formed by revolving this region about the line $x = -2$.
   d. Find the centroid of the region.
58. Consider the region bounded by the graphs of $y = \tan^{-1} x$, $y = 0$, and $x = 1$.
   a. Find the area of the region.
   b. Find the volume of the solid formed by revolving this region about the y-axis.
59. **Average value** A retarding force, symbolized by the dashpot in the accompanying figure, slows the motion of the weighted spring so that the mass's position at time t is
$$y = 2e^{-t}\cos t, \qquad t \ge 0.$$

(33) $u = \ln x \Rightarrow \begin{cases} x = e^u \\ dx = e^u\,du \end{cases}$
$du = \frac{1}{x}\,dx$

$\Rightarrow \int e^{2u}u^2\,du \Rightarrow$

$= \frac{1}{2}u^2e^{u} - \frac{2u}{4}e^{u} + \frac{2e^{u}}{8} + c$

u^2 | e^{2u}
$2u$ | $\oplus\ \frac{1}{2}e^{2u}$
2 | $\ominus\ \frac{1}{4}e^{2u}$
0 | $\oplus\ \frac{1}{8}e^{2u}$

(39) $u = x^2 \Rightarrow du = 2x\,dx$
$dv = \sqrt{x^2+1}\;x\,dx$
$v = \frac{1}{3}(x^2+1)^{\frac{3}{2}}$
by parts

(39) $u = x^2$ $\qquad$ $dv = \sqrt{x^2+1} \cdot x \, dx$

$du = 2x\,dx$ $\qquad$ $v = \frac{1}{3}(x^2+1)^{3/2}$

$$\Rightarrow \frac{1}{3} x^2 (x^2+1)^{3/2} - \int \frac{1}{3}(x^2+1)^{3/2} \cdot 2x \, dx$$

$$\frac{1}{3} x^2 (x^2+1)^{3/2} - \frac{2}{15}(x^2+1)^{5/2} + C$$

where $\frac{d}{dx}\left(\frac{2}{15}(x^2+1)^{5/2}\right) = \frac{2}{15} \cdot \frac{5}{2}(2x) \cdot (x^2+1)^{3/2}$.

(46) $\int \sqrt{x}\, e^{\sqrt{x}} \, dx$

Let $u = \sqrt{x}$

$du = \frac{1}{2\sqrt{x}} dx$

$\Rightarrow dx = 2u\,du$

$$\Rightarrow \int u e^u \cdot (2u\,du) = \int 2u^2 e^u \, du = 2[u^2 e^u - 2u e^u + 2e^u] + C$$

$$= 2[x e^{\sqrt{x}} - 2\sqrt{x} e^{\sqrt{x}} + 2e^{\sqrt{x}}] + C$$

u^2		e^u
$2u$	$+$	e^u
2	$-$	e^u
0	$+$	e^u

8.2 Trigonometric Integrals:

The purpose is to find $\int \sin^m x \cos^n x \, dx$:

n anything even or odd

<u>Case 1</u> If m is odd, we write $m = 2k+1$

then use $\sin^2 x = 1 - \cos^2 x$

$\rightsquigarrow \sin^m x = (\sin^2 x)^k \sin x = (1 - \cos^2 x)^k \sin x.$

Then we write $\boxed{\sin x \, dx}$ as $\boxed{-d(\cos x)}$

<u>Example:</u> Evaluate $\int \sin^3 x \cos^2 x \, dx.$

$\sin^3 x = \sin^2 x \cdot \sin x = (1 - \cos^2 x) \sin x$

$\rightsquigarrow \int \sin^3 x \cos^2 x \, dx = \int (1 - \cos^2 x) \cos^2 x \, \sin x \, dx$

$= \int (1 - \cos^2 x) \cos^2 x \, (-d(\cos x))$

Let $u = \cos x$

$= \int \cos^2 x - \cos^4 x \, (-d \cos x) =$

$= -\int u^2 - u^4 \, du$

$= -\frac{u^3}{3} + \frac{u^5}{5} + C = -\frac{\cos^3 x}{3} + \frac{\cos^5 x}{5} + C$

Case 2: If m is even & n is odd

we write $n = 2k+1$ & use:

$$\cos^2 x = 1 - \sin^2 x$$

$$\Rightarrow \quad \cos^n x = (\cos^2 x)^k \cos x = (1 - \sin^2 x)^k \cos x$$

then use: $\cos x \, dx \rightsquigarrow d(\sin x)$.

Example: Evaluate $\int \cos^5 x \, dx$.

here $\underline{m = 0}$ (even) & $n = 5$ (odd)

$$\rightsquigarrow \quad \cos^5 x = (\cos^2 x)^2 \cos x$$

$$\rightsquigarrow \quad \int \cos^5 x \, dx = \int (1 - \sin^2 x)^2 \cos x \, dx$$

Let $u = \sin x$

$$= \int (1 - \sin^2 x)^2 \, d \sin x$$

$$= \int (1 - u^2)^2 \, du = \int 1 - 2u^2 + u^4 \, du$$

$$= u - \frac{2u^3}{3} + \frac{u^5}{5} + C$$

$$= \sin x - \frac{2\sin^3 x}{3} + \frac{\sin^5 x}{5} + C$$

Case 3 If both m & n are even

then: $\sin^2 x = \dfrac{1-\cos 2x}{2}$, $\cos^2 x = \dfrac{1+\cos 2x}{2}$

Example: $\int \sin^2 x \cos^4 x \, dx$.

$$\int \sin^2 x \cos^4 x \, dx = \int \left(\frac{1-\cos 2x}{2}\right)\left(\frac{1+\cos 2x}{2}\right)^2 dx$$

$$= \frac{1}{8}\int (1-\cos 2x)(1+2\cos 2x+\cos^2 2x)\, dx.$$

$$= \frac{1}{8}\int (1+\cos 2x-\cos^2 2x-\cos^3 2x)\, dx.$$

$$= \frac{1}{8}\left[x+\frac{\sin 2x}{2}-\int (\cos^2 2x+\cos^3 2x)\, dx.\right]$$

m & n are even

- Need to Evaluate $\int \cos^2 2x \, dx + \int \cos^3 2x \, dx$

$\boxed{1}$ $\int \cos^2 2x \, dx = \int \dfrac{1+\cos 4x}{2} dx = \dfrac{1}{2}\left(x+\dfrac{\sin 4x}{4}\right).$

m even, n odd.

$\boxed{2}$ $\int \cos^3 2x \, dx = \int (1-\sin^2 2x)\cos 2x \, dx.$ $\quad u = \sin 2x$, $du = 2\cos 2x \, dx$

$$= \int (1-u^2)\cdot\frac{1}{2}du = \frac{1}{2}\left(u-\frac{u^3}{3}\right) = \frac{1}{2}\left(\sin 2x-\frac{\sin^3 2x}{3}\right)$$

then The whole Integration is:

$$= \frac{1}{8}\left[x+\cancel{\frac{\sin 2x}{2}}-\frac{1}{2}x-\frac{1}{8}\sin 4x-\cancel{\frac{1}{2}\sin 2x}+\frac{\sin^3 2x}{6}\right]+C$$

$$= \frac{1}{16}\left[x-\frac{1}{4}\sin 4x+\frac{1}{3}\sin^3 2x\right]+C$$

Eliminating Square Roots:

Example: Evaluate $\int_0^{\frac{\pi}{4}} \sqrt{1+\cos 4x}\; dx$

Recall: $\cos 2x = 2\cos^2 x - 1 \Rightarrow \cos^2 x = \frac{\cos 2x + 1}{2}$

$\Rightarrow 2\cos^2 2x = \cos 4x + 1$, So we can substitute.

$$\int_0^{\frac{\pi}{4}} \sqrt{1+\cos 4x}\; dx = \int_0^{\frac{\pi}{4}} \sqrt{2}\sqrt{\cos^2 2x}\; dx = \int_0^{\frac{\pi}{4}} \sqrt{2}\; \underbrace{|\cos 2x|}_{\substack{\geq 0 \\ \text{on } [0,\frac{\pi}{4}]}}\; dx$$

$$= \int_0^{\frac{\pi}{4}} \sqrt{2}\cos 2x\; dx = \sqrt{2}\; \frac{\sin 2x}{2}\Big]_0^{\frac{\pi}{4}} = \frac{\sqrt{2}}{2}$$

$\cos 2x = 2\cos^2 x - 1$

$\cos 4x = 2\cos^2 2x - 1$

Integrals of Powers of tan x & sec x:

Example: Evaluate $\int \tan^4 x\; dx$.

$$\int \tan^4 x\; dx = \int \tan^2 x\; \underline{\tan^2 x}\, dx = \int \tan^2 x\,(\sec^2 x - 1)\; dx$$

$$= \int \tan^2 x \sec^2 x\, dx - \int \tan^2 x \,.\, dx$$

$$= \int \tan^2 x\; \sec^2 x\, dx - \int (\sec^2 x - 1)\; dx$$

$$= \int \tan^2 x\; \sec^2 x\, dx - \int \sec^2 x\, dx + \int 1\; dx$$

by parts

In $\int \tan^2 x \sec^2 x \, dx$:

Let $u = \tan x$ $\quad du = \sec^2 x \, dx$

then. $\int u^2 \, du = \frac{1}{3} u^3 + C_1$ فش داعي

then: The whole Integral becomes:

$$\int \tan^4 x \, dx = \frac{1}{3} \tan^3 x - \tan x + x + C_1$$

<u>Example</u>: Evaluate $\int \sec^3 x \, dx$

Integrate by parts:

$u = \sec x$ $\quad dv = \sec^2 x \, dx$

$du = \sec x \tan x \, dx$ $\quad v = \tan x$

$$\Rightarrow \int \sec^3 x \, dx = \sec x \tan x - \int \tan^2 x \sec x \, dx$$

$$= \sec x \tan x - \int (\sec^2 x - 1) \sec x \, dx$$

$$= \sec x \tan x - \int \sec^3 x + \int \sec x \, dx$$

$$\Rightarrow 2 \int \sec^3 x \, dx = \sec x \tan x + \int \sec x \, dx$$

$$\Rightarrow \int \sec^3 x \, dx = \frac{\sec x \tan x}{2} + \frac{1}{2} \ln|\sec x + \tan x| + C_1$$

Products of Sines and Cosines:

$$\sin mx \sin nx = \frac{1}{2}\left[\cos(m-n)x - \cos(m+n)x\right],$$

$$\sin mx \cos nx = \frac{1}{2}\left[\sin(m-n)x + \sin(m+n)x\right],$$

$$\cos mx \cos nx = \frac{1}{2}\left[\cos(m-n)x + \cos(m+n)x\right].$$

<u>Example:</u> Evaluate $\int \sin 3x \cos 5x \, dx$.

$$\int \sin 3x \cos 5x \, dx = \frac{1}{2}\int \left[\sin(-2x) + \sin(8x)\right] dx$$

$$= \frac{1}{2}\left[+\frac{\cos 2x}{2} - \frac{\cos 8x}{8}\right] + C$$

$$= \frac{\cos 2x}{4} - \frac{\cos 8x}{16} + C$$

(8.2) (19) $\int 16 \sin^2 x \cos^2 x \, dx$

$= \int 16 \left(\frac{1-\cos 2x}{2} \right)\left(\frac{1+\cos 2x}{2} \right) dx$

$= \int 4 \, (1-\cos^2 2x) \, dx = 4 \int \left(1 - \left[\frac{1+\cos 4x}{2} \right]\right) dx$

$= 4x - 2x - \frac{1}{2} \sin 4x + C_1 = 2x - \frac{1}{2} \sin 4x + C_1$

(34) $\int \sec x \tan^2 x \, dx = \int \sec x \tan x \tan x \, dx$

Let

$u = \tan x \qquad dv = \sec x \tan x \, dx$

$du = \sec^2 x \, dx \qquad v = \sec x.$

$\Rightarrow = \tan x \sec x - \int \sec^3 x \, dx$

$= \tan x \sec x - \int \sec^2 x \sec x \, dx$

$= \tan x \sec x - \int (\tan^2 x + 1) \sec x \, dx$

$= \tan x \sec x - \int \tan^2 x \sec x \, dx + \int \sec x \, dx$

$= \sec x \tan x - \ln|\sec x + \tan x| - \int \tan^2 x \overset{\sec x}{\uparrow} dx.$

$\Rightarrow 2 \int \sec x \tan^2 x \, dx = \sec x \tan x - \ln|\sec x + \tan x| + C_1$

$\Rightarrow \int \sec x \tan^2 x \, dx = \frac{1}{2} \sec x \tan x - \frac{1}{2} \ln|\sec x + \tan x| + C_1$

(52) $\int \sin 2x \cos 3x \, dx$

$= \frac{1}{2} \int (\underbrace{\sin(-x)}_{\text{odd}} + \sin 5x) \, dx = \frac{1}{2}\cos x - \frac{1}{10}\cos 5x + C$

(64) $\int \frac{\sin^3 x}{\cos^4 x} dx = \int \frac{\sin^2 x \sin x}{\cos^4 x} dx$

$= \int \frac{(1-\cos^2 x)\sin x}{\cos^4 x} dx = \int \frac{\sin x}{\cos^4 x} dx - \int \frac{\cos^2 x \sin x}{\cos^4 x} dx$

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$= \int \sec^3 x \tan x \, dx - \int \sec x \tan x \, dx$

$= \int \sec^2 x \sec x \tan x \, dx - \int \sec x \tan x \, dx$

$= \frac{1}{3} \sec^3 x - \sec x + C$

35 Let $u = \sec x$
$du = \sec x \tan x \, dx$

(67) $\int x \sin^2 x \, dx = \int x \left(\frac{1-\cos 2x}{2}\right) dx$

$= \frac{1}{2} \int x \, dx - \frac{1}{2} \int x \cos 2x \, dx$ $\quad \left[\begin{array}{ll} u = x & dv = \cos 2x \, dx \\ du = dx & v = \frac{1}{2}\sin 2x \end{array} \right.$

$= \frac{1}{2} \frac{x^2}{2} - \frac{1}{2} \left[\frac{1}{2} x \sin 2x - \frac{1}{2} \int \sin 2x \, dx \right]$

$= \frac{1}{4} x^2 - \frac{1}{4} x \sin 2x - \frac{1}{8} \cos 2x + C$