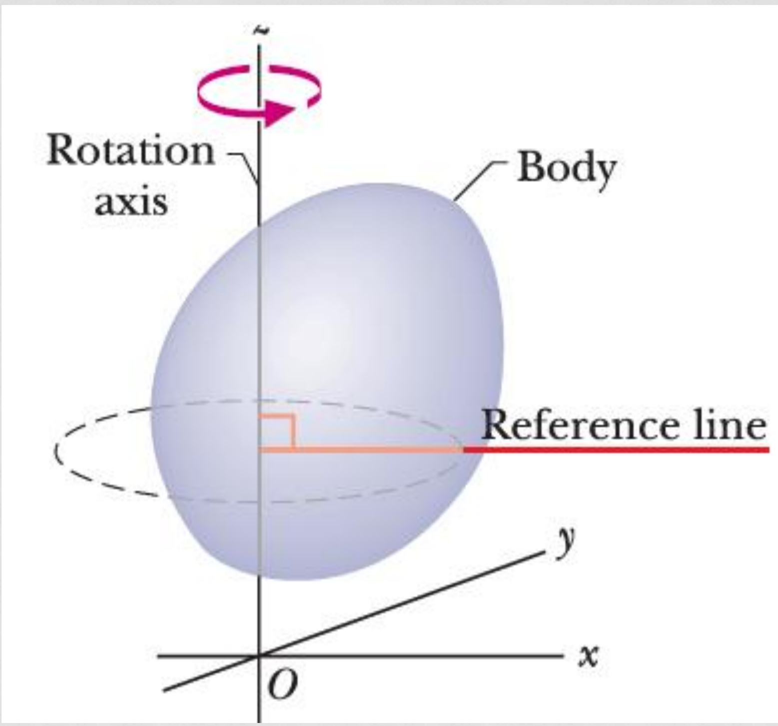
Chapter 10

Rotation

10-1 Rotational Variables

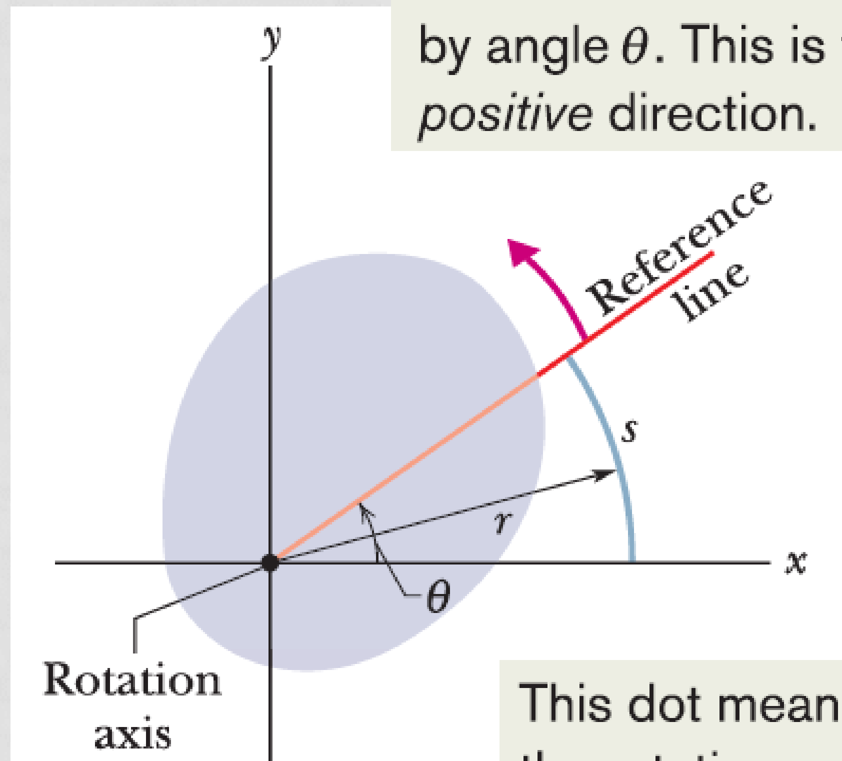
- Translational motion: object moves a long a straight or curved line.
- Rotational motion: object turns about an axis
- In rotational motion, we will need new quantities to express them
  - Torque
  - Rotational inertia
- A **rigid body** rotates as a unit, locked together
- We look at rotation about a **fixed axis**
- These requirements exclude from consideration:
  - The Sun, where layers of gas rotate separately
  - A rolling bowling ball, where rotation and translation occur

- The fixed axis is called the **axis of rotation**
- The **Angular Position** θ of this line (and of the object) is taken relative to a fixed direction, the **zero angular position**



This reference line is part of the body and perpendicular to the rotation axis. We use it to measure the rotation of the body relative to a fixed direction.

The body has rotated *counterclockwise* by angle θ. This is the *positive* direction.



This dot means that the rotation axis is out toward you.

- θ is Measured in **radians** (rad): dimensionless

$$\theta = \frac{s}{r} \quad \text{(radian measure)}$$

where s is the arc length of a circular path of radius r and angle θ.

$$1 \text{ rev} = 360° = \frac{2\pi r}{r} = 2\pi \text{ rad}$$

- Do not reset θ to zero after a full rotation ($\theta = 2\pi\ rad$)
- We know all there is to know about the kinematics of rotation if we have $\theta(t)$ for an object
- Define Angular Displacement as (a body rotates about a rotation axis, changing its angular position from θ_1 to θ_2):

$$\Delta\theta = \theta_2 - \theta_1$$

- *"Clocks are negative"*:

An angular displacement in the counterclockwise direction is positive, and one in the clockwise direction is negative.

Checkpoint 1

A disk can rotate about its central axis like a merry-go-round. Which of the following pairs of values for its initial and final angular positions, respectively, give a negative angular displacement: (a) −3 rad, +5 rad, (b) −3 rad, −7 rad, (c) 7 rad, −3 rad?

$$\Delta\theta = \theta_2 - \theta_1$$

Answer: Choices (b) and (c)

(a) $\Delta\theta = \theta_2 - \theta_1 = (+5\text{ rad}) - (-3\text{ rad}) = +8\text{ rad}$

(b) a) $\Delta\theta = \theta_2 - \theta_1 = (-7\text{ rad}) - (-3\text{ rad}) = -4\text{ rad}$

(c) a) $\Delta\theta = \theta_2 - \theta_1 = (-3\text{ rad}) - (7\text{ rad}) = -10\text{ rad}$

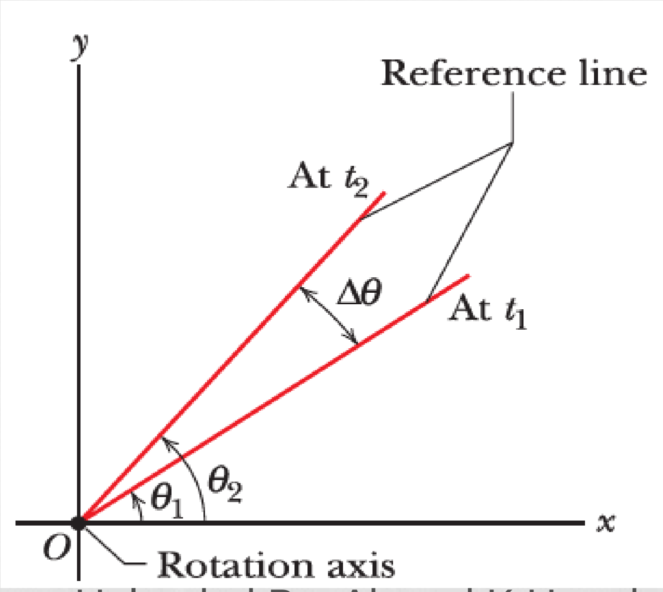
- **Average Angular Velocity**: angular displacement during a time interval

$$\omega_{avg} = \frac{\theta_2 - \theta_1}{t_2 - t_1} = \frac{\Delta\theta}{\Delta t}$$

- **Instantaneous Angular Velocity**: limit as $\Delta t \to 0$

$$\omega = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt}$$

- If the body is rigid, these equations hold for all points on the body
- Magnitude of Angular Velocity = **Angular Speed**



- **Average angular acceleration:** angular velocity change during a time interval

$$\alpha_{avg} = \frac{\omega_2 - \omega_1}{t_2 - t_1} = \frac{\Delta\omega}{\Delta t}$$

- **Instantaneous angular acceleration**: limit as $\Delta t \rightarrow 0$

$$\alpha = \lim_{\Delta t \to 0} \frac{\Delta\omega}{\Delta t} = \frac{d\omega}{dt}$$

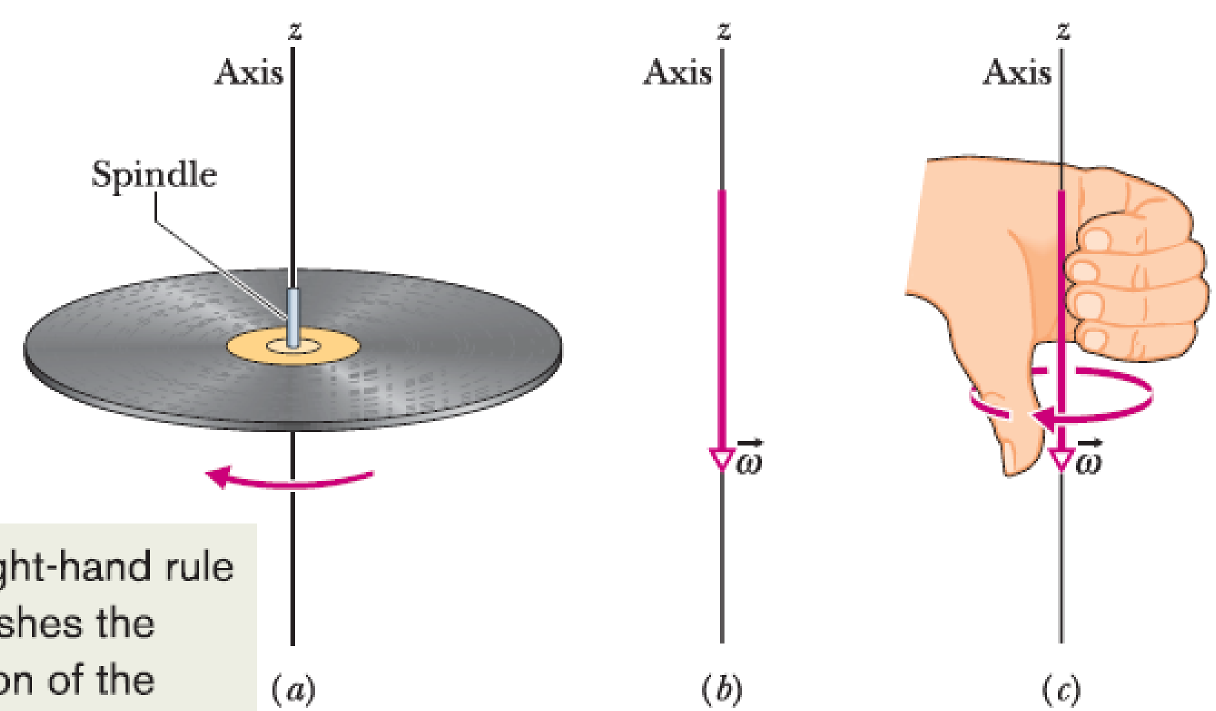
- If the body is rigid, these equations hold for all points on the body

	Translational motion	Rotational motion
position	$r\ (m)$	$\theta\ (rad\ OR\ degree)$
velocity	$v\ (m/s)$	$\omega\ (rad/s\ OR\ rpm)$
acceleration	$a\ (m/s^2)$	$\alpha\ (rad/s^2)$

$$\theta(t) \rightarrow \omega(t) \rightarrow \alpha(t) \qquad Differentiation$$

$$\theta(t) \leftarrow \omega(t) \leftarrow \alpha(t) \qquad Integration$$

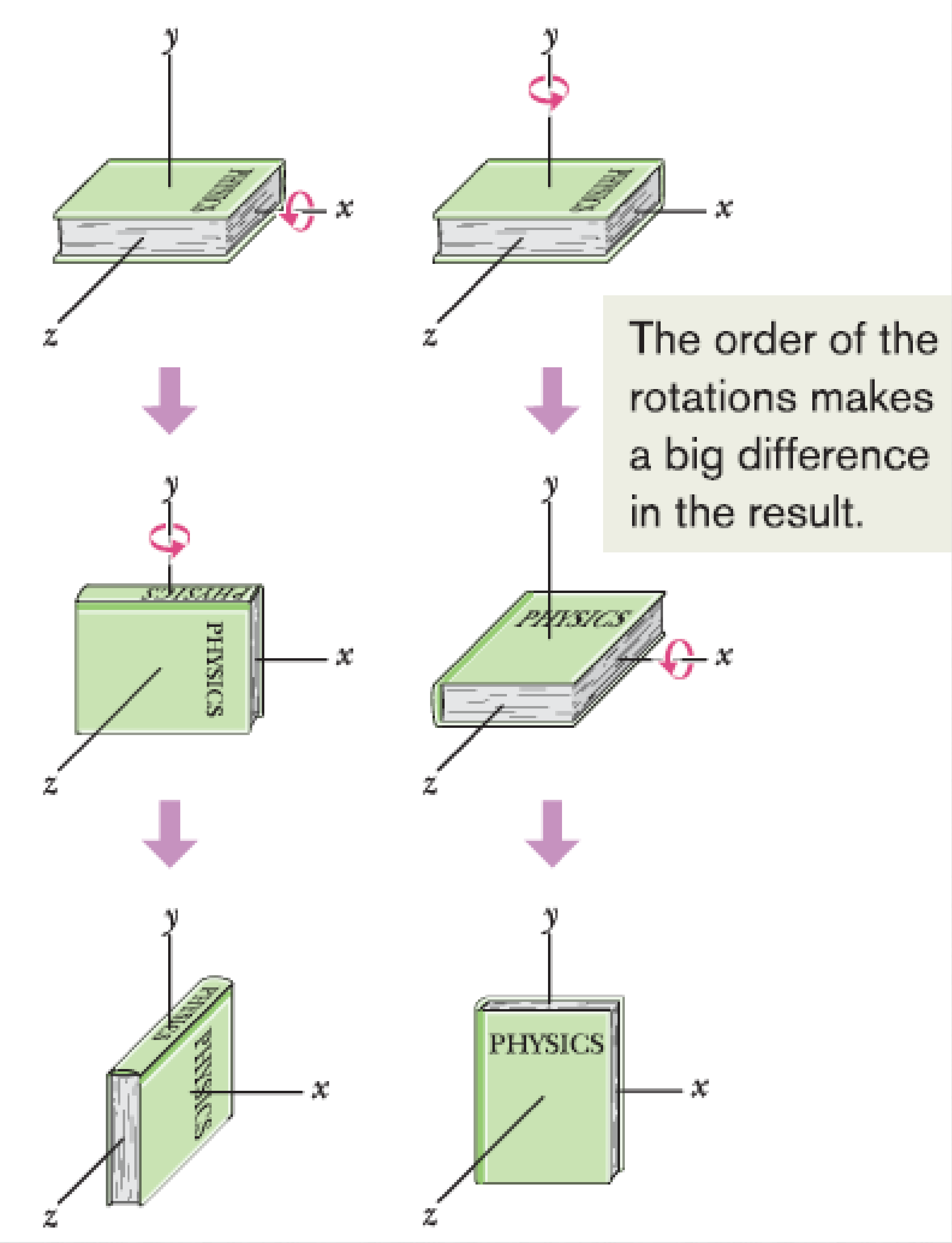
- Angular Velocity and Angular Acceleration $(\omega, \omega_{avg}, \alpha\ and \alpha_{avg})$ are vectors with directions given by a right-hand rule.
- They are positive for counterclockwise rotation and negative for clockwise rotation.
- If the body rotates around an axis, then the vector points along the axis of rotation



This right-hand rule establishes the direction of the angular velocity vector.

- Angular displacements can *not be treated as* vectors, because the order of rotation matters for rotations around different axes

Rule of vector addition: If you add two vectors, the order in which you add them does not matter.
Angular displacements fail this test!

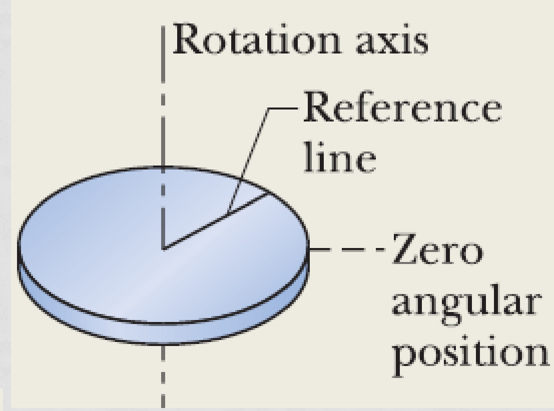


Sample Problem 10.01 Angular velocity derived from angular position

The disk in Fig. 10-5*a* is rotating about its central axis like a merry-go-round. The angular position $\theta(t)$ of a reference line on the disk is given by

$$\theta = -1.00 - 0.600t + 0.250t^2,$$

with t in seconds, θ in radians, and the zero angular position as indicated in the figure.



(a) Graph the angular position of the disk versus time from $t = -3.0$ s to $t = 5.4$ s. Sketch the disk and its angular position reference line at $t = -2.0$ s, 0 s, and 4.0 s, and when the curve crosses the t axis.

For $t = -2.0$ s,

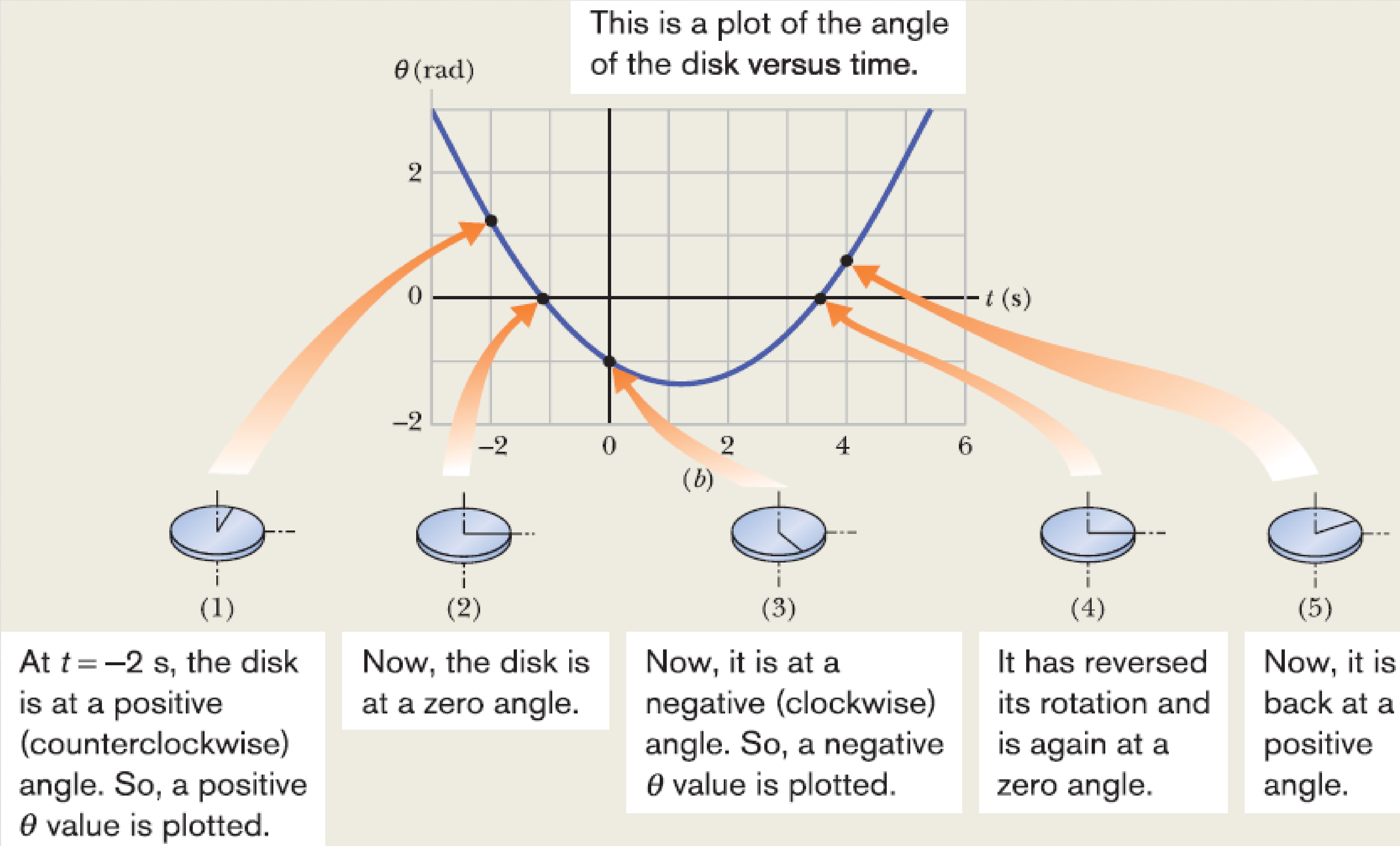
$$\theta = -1.00 - (0.600)(-2.0) + (0.250)(-2.0)^2$$

$$= 1.2 \text{ rad} = 1.2 \text{ rad} \frac{360°}{2\pi \text{ rad}} = 69°$$

This means that at $t = -2.0$ s the reference line on the disk is rotated counterclockwise from the zero position by angle 1.2 rad = 69° (counterclockwise because θ is positive).

$t = 0, \theta = -1\ rad = -57^{\circ}$ (clockwise by 57°)

$t = 4\ s, \theta = +0.6\ rad = 34^{\circ}$ (counterclockwise by 34°)



$$\theta = 0 \;\rightarrow\rightarrow\quad \theta = -1 - 0.6\,t + 0.25\,t^2 = 0$$

$$t = \frac{+0.6 \pm \sqrt{(-0.6)^2 - 4(0.25)(-1)}}{2(0.25)} = 3.53\text{ s or } -1.13\text{ s}$$

(b) At what time t_{min} does $\theta(t)$ reach the minimum value ? What is that minimum value?

To find the extreme value (here the minimum) of a function, we take the first derivative of the function and set the result to zero.

$$\frac{d\theta}{dt} = -0.600 + 0.500t = 0$$

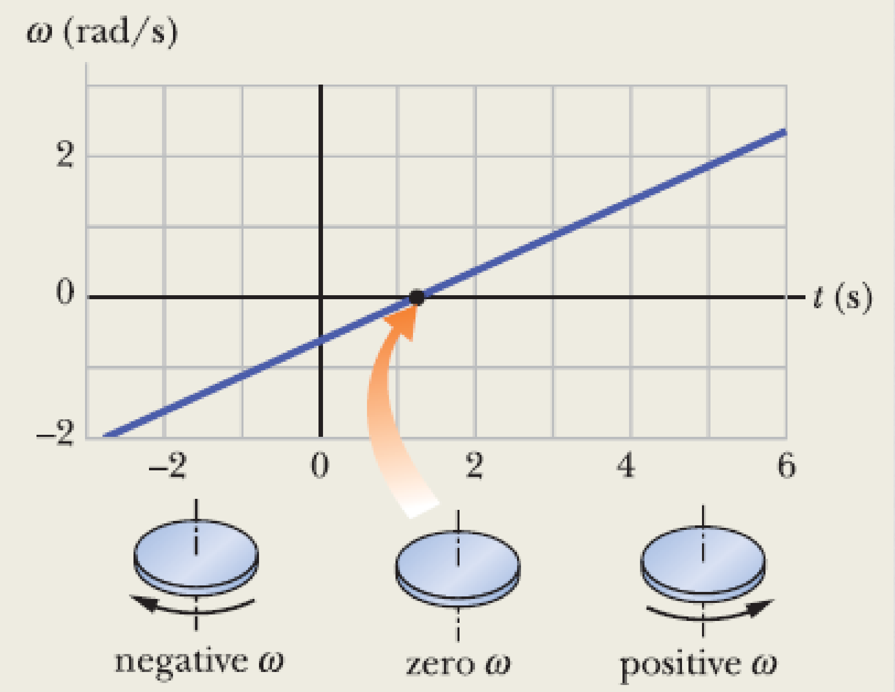
$$t_{min} = 1.20 \text{ s.}$$

$$\theta = -1.36 \text{ rad} \approx -77.9°$$

(c) Graph the angular velocity ω of the disk versus time

$$\omega = -0.600 + 0.500t$$

The angular velocity is initially negative and slowing, then momentarily zero during reversal, and then positive and increasing.



Sample Problem 10.02 Angular velocity derived from angular acceleration

A child's top is spun with angular acceleration

$$\alpha = 5t^3 - 4t,$$

with t in seconds and α in radians per second-squared. At $t = 0$, the top has angular velocity 5 rad/s, and a reference line on it is at angular position $\theta = 2$ rad.

(a) Obtain an expression for the angular velocity $\omega(t)$ of the top.

$$\omega = \int (5t^3 - 4t)\, dt = \tfrac{5}{4}t^4 - \tfrac{4}{2}t^2 + C.$$

$$at\ t = 0\ s, \omega = 5\, rad/s \rightarrow\rightarrow 5\, rad/s = 0 - 0 + C \rightarrow\rightarrow C = 5\, rad/s$$

$$\omega = \frac{5}{4}t^4 - 2t^2 + 5\, rad/s$$

(b) Obtain an expression for the angular position $\theta(t)$ of the top.

$$\theta = \int \omega\, dt = \int \left(\tfrac{5}{4}t^4 - 2t^2 + 5\right) dt$$

$$= \tfrac{1}{4}t^5 - \tfrac{2}{3}t^3 + 5t + C$$

$$at\ t = 0\ s, \theta = 2\, rad \rightarrow\rightarrow 2\ \text{rad} = C$$

$$\rightarrow\rightarrow C = 2 rad$$

$$\theta = \frac{1}{4}t^5 - \frac{2}{3}t^3 + 5t + 2$$

10-2 Rotation with Constant Angular Acceleration

- The same equations hold as for *constant linear acceleration*, with changing linear quantities to angular ones

Linear Equation	Missing Variable		Angular Equation
$v = v_0 + at$	$x - x_0$	$\theta - \theta_0$	$\omega = \omega_0 + \alpha t$
$x - x_0 = v_0 t + \frac{1}{2}at^2$	v	ω	$\theta - \theta_0 = \omega_0 t + \frac{1}{2}\alpha t^2$
$v^2 = v_0^2 + 2a(x - x_0)$	t	t	$\omega^2 = \omega_0^2 + 2\alpha(\theta - \theta_0)$
$x - x_0 = \frac{1}{2}(v_0 + v)t$	a	α	$\theta - \theta_0 = \frac{1}{2}(\omega_0 + \omega)t$
$x - x_0 = vt - \frac{1}{2}at^2$	v_0	ω_0	$\theta - \theta_0 = \omega t - \frac{1}{2}\alpha t^2$

Checkpoint 2

In four situations, a rotating body has angular position $\theta(t)$ given by (a) $\theta = 3t - 4$, (b) $\theta = -5t^3 + 4t^2 + 6$, (c) $\theta = 2/t^2 - 4/t$, and (d) $\theta = 5t^2 - 3$. To which situations do the angular equations of Table 10-1 apply?

Constant Angular Acceleration: $\alpha = \text{constant}$

$$\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}$$

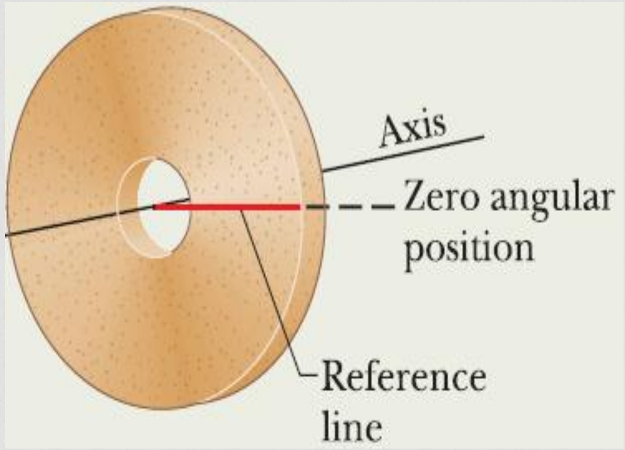
(a) $\omega = 3\ \text{rad/s},\ \alpha = 0$
(b) $\omega = -15t^2 + 8t,\ \alpha = -30\,t + 8$
(c) $\theta = 2t^{-2} - 4t^{-1}, \omega = -4t + 4t^{-2},\ \alpha = -4 - 8t^{-3}$
(d) $\omega = 10t, \alpha = 10\ \text{rad/s}^2$

Answer: Situations (a) and (d); the others do not have constant angular acceleration.

Sample Problem 10.03 Constant angular acceleration, grindstone

A grindstone (Fig. 10-8) rotates at constant angular acceleration $\alpha = 0.35\ \text{rad/s}^2$. At time $t = 0$, it has an angular velocity of $\omega_0 = -4.6$ rad/s and a reference line on it is horizontal, at the angular position $\theta_0 = 0$.

(a) At what time after $t = 0$ is the reference line at the angular position $\theta = 5.0$ rev?



$$\theta - \theta_0 = \omega_0 t + \tfrac{1}{2}\alpha t^2 \qquad \theta = 5.0\ \text{rev} = 10\pi\ \text{rad}$$

$$10\pi\ \text{rad} = (-4.6\ \text{rad/s})t + \tfrac{1}{2}(0.35\ \text{rad/s}^2)t^2 \qquad t = 32\ \text{s}$$

Clockwise = negative
Counterclockwise = positive

Description: The wheel is initially rotating in the negative (clockwise) direction with angular velocity $\omega_0 = -4.6$ rad/s, but its angular acceleration α is positive. This initial opposition of the signs of angular velocity and angular acceleration means that the wheel slows in its rotation in the negative direction, stops, and then reverses to rotate in the positive direction. After the reference line comes back through its initial orientation of $\theta = 0$, the wheel turns an additional 5.0 rev by time $t = 32$ s.

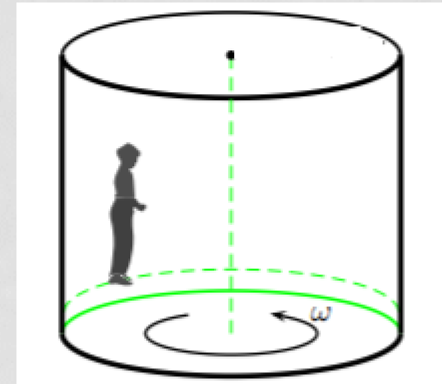
(c) At what time t does the grindstone momentarily stop?

$$t = \frac{\omega - \omega_0}{\alpha}$$

$$t = \frac{0 - (-4.6\ \text{rad/s})}{0.35\ \text{rad/s}^2} = 13\ \text{s}$$

Sample Problem 10.04 Constant angular acceleration, riding a Rotor

While you are operating a Rotor (a large, vertical, rotating cylinder found in amusement parks), you spot a passenger in acute distress and decrease the angular velocity of the cylinder from 3.40 rad/s to 2.00 rad/s in 20.0 rev, at constant angular acceleration. (The passenger is obviously more of a "translation person" than a "rotation person.")



(a) What is the constant angular acceleration during this decrease in angular speed?

$$t = \frac{\omega - \omega_0}{\alpha}$$

$$\theta - \theta_0 = \omega_0\left(\frac{\omega - \omega_0}{\alpha}\right) + \tfrac{1}{2}\alpha\left(\frac{\omega - \omega_0}{\alpha}\right)^2$$

$$\alpha = \frac{\omega^2 - \omega_0^2}{2(\theta - \theta_0)} = \frac{(2.00\ \text{rad/s})^2 - (3.40\ \text{rad/s})^2}{2(125.7\ \text{rad})} = -0.0301\ \text{rad/s}^2$$

$$20\ \text{rev} = 125.7\ \text{rad}$$

(b) How much time did the speed decrease take?

$$t = \frac{\omega - \omega_0}{\alpha} = \frac{2.00\ \text{rad/s} - 3.40\ \text{rad/s}}{-0.0301\ \text{rad/s}^2} = 46.5\ \text{s}.$$

10-3 Relating the Linear and Angular Variables

- Linear and angular variables are related by *r*, perpendicular distance from the rotational axis
- Position (note θ *must* be in radians): $s = \theta r$
- Speed (note ω *must* be in radian measure): $v = \omega r$

Note: All points within the rigid body have the same angular speed ω, points with greater radius r have greater linear speed$v$.

- We can express the Period (uniform circular motion) in radian measure:

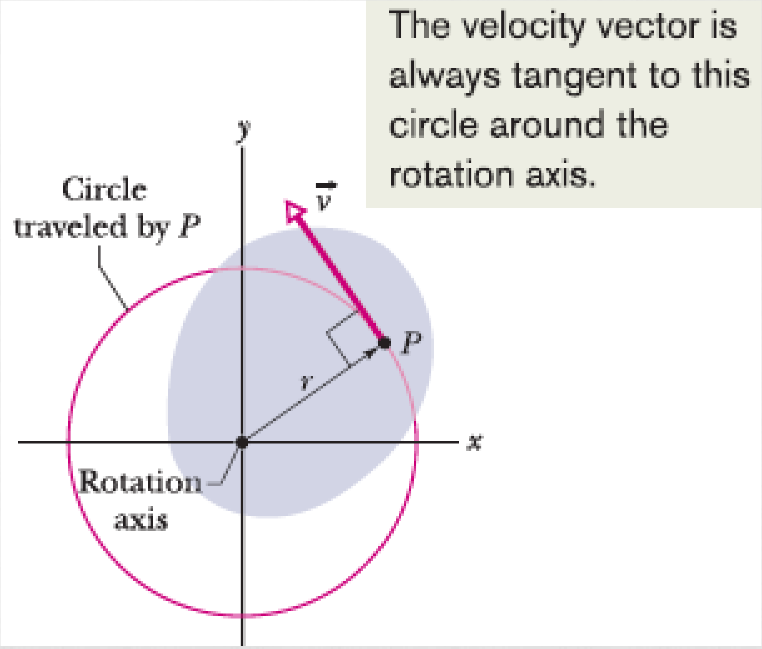
$$T = \frac{2\pi}{\omega}$$

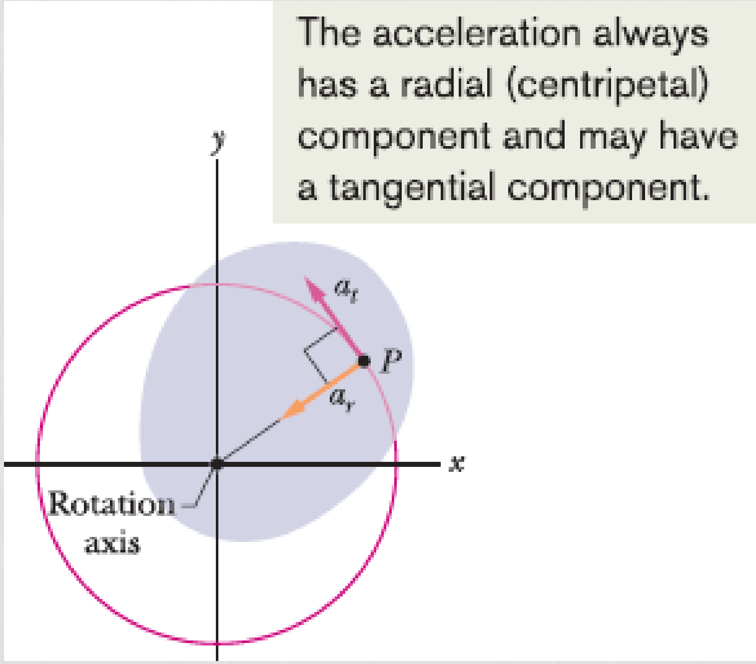
- Tangential acceleration (radians):

$$a_t = \alpha r$$

- Radial acceleration in terms of angular velocity (radians):

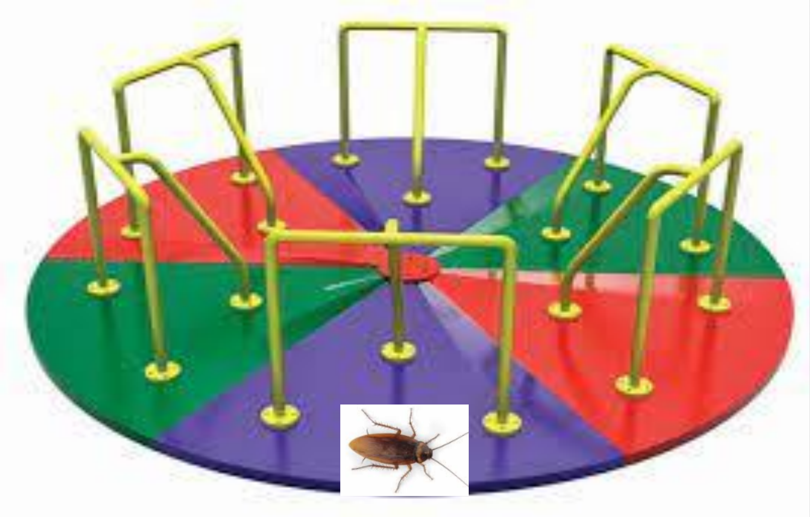
$$a_r = \frac{v^2}{r} = \omega^2 r$$





Checkpoint 3

A cockroach rides the rim of a rotating merry-go-round. If the angular speed of this system (*merry-go-round + cockroach*) is constant, does the cockroach have (a) radial acceleration and (b) tangential acceleration? If ω is decreasing, does the cockroach have (c) radial acceleration and (d) tangential acceleration?



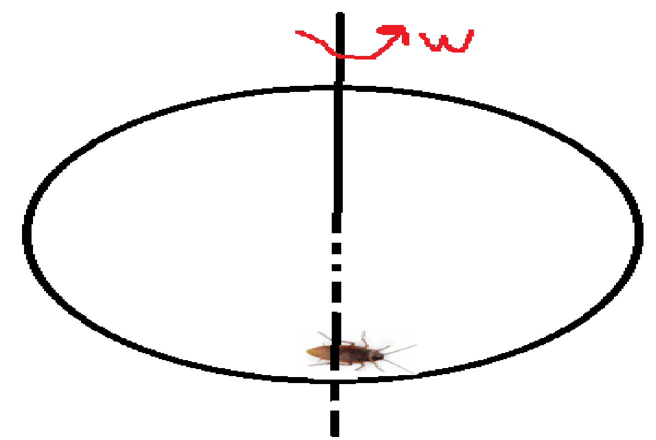
Answer:

(a) yes, since $a_r = \omega^2 r$

(b) No, since $a_t = \alpha r$ and the angular acceleration α is zero (No changing in the angular speed)

(c) yes

(d) yes



10-4 Kinetic Energy of Rotation

- A rotating rigid body (collection of particles with different speed):

$$K = \tfrac{1}{2}m_1v_1^2 + \tfrac{1}{2}m_2v_2^2 + \tfrac{1}{2}m_3v_3^2 + \cdots$$

$$K = \sum \tfrac{1}{2}m_iv_i^2$$

- different linear velocities (same angular velocity for all particles but possibly different radii)

$$K = \sum \tfrac{1}{2}m_i(\omega r_i)^2 = \tfrac{1}{2}\left(\sum m_i r_i^2\right)\omega^2$$

We call the quantity in parentheses on the right side the **Rotational Inertia**, or **Moment of Inertia**, I

→This is a constant for a rigid object and given rotational axis

→Caution: the axis for I must always be specified

- Rotational Inertia (system of discrete rotating particles):

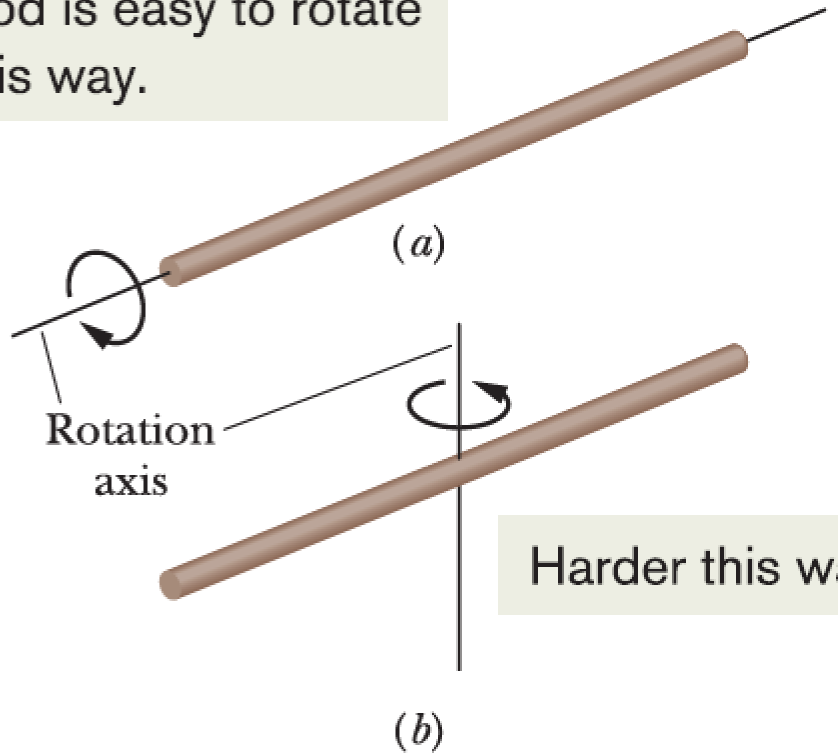
$$I = \sum m_i r_i^2 \quad \text{(rotational inertia)}$$

SI unit for I is $kg \cdot m^2$

- The kinetic energy of a rigid body rotating about a fixes axis is given by:

$$K = \frac{1}{2} I \omega^2 \quad \text{(radian measure)}$$

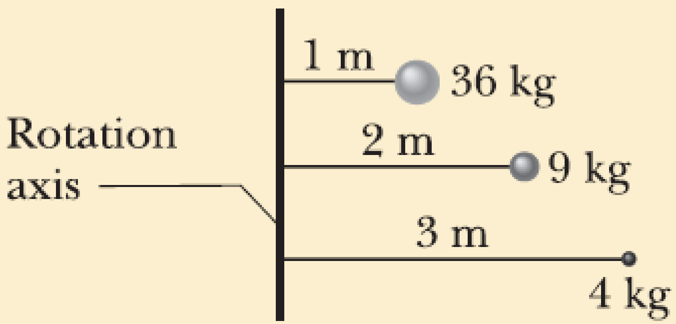
- Rotational inertia corresponds to how difficult it is to change the state of rotation (speed up, slow down or change the axis of rotation)



A long rod is much easier to rotate about (*a*) its central (longitudinal) axis than about (*b*) an axis through its center and perpendicular to its length. The reason for the difference is that the mass is distributed closer to the rotation axis in (*a*) than in (*b*).

Checkpoint 4

The figure shows three small spheres that rotate about a vertical axis. The perpendicular distance between the axis and the center of each sphere is given. Rank the three spheres according to their rotational inertia about that axis, greatest first.



Answer: **They are all equal!**

$$I = \sum m_i r_i^2 \quad \text{(rotational inertia)}$$

10-5 Calculating the Rotational Inertia

- system of discrete rotating particles:

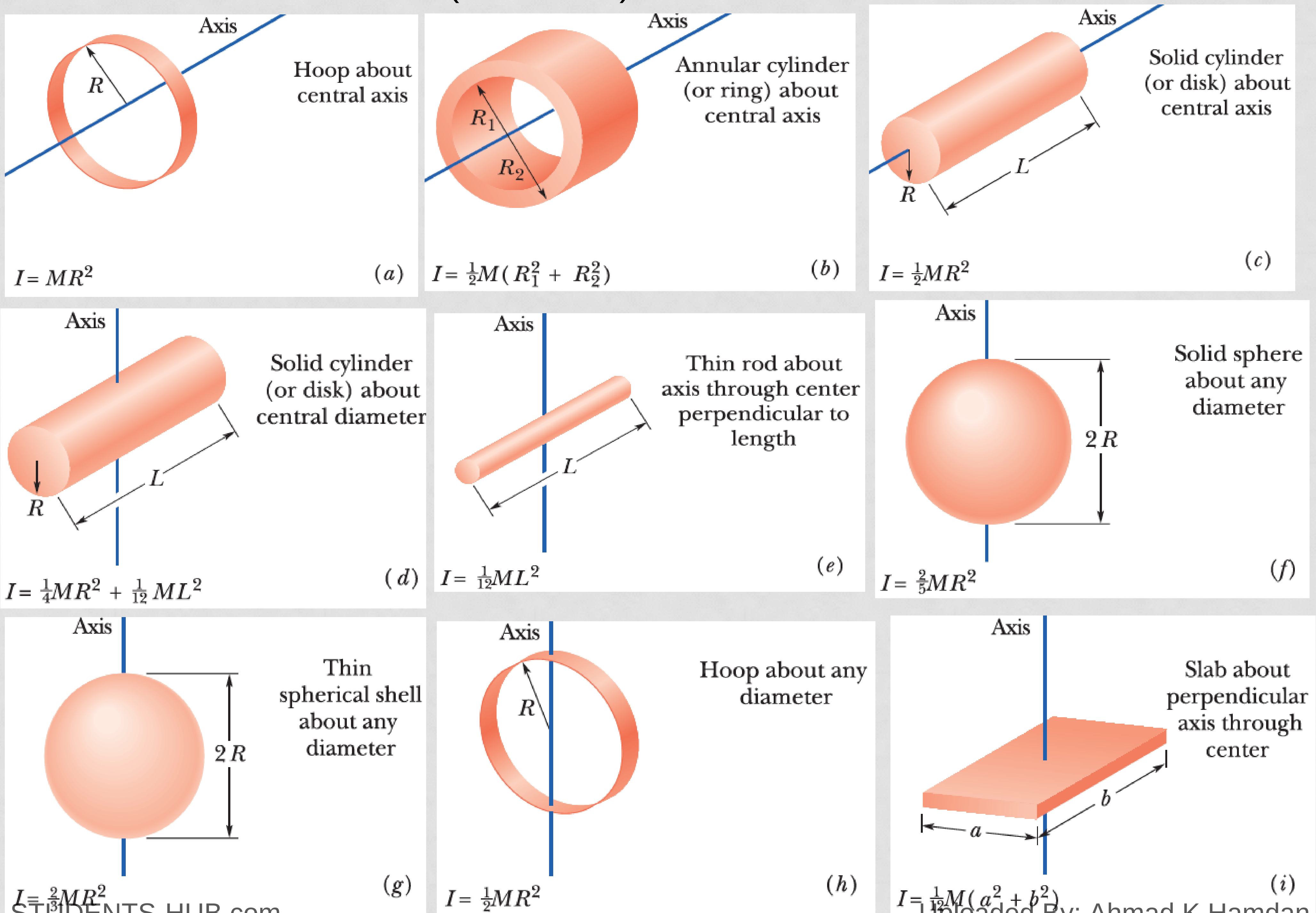
$$I = \sum m_i r_i^2 \quad \text{(rotational inertia)}$$

- A body with continuously distributed mass:

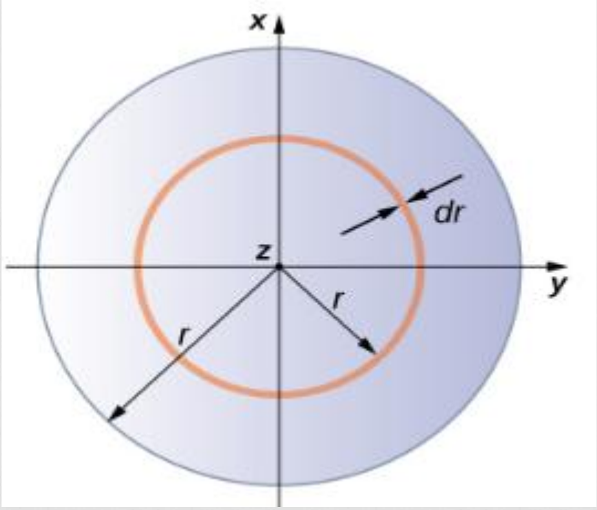
$$I = \int r^2\, dm \quad \text{(rotational inertia, continuous body)}$$

- Note: r and r_i represent the perpendicular distance from the axis of rotation to each mass element in the body.
- But there is a set of common shapes for which values have already been calculated (Table 10-2) for common axes

Some Rotational Inertias (Table 10-2):



Moment of inertia of a uniform thin disk about a rotational axis through the com: (The Disk made up of a series of thin rings)



$I = \int r^2 dm,$ use $\rho = \frac{M}{V}$

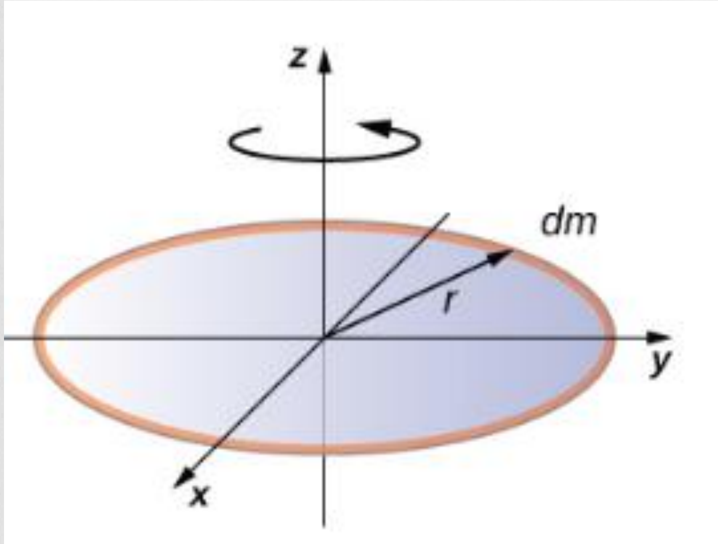
$$I = \rho \int r^2 dV$$

$$I = \rho \int r^2 dV = \rho \int r^2 (2\pi r h\, dr)$$

$$I = \frac{M}{V_{Disk}} \int_0^R r^2 (2\pi r h\, dr) = \frac{M}{\pi R^2 h} (2\pi h) \int_0^R r^3 dr$$

$$I = \frac{2M}{R^2} \frac{R^4}{4} = \frac{1}{2} M R^2$$

$$\boldsymbol{I_{Disk,com} = \frac{1}{2} M R^2}$$

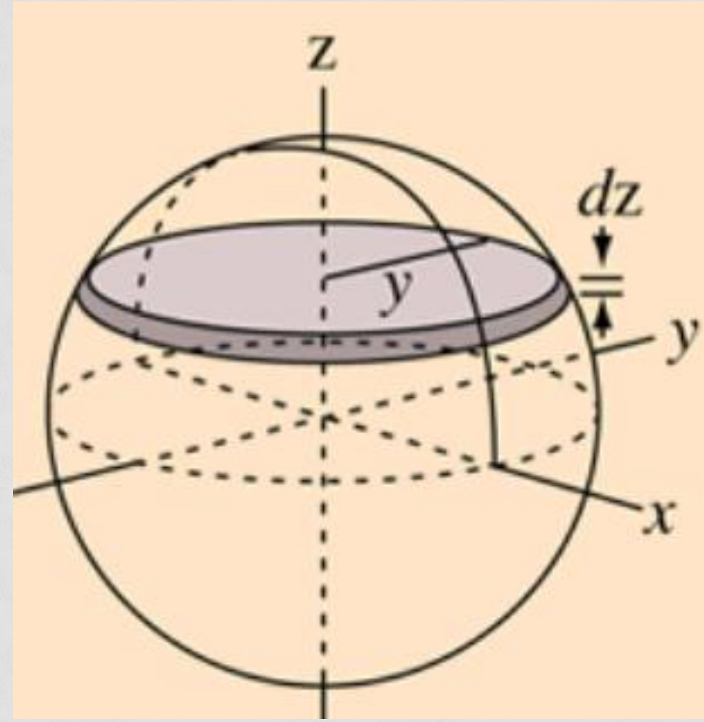


Moment of inertia of a sphere about a rotational axis through the com:

(The sphere made up of infinitesimally thin disks about the z axis)

$$I_{Disk} = \frac{1}{2}MR^2$$

$$dI = \frac{1}{2}y^2 dm = \frac{1}{2}y^2 \rho dV = \frac{1}{2}y^2 \rho(\pi y^2 dz)$$



$$I = \frac{1}{2}\rho\pi \int_{-R}^{R} y^4 dz = \frac{1}{2}\rho\pi \int_{-R}^{R} (R^2 - z^2)^2 dz$$

$$I = \rho\pi \int_{0}^{R} (R^4 - 2R^2z^2 + z^4) dz = \frac{8}{15}\rho\pi R^5$$

Use $\rho = \frac{M}{v_{sphere}} = \frac{M}{\frac{4}{3}\pi R^3}$

$$I = \frac{8}{15}\left(\frac{M}{\frac{4}{3}\pi R^3}\right)\pi R^5 = \frac{2}{5}MR^2$$

$$\boldsymbol{I_{sphere,com} = \frac{2}{5}MR^2}$$

- **Parallel-axis Theorem**:
  If we know the moment of inertia for the center of mass axis, we can find the moment of inertia for a parallel axis

$$I = I_{com} + Mh^2$$

Notes:

- The axes *must* be parallel, and the first *must* go through the center of mass
- h is the perpendicular distance between the two axes
- This does *not* relate the moment of inertia for two arbitrary axes

Proof of the Parallel-Axis Theorem

$$I = \int r^2\, dm = \int [(x-a)^2 + (y-b)^2]\, dm$$

The moment of inertia about an axis through P

$$I = \int (x^2 + y^2)\, dm - 2a \int x\, dm - 2b \int y\, dm + \int (a^2 + b^2)\, dm$$

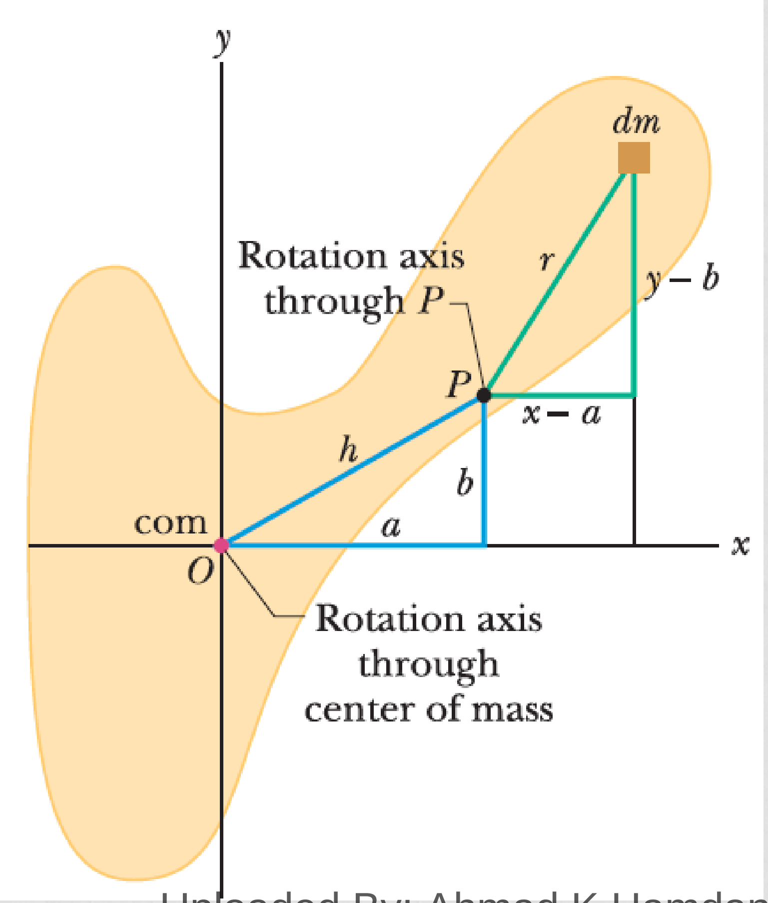
The coordinates of the center of mass (multiplied by a constant): $x_{com} = y_{com} = 0$

$$I = \int (x^2 + y^2)\, dm + \int (a^2 + b^2)\, dm$$

$$I = \int (x^2 + y^2)\, dm + \int (a^2 + b^2)\, dm$$

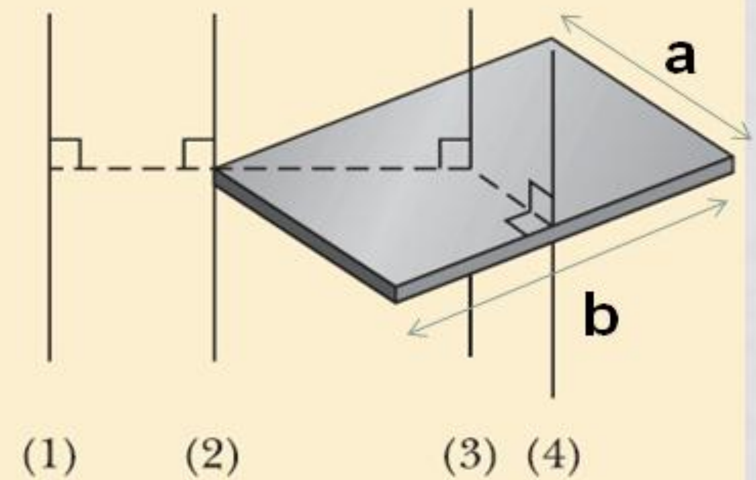
$$I = \int R^2\, dm + \int h^2\, dm$$

$$I = I_{com} + Mh^2$$

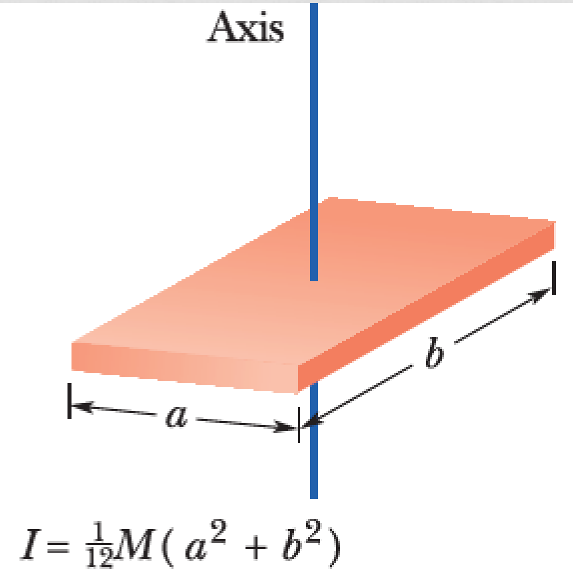


Checkpoint 5

The figure shows a book-like object (one side is longer than the other) and four choices of rotation axes, all perpendicular to the face of the object. Rank the choices according to the rotational inertia of the object about the axis, greatest first.



Answer: (1), (2), (4), (3)



By Parallel-axis Theorem: $I = I_{com} + Mh^2$

$$I_3 = \frac{1}{12}\,\mathrm{M}\,(a^2 + b^2) = I_{com}$$

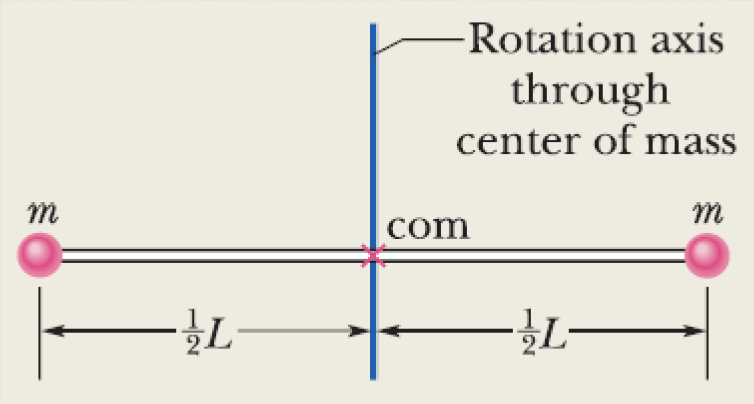
$$I_4 = I_{com} + Mh^2 = \frac{1}{12}\,\mathrm{M}\,(a^2 + b^2) + M\,\frac{a^2}{4}$$

$$I_2 = I_{com} + Mh^2 = \frac{1}{12}\,\mathrm{M}\,(a^2 + b^2) + M\,\frac{(a^2+b^2)}{4} = \frac{1}{3}\,\mathrm{M}\,(a^2 + b^2)$$

Sample Problem 10.06 Rotational inertia of a two-particle system

A rigid body consisting of two particles of mass m connected by a rod of length L and negligible mass.

1. The rotational inertia I_{com} about an axis through the center of mass, perpendicular to the rod:

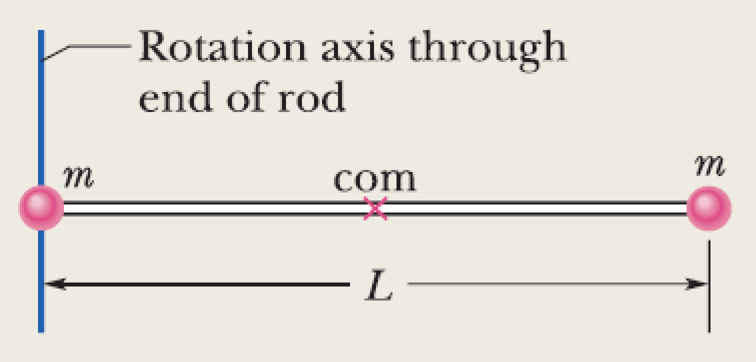


$$I = \sum m_i r_i^2 = (m)(\tfrac{1}{2}L)^2 + (m)(\tfrac{1}{2}L)^2$$

$$I = \tfrac{1}{2}mL^2$$

2. The rotational inertia I of the body bout an axis through the left end of the rod and parallel to the first axis :

$$I = m(0)^2 + mL^2 = mL^2$$



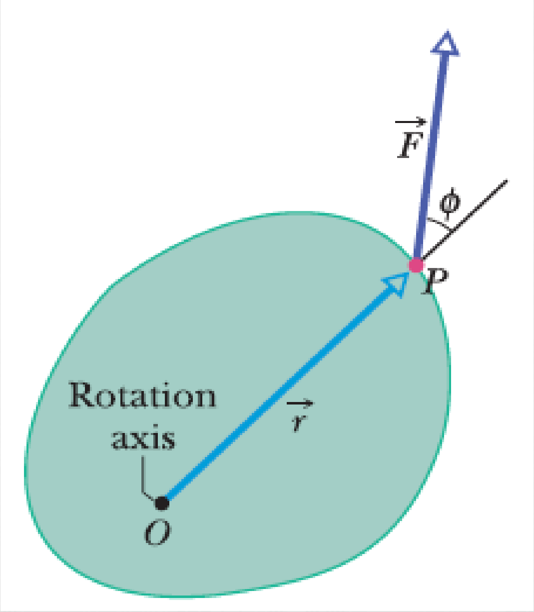
By Parallel-axis Theorem:

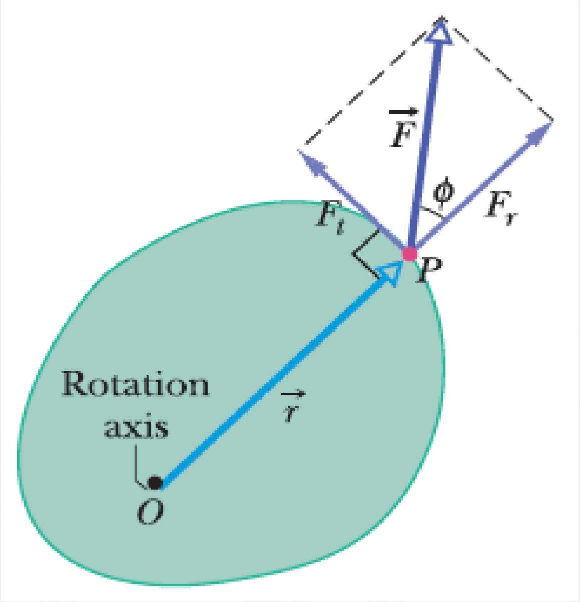
$$I = I_{com} + Mh^2 = \tfrac{1}{2}mL^2 + (2m)(\tfrac{1}{2}L)^2$$

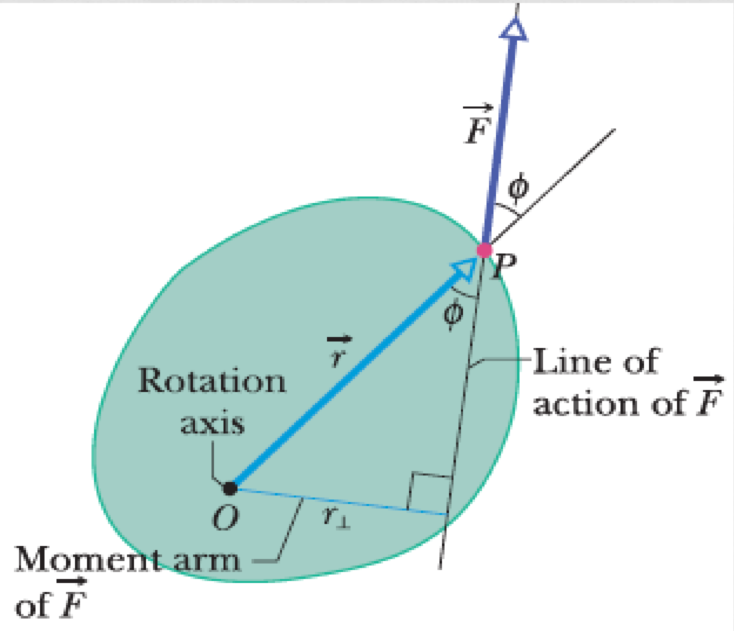
$$I = mL^2$$

10-6 Torque

- The force necessary to rotate an object depends on the angle of the force and where it is applied.
- We can resolve the force into components to see how it affects rotation.







The torque due to this force causes rotation around this axis

But actually only the *tangential* component of the force causes the rotation.

You calculate the same torque by using this moment arm distance and the full force magnitude.

- **Torque** takes these factors into account:

$$\tau = (r)(F \sin \phi)$$

- A line extended through the applied force is called the **line of action** of the force.
- The perpendicular distance from the line of action to the rotational axis is called the **moment arm.**
- The unit of torque is the newton-meter, $N.m$
- Note that 1 J = 1 N.m, but torques are never expressed in joules, torque is not energy
- Torque is a vector quantity.

- Again, torque is positive if it would cause a counterclockwise rotation, otherwise negative

"Clocks are negative"

- For several torques, the **net torque** or **resultant torque** is the sum of individual torques (τ_{net}).

Checkpoint 6

The figure shows an overhead view of a meter stick that can pivot about the dot at the position marked 20 (for 20 cm). All five forces on the stick are horizontal and have the same magnitude. Rank the forces according to the magnitude of the torque they produce, greatest first.

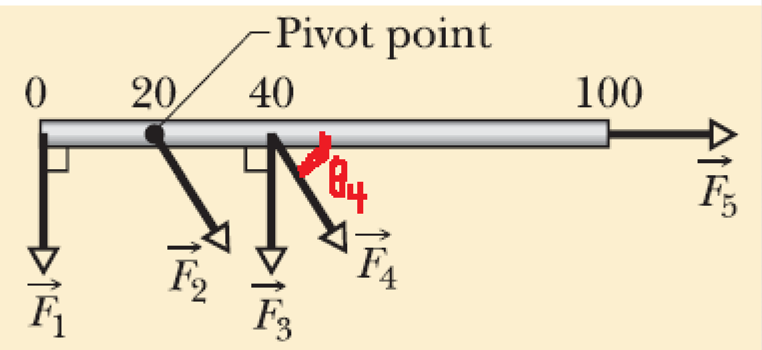
Answer: $F_1, F_3 > F_4 > F_2, F_5$

$$\tau_1 = \tau_3 = (0.2\ m)(F \sin 90^\circ) = 0.2\ F$$

$$\tau_5 = (0.8\ m)(F \sin 0^\circ) = 0$$

$$\tau_2 = (0\ m)(F\ sin\ \theta) = 0$$

$$\tau_4 = (0.2\ m)(F \sin \theta_4)$$



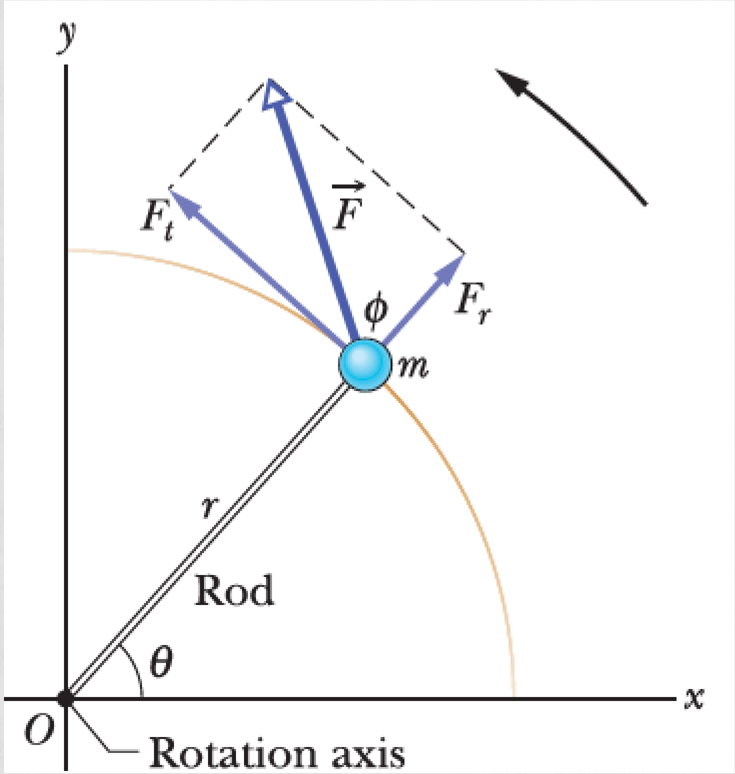
$$\tau = (r)(F \sin \phi)$$

10-7 Newton's Second Law for Rotation

- Rewrite $\vec{F} = m\vec{a}$ with rotational variables:

$$\tau_{\text{net}} = I\alpha$$

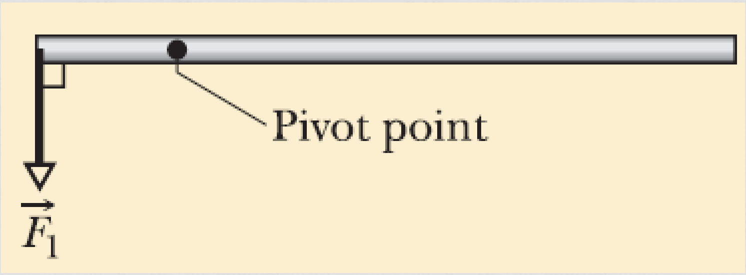
- It is torque that causes angular acceleration.



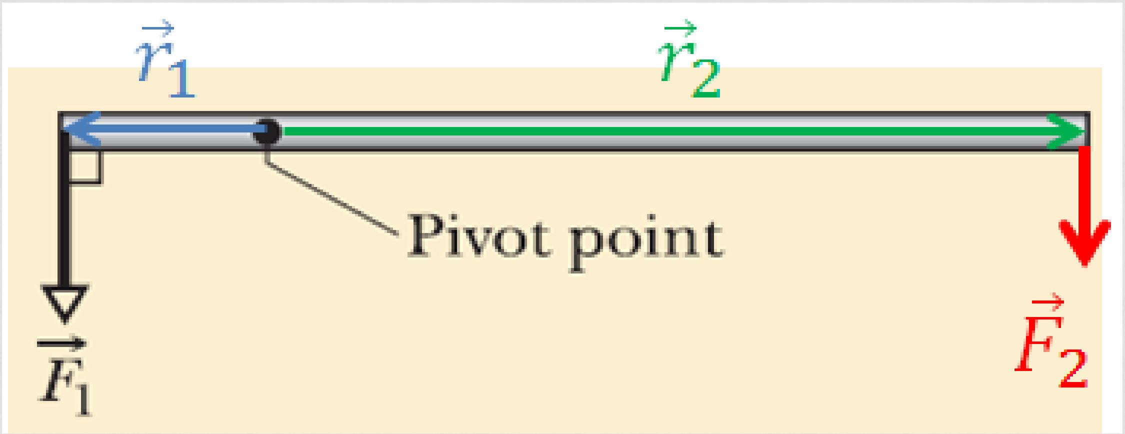
The torque due to the tangential component of the force causes an angular acceleration around the rotation axis.

Checkpoint 7

The figure shows an overhead view of a meter stick that can pivot about the point indicated, which is to the left of the stick's midpoint. Two horizontal forces, $\vec{F}_1$ and $\vec{F}_2$, are applied to the stick. Only $\vec{F}_1$ is shown. Force $\vec{F}_2$ is perpendicular to the stick and is applied at the right end. If the stick is not to turn, (a) what should be the direction of $\vec{F}_2$, and (b) should F_2 be greater than, less than, or equal to F_1?



$$\tau_{net} = I\alpha$$



The stick is not to turn, the angular acceleration is zero so the net torque is zero.

$$\tau_1 = r_1(F_1 \sin 90^\circ) = r_1 F_1$$, attempts counterclockwise rotation

τ_2 must be equal to τ_1 in magnitude but attempts rotation in the opposite direction.

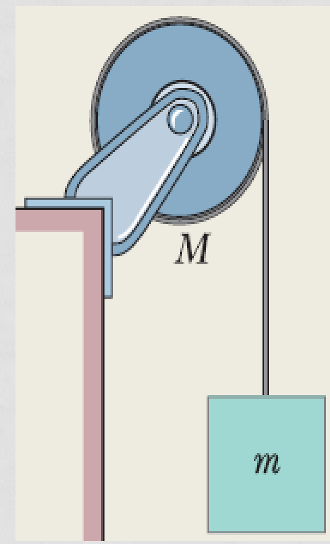
$$\tau_2 = r_2(F_2 \sin 90^\circ) = r_2 F_2$$, clockwise rotation, $r_1 F_1 = r_2 F_2$

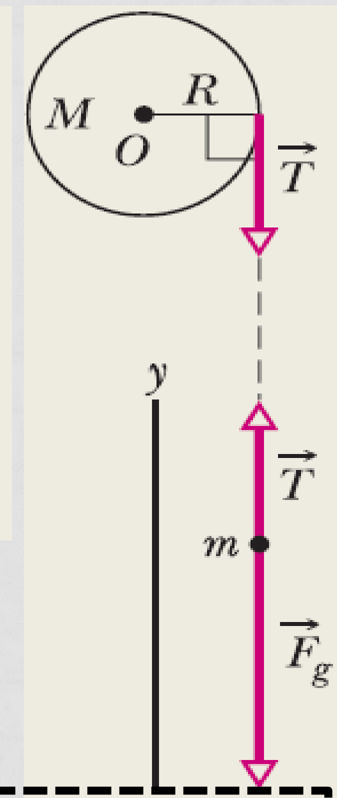
(a) F_2 should point downward. (clockwise rotation)

(b) should have a smaller magnitude than F_1

Sample Problem 10.10 Newton's second law, rotation, torque, disk

Figure 10-19*a* shows a uniform disk, with mass $M = 2.5$ kg and radius $R = 20$ cm, mounted on a fixed horizontal axle. A block with mass $m = 1.2$ kg hangs from a massless cord that is wrapped around the rim of the disk. Find the acceleration of the falling block, the angular acceleration of the disk, and the tension in the cord. The cord does not slip, and there is no friction at the axle. The falling block causes the disk to rotate.





Newton's second law (Block): $\vec{F} = m\vec{a}$

Forces on block: $T - mg = m(-a)$

Newton's second law (Disk): **Torque on disk:** $\tau_{\text{net}} = I\alpha$ Clockwise Rotation

$$-RT = \tfrac{1}{2}MR^2(-\alpha)$$

Because the cord does not slip, the magnitude a of the block's linear acceleration and the magnitude a_t of the (tangential) linear acceleration of the rim of the disk are equal.

$$a_t = \alpha R = a \rightarrow\rightarrow \alpha = \frac{a}{R}$$

$$T = \tfrac{1}{2}Ma$$

$$a = g\frac{2m}{M + 2m} = (9.8\ \text{m/s}^2)\frac{(2)(1.2\ \text{kg})}{2.5\ \text{kg} + (2)(1.2\ \text{kg})} = 4.8\ \text{m/s}^2.$$

$$T = \tfrac{1}{2}Ma = \tfrac{1}{2}(2.5\ \text{kg})(4.8\ \text{m/s}^2) = 6.0\ \text{N}.$$

$$\alpha = \frac{a}{R} = \frac{4.8\ \text{m/s}^2}{0.20\ \text{m}} = 24\ \text{rad/s}^2$$

10-8 Work and Rotational Kinetic Energy

- The rotational work-kinetic energy theorem states:

$$\Delta K = K_f - K_i = \frac{1}{2}I\omega_f^2 - \frac{1}{2}I\omega_i^2 = W$$

- The work done in a rotation about a fixed axis can be calculated by:

$$W = \int_{\theta_i}^{\theta_f} \tau\, d\theta$$

- Which, for a constant torque, reduces to: $W = \tau(\theta_f - \theta_i)$

- We can relate work to power with the equation:

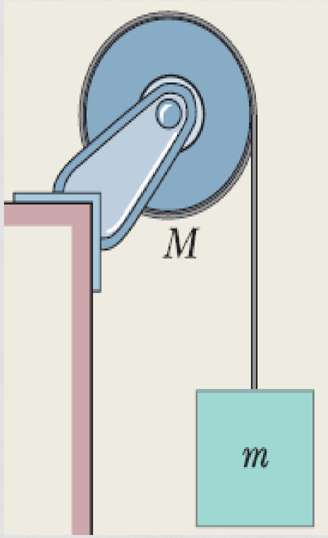
$$P = \frac{dW}{dt} = \tau\omega$$

Table 10-3 Some Corresponding Relations for Translational and Rotational Motion

Pure Translation (Fixed Direction)		Pure Rotation (Fixed Axis)	
Position	x	Angular position	θ
Velocity	$v = dx/dt$	Angular velocity	$\omega = d\theta/dt$
Acceleration	$a = dv/dt$	Angular acceleration	$\alpha = d\omega/dt$
Mass	m	Rotational inertia	I
Newton's second law	$F_{net} = ma$	Newton's second law	$\tau_{net} = I\alpha$
Work	$W = \int F\,dx$	Work	$W = \int \tau\,d\theta$
Kinetic energy	$K = \frac{1}{2}mv^2$	Kinetic energy	$K = \frac{1}{2}I\omega^2$
Power (constant force)	$P = Fv$	Power (constant torque)	$P = \tau\omega$
Work–kinetic energy theorem	$W = \Delta K$	Work–kinetic energy theorem	$W = \Delta K$

Sample Problem 10.11 Work, rotational kinetic energy, torque, disk

Let the disk in Fig. 10-19 start from rest at time $t = 0$ and also let the tension in the massless cord be 6.0 N and the angular acceleration of the disk be $-24\ \text{rad/s}^2$. What is its rotational kinetic energy K at $t = 2.5$ s?



$w_0 = 0$; start from rest $(K_i = 0)$, $T = 6.0\ N$

$\alpha = -24\ rad/s^2$; Clockwise rotation

Disk: $M = 2.5\ kg$
R= 20 cm

1. Rotational Kinetic Energy: $K = \frac{1}{2} I w^2$

Disk rotational inertia (com rotational axis):

$$I = \frac{1}{2} M R^2 = \frac{1}{2}(2.5\ kg)(0.2\ m)^2 = 0.05\ kg.m^2$$

To find the angular velocity at $t = 2.5\ s$, we can apply constant angular acceleration equations:

$w = w_0 + \alpha t = 0 + (-24\ rad/s^2)(2.5s) = -60\, rad/s$

$$K = \frac{1}{2} I w^2 = \frac{1}{2}(0.05\ kg.m^2)(-60\, rad/s)^2 = 90\ J$$

2. We can use also work-kinetic energy theorem:

$$\Delta K = K_f - K_i = W \to\to K_f = \text{W}$$

$W = \tau(\theta_f - \theta_i)$; constant torque

To find the angular displacement $\theta_f - \theta_i$, we can apply constant angular acceleration equation:

$$\theta_f - \theta_i = w_0 t + \frac{1}{2}\alpha t^2 \quad \to\to \theta_f - \theta_i = \frac{1}{2}\alpha t^2 = \frac{1}{2}(-24\ rad/s^2)(2.5\ s)^2 = -75\, rad$$

$$\tau = -RT = -(0.2\ m)(6.0\ N) = -1.2\ N.m$$

$$K_f = \text{W} = \tau(\theta_f - \theta_i) = (-1.2\ N.m)(-75\, rad) = 90\ J$$

••4 The angular position of a point on a rotating wheel is given by $\theta = 2.0 + 4.0t^2 + 2.0t^3$, where θ is in radians and t is in seconds. At $t = 0$, what are (a) the point's angular position and (b) its angular velocity? (c) What is its angular velocity at $t = 4.0$ s? (d) Calculate its angular acceleration at $t = 2.0$ s. (e) Is its angular acceleration constant?

$$\theta = 2 + 4t^2 + 2t^3$$

a) $\theta(t = 0) = 2\ rad$

b) $w = \frac{d\theta}{dt} = 8t + 6t^2$

$$w(\mathrm{t} = 0) = 0$$

c) $w(\mathrm{t} = 4\ s) = 128\ rad/s$

d) $\alpha = \frac{dw}{dt} = 8 + 12\ t$

$$\alpha(t = 2\ s) = 32\ rad/s^2$$

e) α is not constant (depends on time)

•9 A drum rotates around its central axis at an angular velocity of 12.60 rad/s. If the drum then slows at a constant rate of 4.20 rad/s^2, (a) how much time does it take and (b) through what angle does it rotate in coming to rest?

$\alpha = -4.2\ rad/s^2, w_0 = 12.6\ rad/s$

The drum is coming to rest: $w_f = 0$

a) By using constant angular acceleration equations:

$$w_f = w_0 + \alpha t \rightarrow\rightarrow t = \frac{-w_0}{\alpha} = \frac{-12.6\ rad/s}{-4.2\ rad/s^2} = 3.0\ s$$

b) $\theta_f - \theta_i = w_0 t + \frac{1}{2}\alpha t^2$

$$\theta_f - \theta_i = (12.6\ rad/s)(3.0\ s) + \frac{1}{2}(-4.2\ rad/s^2)(3.0\ s)^2 = 18.9\ rad$$

$$\Delta\theta = 18.9\ rad = 3.0\ revolutions$$

••28 In Fig. 10-31, wheel A of radius $r_A = 10$ cm is coupled by belt B to wheel C of radius $r_C = 25$ cm. The angular speed of wheel A is increased from rest at a constant rate of 1.6 rad/s^2. Find the time needed for wheel C to reach an angular speed of 100 rev/min, assuming the belt does not slip. (*Hint:* If the belt does not slip, the linear speeds at the two rims must be equal.)

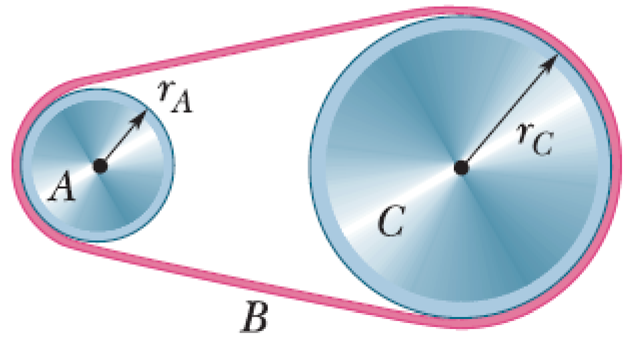


Figure 10-31 Problem 28.

$\alpha_A = 1.6\, rad/s^2$

The belt does not slip, a point on the rim of wheel C has the same tangential acceleration as a point on the rim of wheel A: $\alpha_A r_A = \alpha_C r_C$

$$\alpha_C = \frac{\alpha_A r_A}{r_C} = \frac{(1.6\, rad/s^2)(10\, cm)}{(25\, cm)} = 0.64\, rad/s^2$$

Wheel C: $\alpha_C = 0.64\, rad/s^2$, $w_0 = 0$ (starts from rest)

$$w_f = 100\ rev/min = 10.47\, rad/s$$

By using constant angular acceleration equations:

$$w_f = w_0 + \alpha t \rightarrow\rightarrow t = \frac{w_f}{\alpha} = \frac{10.47\, rad/s}{0.64\, rad/s^2} = 16.4\, s$$

•38 Figure 10-35 shows three 0.0100 kg particles that have been glued to a rod of length $L = 6.00$ cm and negligible mass. The assembly can rotate around a perpendicular axis through point O at the left end. If we remove one particle (that is, 33% of the mass), by what percentage does the rotational inertia of the assembly around the rotation axis decrease when that removed particle is (a) the innermost one and (b) the outermost one?

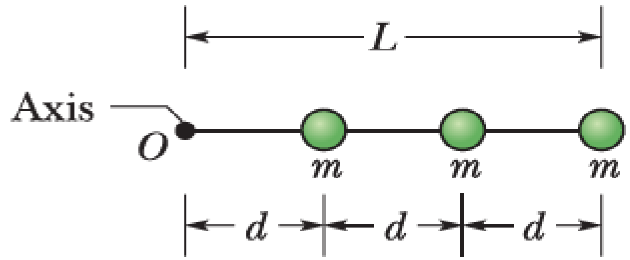


Figure 10-35 Problems 38 and 62.

$$I = \sum m_i r_i^2 \qquad \text{(rotational inertia)}$$

$$I = md^2 + m(2d)^2 + m(3d)^2 = 14\,md^2$$

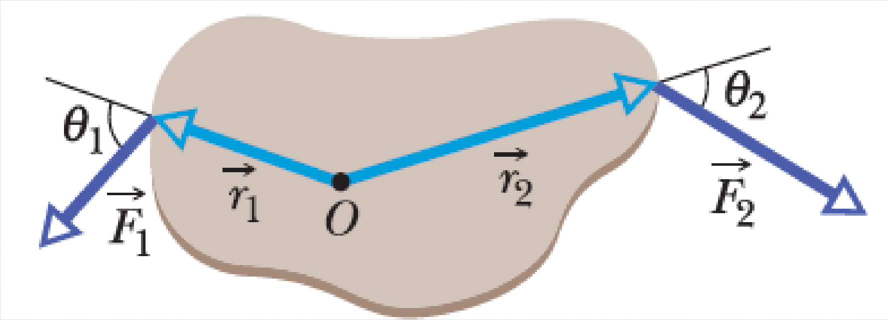
a) The innermost particle is removed: $I_1 = m(2d)^2 + m(3d)^2 = 13\,md^2$

$$\text{Percentage of decreasing} = \frac{|I_1 - I|}{I} = \frac{|13\,md^2 - 14\,md^2|}{14\,md^2} = 0.07 = 7\%$$

b) The outermost particle is removed: $I_2 = md^2 + m(2d)^2 = 5\,md^2$

$$\text{Percentage of decreasing} = \frac{|I_2 - I|}{I} = \frac{|5\,md^2 - 14\,md^2|}{14\,md^2} = 0.64 = 64\%$$

•45 SSM ILW The body in Fig. 10-39 is pivoted at O, and two forces act on it as shown. If $r_1 = 1.30$ m, $r_2 = 2.15$ m, $F_1 = 4.20$ N, $F_2 = 4.90$ N, $\theta_1 = 75.0°$, and $\theta_2 = 60.0°$, what is the net torque about the pivot?



τ_1 tends to cause counterclockwise rotation , τ_1 is positive.

τ_2 tends to cause clockwise rotation , τ_2 is negative.

$$\tau_1 = +r_1(F_1 \sin 75^\circ) = +1.3\text{ m}\,(4.2\,N \sin 75^\circ) = +5.27\,N.m$$

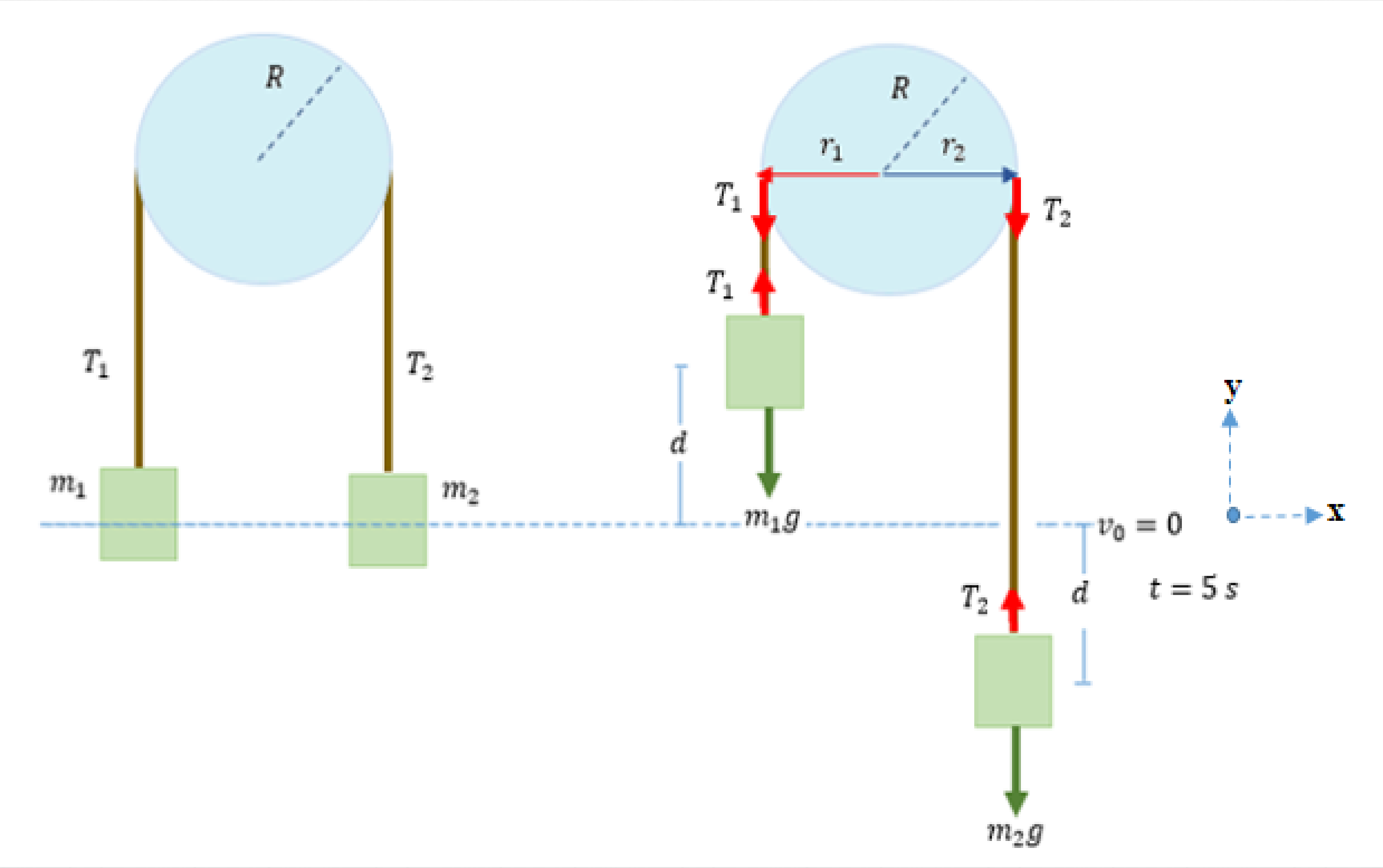
$$\tau_2 = -r_2(F_2 \sin 60^\circ) = -2.15\text{ m}\,(4.9\,N \sin 60^\circ) = -9.12\,N.m$$

$$\tau_{net} = +r_1(F_1 \sin 75^\circ) - r_2(F_2 \sin 60^\circ) = +5.27\,N.m - 9.12\,N.m$$

$$\tau_{net} = -3.85\,N.m$$

α is negative so the body will rotate clockwise

10-51) In the below figure, block 1 has mass m_1=460 g, block 2 has mass m_2=500 g, and the pulley, which is mounted on a horizontal axle with negligible friction, has radius *R* =5.00 cm. When released from rest, block 2 falls 75.0 cm in 5.00 s without the cord slipping on the pulley. (a) What is the magnitude of the acceleration of the blocks? What are (b) tension T_2 and (c) tension T_1? (d) What is the magnitude of the pulley's angular acceleration? (e) What is its rotational inertia?



$$m_1 = 460g = 0.46\ kg$$
$$m_2 = 500g = 0.5\ kg$$
$$R = 5\ cm = 0.05\ m$$
$$d = 75\ cm = 0.75\ m$$
$$v_0 = 0$$
$$t = 5\ s$$

a) The magnitude of the acceleration of the blocks

For block 2:

$$y - y_0 = v_0 t + \frac{1}{2} a t^2$$

$$-0.75 = 0 + \frac{1}{2} a (5^2)$$

$$a_2 = -0.06\ m/s^2$$

Note: the negative sign indicates that the acceleration of block2 is downward, thus

$$\vec{a}_2 = -(0.06\, m/s^2)\hat{j}$$

Block1 has the same acceleration but in the opposite direction of block 2. This is because the two blocks moved the same distance d without the cord slipping on the pulley. This means that

$$\vec{a}_1 = +(0.06\, m/s^2)\hat{j}$$

The magnitude of acceleration of the two block is similar, it is $0.06\ m/s^2$

b) Tension T_2

Applying Newton’s second low on bock2

$$T_2 - m_2 g = m_2(-a_2)$$

$$T_2 = m_2 g - m_2 a_2$$

$$T_2 = (0.5)(9.8) - (0.5)(0.06) = 4.87\ N$$

c) Tension T_1

Applying Newton’s second low on bock1

$$T_1 - m_1 g = m_1 a_1$$

$$T_1 = m_1 g + m_1 a_1$$

$$T_1 = (0.46)(9.8) + (0.46)(0.06) = 4.54\ N$$

d) What is the magnitude of the pulley's angular acceleration?
The angular acceleration α of the pulley is

$$\alpha = \frac{a_t}{R} = \frac{0.06}{0.05} = 1.2\ rad/s^2$$

e) What is the pulley's rotational inertia?
Now applying Rotational Newton's second law on the pulley

$$\vec{\tau}_{net} = I_{pulley}\vec{\alpha}$$

$\vec{\tau}_1 + \vec{\tau}_2 = I\vec{\alpha}$ ……………….. (1)

Note: $|\vec{\tau}_1| = |\vec{r}_1 \times \vec{T}_1| = r_1 T_1 \sin 90^{\circ} = RT_1 = (0.05)(4.54) = 0.227\ N.m$

$|\vec{\tau}_2| = |\vec{r}_2 \times \vec{T}_1| = r_2 T_2 \sin 90^{\circ} = RT_2 = (0.05)(4.87) = 0.2435\ N.m$

Using the right hand rule: The direction of $\vec{\tau}_1$ out of the page (counterclockwise (+))
The direction of $\vec{\tau}_2$ into the page (clockwise (-))

Note: the direction of the angular acceleration $\vec{\alpha}$ is the same as the direction of $\vec{\tau}_{net}$, thus , it is into the page also as $|\vec{\tau}_2| > |\vec{\tau}_1|$

Substituting in eq.(1) above you get

$$+0.227 - 0.2435 = I(-1.2)$$

$$I = \frac{+0.23 - 0.2435}{-1.2} = 0.0138\ Kg.m^2$$

•61 A 32.0 kg wheel, essentially a thin hoop with radius 1.20 m, is rotating at 280 rev/min. It must be brought to a stop in 15.0 s. (a) How much work must be done to stop it? (b) What is the required average power?

$M = 32.0\ kg$
$R = 1.2\ m$

a) By using work-kinetic energy theorem:

$$\Delta K = K_f - K_i = W$$

$$w_0 = 280\, rev/min = 29.31\, rad/s\,, w_f = 0; K_f = 0$$

Hoop(wheel) rotational inertia (com rotational axis):

$$I = MR^2 = (32\ kg)(1.2\ m)^2 = 46.08\ kg.m^2$$

$$W = \Delta K = K_f - K_i = -K_i$$

$$W = -\frac{1}{2} I {w_0}^2 = -\frac{1}{2}(46.08\ kg.m^2)(29.31\ rad/s)^2 = -19.79\ kJ$$

b) The average power:

$$P = \frac{W}{\Delta t} = \frac{19.79 \times 10^3 J}{15\ s} = 1.32\ kW$$

