Ch.3 Vector spaces

By: Rawan AlFares

Vector spaces 3.1

Df:- A vector space V is a set of elements together with the operations of addition and scalar multiplication such that the following conditions are satisfied:

C1:- if $x \in V$ and α is scalar, then $\alpha x \in V$
"closed under scalar multiplication".

C2: if $x, y \in V$, then $x+y \in V$, "closed under addition".

* أهم إشي نبدأ بهدول 2 conditions.
وإذا شرط واحد من 10 ما انطبق يبقى مابكون vector

A1: $x+y = y+x \quad , \forall x, y \in V$.
A2: $(x+y)+z = x+(y+z)$
A3: $\exists$ an element $0 \in V$. such that $x+0 = 0+x = x \quad , \forall x \in V$. $\quad 0 = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$
A4: $\forall x \in V, \exists \; -x \in V$ such that $x + -x = 0$. معكوسه نظير جمعي
A5:- $\alpha(x+y) = \alpha x + \alpha y$, for each α scalar and $x, y \in V$.
A6:- $(\alpha+\beta)x = \alpha x + \beta x$, for each α, β scalars and $x \in V$.
A7:- $(\alpha\beta)x = \alpha(\beta x)$
A8:- $1 . x = x$. النظير الضربي

*** Notation:** $(V, +, \cdot)$

Examples for vector space:-

1. $(\mathbb{R}, +, \cdot)$ "$\mathbb{R}$ with usual addition and multiplication is vector space.

2. $V = \mathbb{R}^2$ with usual $(+)$ and $(\cdot)$ is a vector space where $(a,b) + (c,d) = (a+c, b+d)$
$\alpha(a,b) = (\alpha a, \alpha b)$.

Proof: let α be scalar and $x = (a,b)$, $y = (c,d)$, $z = (e,f) \in \mathbb{R}^2$. then

C1:- $\alpha x = \alpha(a,b) = (\alpha a, \alpha b) \in \mathbb{R}^2$, is closed under scalar multiplication.

C2: $x+y = (a,b)+(c,d) = (a+c, b+d) \in \mathbb{R}^2$, is closed under addition.

دائماً نبدأ نفحص هدول الشرطين
addition
scalar multiplication

A1:- $x+y = (a,b) + (c,d)$
$= (a+c, b+d) = (c+a, d+b)$
$= (c,d) + (a,b)$
$= y + x$

A2:- $x+y+z = ((a,b)+(c,d)) + (e,f) = (a,b) + ((c,d) + (e,f))$

A3: $x+0 = (a,b) + (0,0) = (a+0, b+0) = (a,b) = x \quad , \forall x \in V$
$0 = (0,0) \in \mathbb{R}^2$

A4:- $\forall x = (a,b) \in \mathbb{R}^2$, $\exists \; -x = (-a,-b) \in \mathbb{R}^2 \;|\; (a,b)+(-a,-b) = (0,0)$.

A5:- $\alpha(x+y) = \alpha[(a,b)+(c,d)] = \alpha[(a+c, b+d)]$
$= ((\alpha a + \alpha c), (\alpha b + \alpha d)) = \alpha(a,b) + \alpha(c,d) = \alpha x + \alpha y$

A_6:- $(\alpha+\beta)X = (\alpha+\beta)(a,b) = \alpha(a,b)+\beta(a,b)$

A_7:- $(\alpha\beta)X = \alpha(\beta X) = \alpha(\beta(a,b))$
$= \alpha(\beta a, \beta b)$
$= (\alpha\beta)a, (\alpha\beta)b)$
$= \alpha\beta(a,b)$ ✱

A_8:- $1X = 1(a,b) = (1.a, 1.b) = (a,b)$.

So $\mathbb{R}^2$ is a vector space.

Rmk:- $(\mathbb{R}^n, +, \cdot)$ is vector space.

3. $M_{m\times n} = \mathbb{R}^{m\times n}$ is the set of all $m\times n$ matricies with real entries under addition and Scalar multiplication is a vector space.

4. The set of all real valued functions under (+) and (·) :- $\left\{ \begin{array}{l} (f+g)(x) = f(x)+g(x) \\ (\alpha f)(x) = \alpha f(x) \end{array} \right\}$ is a vector space.

✱ the Zero Polynomial is $O(x) = 0$ of degree Zero.

5. $C[a,b] = \{ f: [a,b] \longrightarrow \mathbb{R} : f$ is Continuoes on $[a,b] \}$. under addition and Scalar multiplication of functions : $(f+g)(x) = f(x)+g(x)$, $\alpha f(x)$ is a vector Space.
* اذا اقتران المتصل على فترة.

6. $C^{(n)}[a,b] = \{ f: [a,b] \longrightarrow \mathbb{R} : f^{(n)}$ is Continuoes on $[a,b] \}$ under addition and Scalar multiplication of functions : $(f+g)(x) = f(x)+g(x)$, $(\alpha f)(x) = \alpha f(x)$ is a vector space.
* المشتقة (n) متصلة على فترة

7. $P_n = \{ f(x) = a_{n-1} x^{n-1} + \dots + a_1 x + a_0$, $x<n$, $\dots a_n \in \mathbb{R} \}$, is a vector Space
* اقتران كثير الحدود (degree<n).
مثلاً P_4 ← كل الاقترانات الي درجتهم أقل من 4 $(0 \to 3)$.

ex. $P_3 = \{ f(x) = ax^2+bx+c$, $a,b,c \in \mathbb{R} \}$
$P_2 = \{ f(x) = ax+b$, $a,b \in \mathbb{R} \}$

not 8. $\mathbb{Q} = \{ x = \frac{a}{b} : a,b$ are integers, $b \neq 0$ is not a vector space.

Counter ex:- $x = \frac{2}{5} \in \mathbb{Q}$, $\alpha = \sqrt{2}$ Scalar but,
$\alpha x = \sqrt{2} \cdot \frac{2}{5} = \frac{2\sqrt{2}}{5} \notin \mathbb{Q}$

not 9. $\mathbb{Z} = \{ 0, \mp 1, \mp 2, \dots \}$ is not a vector space. under usual addition and Scalar multiplication

✱ Counter example:- $x = 2$, $\alpha = \frac{2}{3} \longrightarrow \alpha x = \frac{4}{3} \notin \mathbb{Z}$, So it's not Vector Space.

not 10. $V = \{ f(x) : \deg(f) = 3 \}$ is not a vector Space under usual $+, \cdot$ of functions.
degree of functions.

Counter ex:- $f(x) = 1 - x^3$, $g(x) = 1 + x + x^3$, $f(x) + g(x) = 2 + x \notin f(x)$

لأنو اقتران مش من الدرجة الثالثة.

not 11. $V = \{ (1, y) : y \in \mathbb{R} \}$ under usual addition and Scalor multiplication is not a vector.

12. $V = \{ (0, y, 0) : y \in \mathbb{R} \}$ under usual addition & scalor multiplication is a vector space.

Theorem

let V be a vector space, then :-

(i) $0V = \vec{0}$, $\forall v \in V$

(ii) if $x + y = \vec{0}$, then $y = -x$.

(III) $-1 \cdot V = -V$, $\forall v \in V$

pf (i) :-

$$0 + 0 = 0$$
$$(0 + 0)v = 0v$$
$$0v + 0v = 0v$$
$$0v + \underbrace{0v - 0v}_{} = \underbrace{0v - 0v}_{}$$
$$0v + \vec{0} = \vec{0}$$
$$0v = \vec{0}$$

#

pf (II) :-

$$\vec{x} + \vec{y} = \vec{0}$$
$$\vec{x} + \vec{y} - \vec{x} = \vec{0} + -\vec{x}$$
$$\underbrace{\vec{x} - \vec{x}}_{} + \vec{y} = -\vec{x}$$
$$0 + \vec{y} = -\vec{x}$$
$$\vec{y} = -\vec{x}$$

#

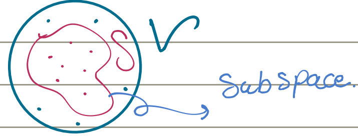
$-1 \cdot v = -v$

pf (III) :- we know that

$$1 + -1 = 0$$
$$(1 + (-1)) \cdot v = 0 \cdot v$$
$$v + -v = \vec{0}$$
$$-1 \cdot [v + -v] = -1 \cdot \vec{0}$$
$$-1 \cdot v + (-)(-v) = \vec{0}$$
$$-1 \cdot v + v = 0$$
$$-1 \cdot \vec{v} = -\vec{v}$$

#

3.2 Subspace



Def:- let V be a vector space and $S \subseteq V$, $S \neq \emptyset$ (S is non empty). S is called a subspace of V if

1. For any $s_1, s_2 \in S$, we have $s_1 + s_2 \in S$.
2. For any $s_1 \in S$ and $\alpha \in \mathbb{R}$, we have $\alpha s_1 \in S$.

Condition for S:-

1) $S_1 + S_2 = S_2 + S_1$
2) $S_1 + (S_2 + S_3) = (S_1 + S_2) + S_3$
⋮
8) $1 . S = S . 1$

Remark:-
if S is subspace of V, then S is a vector space.

EX:- let $S = \left\{ \begin{pmatrix} x \\ 0 \end{pmatrix} : x \in \mathbb{R} \right\}$. Is S is supspace of $\mathbb{R}^2$.

① $S \neq \emptyset$ ✓

② let $s_1 = \begin{bmatrix} x_1 \\ 0 \end{bmatrix}$, $s_2 = \begin{bmatrix} x_2 \\ 0 \end{bmatrix}$, Now, $s_1 + s_2 = \begin{bmatrix} x_1 + x_2 \\ 0 \end{bmatrix} \in S$

③ $s_1 = \begin{bmatrix} x_1 \\ 0 \end{bmatrix} \in S$, $\alpha \in \mathbb{R}$, Now $\alpha s_1 = \alpha \begin{bmatrix} x_1 \\ 0 \end{bmatrix} = \begin{bmatrix} \alpha x_1 \\ 0 \end{bmatrix} \in S$

So S is a subspace of $\mathbb{R}^2$

Remark:-
if S is subspace of V then $0_V \in S$.

Proof:- let S is a subspace of V.

let $s \neq \emptyset$, $x \in S$, ex $\begin{pmatrix} 0 \\ 0 \end{pmatrix} \in S$

* دائمًا افحص الصفر أول إشي

and consider $\alpha = 0$.

$\alpha x = 0 . x \in S$
$= \vec{0} = 0_V \in S$

So, $0_V \in S$.

NOTE:-
* اذا كانت المجموعة Subspace [ضروري جدًا] الـ $\vec{0}_V$ يكون موجود فيها، فلذلك نفحصه أول شي، إذا ما كان موجود، ما بكمل لإنها بتطلع not subspace.

Remark:-
if V is a vector space, $S \subseteq V$. if $0_V \notin S$ then S is not subspace.

EX. $S = \left\{ \begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{R}^2 : a + b = 1 \right\}$, is S a subspace of $\mathbb{R}^2$.

① $S \neq \emptyset$, ex. $\left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$ مستحيل a و b يكونوا صفر

فبالتالي مستحيل نلاقي $\vec{0}_V$

So it's not subspace.

Ex:- $S = \{ A = (a_{ij})_{2\times 2} : \det(A) = 0 \}$ is S subspace of $\mathbb{R}^{2\times 2}$.

1) $S \neq \emptyset$, $0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in S$. $\det(0) = 0$

2) let $A, B \in S$, $\Rightarrow \det(A) = \det(B) = 0$.
$A+B \in S$. $\det(A+B) \overset{?}{=} 0$

counter ex:- $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, $A+B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \rightarrow \det(A+B) = 1 \neq 0$

so S it's not subspace vector.

EX. S is the set of all symmetric $n\times n$ - matricies, is S a subspace of $\mathbb{R}^{n\times n}$

$S = \{ A \in \mathbb{R}^{n\times n} : A \text{ is symmetric} \} = \{ A \in \mathbb{R}^{n\times n} : A^T = A \}$

① $S \neq \emptyset$, $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $A^T = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $A = A^T$ ✓

② let $A, B \in S$,
$A = A^T$, $B = B^T$

$(A+B)^T = A^T + B^T$
$= A + B \in S$

③ let $A \in S$, $\alpha \in \mathbb{R}$,
$A = A^T$

$(\alpha . A)^T = \alpha (A^T)$
$= \alpha A \in S$

Now, S is Subspace.

Ex. let V be a vector, $S=\{\vec{0}\}$. Is a vector space.

1) $S \neq \phi$, $0 \in S$. ✓

2) let $A, B \in S$,

$A=0,\ B=0,\quad A+B = 0+0 = 0 \in S$ ✓

3) let $s_1 \in S$, and $\alpha \in \mathbb{R}$

$s_1 = 0,\quad \alpha s_1 = 0 \cdot s_1 = 0 \cdot 0 = 0 \in S$ ✓

So, S is a vector Space.

Ex. $S = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3 ,\ x_1 = 2x_2 \right\}$, Is S subspace of $\mathbb{R}^3$.

$$\mathbb{R}^3 = \left\{ \begin{matrix} 2x_2 \\ x_2 \\ x_3 \end{matrix} \right\}$$

1) $S \neq \phi$, ex. $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \in \mathbb{R}^3$ ✓

2) let $A, B \in S$

$$A = \begin{bmatrix} 2b_1 \\ b_1 \\ c_1 \end{bmatrix},\quad B = \begin{bmatrix} 2b_2 \\ b_2 \\ c_2 \end{bmatrix}$$

$$A+B = \begin{bmatrix} 2b_1 + 2b_2 \\ b_1 + b_2 \\ c_1 + c_2 \end{bmatrix} = \begin{bmatrix} 2(b_1+b_2) \\ b_1+b_2 \\ c_1+c_2 \end{bmatrix} \in S \ ✓$$

3) let $s_1 \in S$, and $\alpha \in \mathbb{R}$

$$s_1 = \begin{bmatrix} 2x_2 \\ x_2 \\ x_3 \end{bmatrix} = \alpha \begin{bmatrix} 2x_2 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2(\alpha x_2) \\ \alpha x_2 \\ x_3 \end{bmatrix} \in S \ ✓$$

So S is subspace of $\mathbb{R}^3$.

Ex. $S = \{ A \in \mathbb{R}^{2\times 2} :\ a_{21} = -a_{12} \}$, Is S subspace of $\mathbb{R}^{2\times 2}$.

$$\mathbb{R}^{2\times 2} = \begin{bmatrix} a_{11} & a_{12} \\ -a_{12} & a_{22} \end{bmatrix}$$

1) $S \neq \phi$, $S = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ ✓

2) let $A, B \in S$,

$$A = \begin{bmatrix} a_1 & a_2 \\ -a_2 & a_4 \end{bmatrix},\quad B = \begin{bmatrix} b_1 & b_2 \\ -b_2 & b_4 \end{bmatrix}$$

$$A+B = \begin{bmatrix} a_1+b_1 & a_2+b_2 \\ -a_2+-b_2 & a_4+b_4 \end{bmatrix} = \begin{bmatrix} a_1+b_1 & \overset{c}{a_2+b_2} \\ \underbrace{-(a_2+b_2)}_{-c} & a_4+b_4 \end{bmatrix} \in S \ ✓$$

3) let $s_1 \in S$, $\alpha \in \mathbb{R}$, then

$$s_1 = \begin{bmatrix} a_1 & a_2 \\ -a_2 & a_4 \end{bmatrix} \longrightarrow \alpha s_1 = \begin{bmatrix} \alpha a_1 & \alpha a_2 \\ -\alpha a_2 & a_4 \end{bmatrix} \in S \ ✓$$

So, S is Subspace of $\mathbb{R}^{2\times 2}$

EX. $S = \{ P(x) \in P_4 : \deg(P(x)) \text{ is even} \}$, Is S subspace.

$\deg = 0, 2$

راح يكون فيه من deg (0-3) → $P(x)$
ولكن اللي بدنا إياه، إنو أكبر deg
يكون even .

1) $S \neq \emptyset$, $0 \in S$ ✓

2) $P(x), q(x) \in S$, deg $(P(x)$ and $\deg(q(x)))$ is even

$P(x) + q(x) \notin S$

Counter ex: $P(x) = x + x^2 \in S$
$q(x) = x - x^2 \in S.$
$P(x) + q(x) = x + \cancel{x^2} + x \cancel{- x^2}$
$= 2x \notin S$
↳ deg is not even.

So, S is not Subspace.

EX. $S = \{ P(x) \in P_4 : P(1) = 0 \}$, Is S a Subspace of P_4.

$P(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$
$P(1) = a_0 + a_1 + a_2 + a_3 = 0$

1) $S \neq \emptyset$, $0(1) = 0$

2) let $g(x), q(x) \in S$, $g(1) = 0$, $q(1) = 0$

$g(x) + q(x) \overset{??}{\in} S$

$g(1) + q(1) = 0 + 0 = 0 \in S$ ✓

3) let $g(x) \in S$, $\alpha \in \mathbb{R}$

$g(1) = 0$, $\alpha g(x) = \alpha \cdot 0 = 0 \in S$ ✓

So, S is Subspace.

EX. $S = \{ f(x) \in C^2[-1,1] : f''(x) + f(x) = 0 \}$. is Subspace or not ?.

ex. $\sin(x)$, $f' = \cos(x)$, $f''(x) = -\sin(x)$, ✓ $f + f'' = 0$.

$C^n[a,b]$: all functions $f(x)$ such that $f, f', f'', \dots f^{(n)}$ are cont on $[a,b]$.

1) $S \neq \emptyset$, $0 + 0 = 0$ ✓

2) let $f(x), g(x) \in S$,
$f''(x) + f(x) = 0$, $g''(x) + g(x) = 0$

let $f(x) + g(x) = q(x)$

$q(x) \Rightarrow q''(x) + q(x)$
$= (f(x) + g(x))'' + g(x) + f(x)$
$= f''(x) + g''(x) + g(x) + f(x)$
$= f''(x) + f(x) + g''(x) + g(x)$
$= 0 + 0$
$= 0 \in S$ ✓

3) let $q(x) \in S$, $\alpha \in \mathbb{R}$,

$(\alpha q)'' + (\alpha q) = \alpha q''(x) + \alpha q(x)$
$= \alpha (q''(x) + q(x))$
$= \alpha (0) = 0.$

So, S is Subspace of $f(x)$

Null space of matrix :- the set of all Solutions of $A\vec{x}=\vec{0}$.

let $A=(a_{ij})_{m\times n}$ be a matrix
and let $N(A)=\{x\in\mathbb{R}^n : x \text{ is a Solution to } Ax=0\}$.

Is $N(A)$ a subspace of $\mathbb{R}^n$?

(1) $N(A)\neq\emptyset$, since $\vec{0}=\begin{bmatrix}0\\0\\\vdots\\0\end{bmatrix}$ is a solution to $Ax=0$.

(2) let $x,y\in N(A)$
$Ax=0,\quad Ay=0$
$(x+y)\in^{??} N(A)$. $\quad = A(x+y)$
$= Ax+Ay$
$= 0+0 = 0.$

(3) let $x\in N(A)$, $\alpha\in\mathbb{R}$
$Ax=0$
$\alpha x \in ? N(A)$.

$A(\alpha x)=\alpha A(x)=\alpha\cdot 0$
$=0$

So, $N(A)$ is Subspase of $\mathbb{R}^n$

EX. Find the null Space of $A=\begin{bmatrix}1&2&-1&1\\2&-1&1&2\\1&-1&0&1\end{bmatrix}_{3\times4}$

↳ find all Solutions of $Ax=0$.
solve $Ax=0$.

$\xrightarrow{\substack{-2R_1+R_2\\-R_1+R_3}}\left[\begin{array}{cccc|c}1&2&-1&1&0\\0&-5&3&0&0\\0&-3&1&0&0\end{array}\right]\xrightarrow{-\frac{3}{5}R_2+R_3}\begin{bmatrix}\underline{1}&2&-1&1\\0&\underline{-5}&3&0\\0&0&\underline{-\frac{4}{5}}&0\end{bmatrix}$ #3 leading Variables.
#1 free variable

$\alpha=x_4$
$-\frac{4}{5}x_3=0 \rightarrow x_3=0$

$-5x_2+3x_3=0 \rightarrow x_2=0$
$x_1+2x_2-x_3+x_4=0$
$x_1+\alpha=0 \rightarrow \boxed{x_1=-\alpha}$

$X=\begin{bmatrix}-\alpha\\0\\0\\\alpha\end{bmatrix}$

$N(A)=\left\{\begin{pmatrix}-\alpha\\0\\0\\\alpha\end{pmatrix}: \alpha\in\mathbb{R}\right\}$

$N(A)=\left\{\alpha\begin{bmatrix}-1\\0\\0\\1\end{bmatrix}:\alpha\in\mathbb{R}\right\}$

EX. $A = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{bmatrix}$, find the $N(A)$.

$\xrightarrow{-2R_1+R_2} \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & -1 & -2 & 1 \end{bmatrix}$

$X_3 = \alpha$, $X_4 = \beta$

① $-X_2 - 2X_3 + X_4 = 0$
$+X_2 = -2\alpha + \beta$

② $X_1 + X_2 + X_3 = 0$
$X_1 = -\alpha + 2\alpha - \beta$
$X_1 = \alpha - \beta$

$\therefore X = \begin{bmatrix} \alpha - \beta \\ -2\alpha + \beta \\ \alpha \\ \beta \end{bmatrix}$

linear combination of vectors

if $v_1, v_2 \dots v_n \in V$, a sum of the form, $\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n$, where a_i are scalar is called a linear combination of $v_1, v_2 \dots v_n$.

EX. if $v_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $v_2 = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$, $v_3 = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} \in \mathbb{R}^3$

a) A linear Combination. → "بتقدر تجرب بأي أرقام بدك إياهم"

$2v_1 + 3v_2 - 4v_3 = \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} + \begin{bmatrix} 6 \\ -3 \\ 0 \end{bmatrix} + \begin{bmatrix} -12 \\ -4 \\ 4 \end{bmatrix}$

$= \begin{bmatrix} -4 \\ -3 \\ 10 \end{bmatrix}$, is a linear Combination.

b) is $\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$ is a linear Combination of v_1, v_2, v_3? → يعني بحكيلك حل المعادلة

Solve. $\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$ for $\alpha_1, \alpha_2, \alpha_3$.

$\begin{bmatrix} \alpha_1 + 2\alpha_2 + 3\alpha_3 \\ 2\alpha_1 - \alpha_2 + \alpha_3 \\ 3\alpha_1 \quad - \alpha_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$ → Solve the system.

system → Consistant → Yes
system → in Consistant → No

* A set of all linear Combinations of $v_1, \dots v_n$
$= \{\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n , \alpha_i \in \mathbb{R}\}$.
is called Span of $v_1, v_2 \dots v_n$.

$\text{span}(v_1 \dots v_n)$ = The set of all linear Combination of $v_1 \dots v_n$.

$$\text{span}(v_1, \dots, v_n) = \{\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n, \ \alpha_i \in \mathbb{R}\}.$$

- is $v_1 \in \text{span}(v_1, \dots, v_n)$? yes.

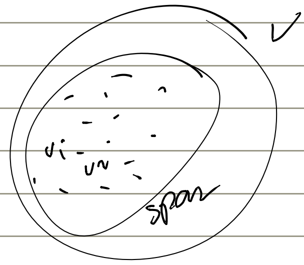
  $v_1 = (1)v_1 + 0v_2 + 0v_3 \dots 0v_n \rightarrow v_1 \in \text{span}(v)$.
- v_2 is span since

  $v_2 = 0v_1 + 1v_2 + 0v_3 + \dots 0v_n$

* $v_i \in \text{Span}(v_1 \dots v_n), \ \forall v_i, \ i = 1 \dots n.$

* Is $0 \in \text{Span}(v_1 \dots v_n)$?

  $0 = 0v_1 + 0v_2 + \dots 0v_n \longrightarrow 0 \in \text{Span}(v_1, \dots v_n)$



let $v_1 \dots v_n \in V$, then $\text{Span}(v_1 \dots v_n)$ is a Subspace of V.

Proof:- $\text{span}(v_1 \dots v_n) = \{\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 + \dots + \alpha_n v_n, \ \alpha_i \text{ Scalars}\}$.

1) $\text{Span}(v_1 \dots v_n) \neq \phi$. $\quad 0 \in \text{Span}(V)$ ✓ $\quad 0 = 0v_1 + 0v_2 + 0v_3 \dots 0v_n$

2) let $s_1, s_2 \in \text{Span}(v_1 \dots v_n)$

$s_1 = \alpha_1 v_1 + \alpha_2 v_2 + \dots \alpha_n v_n.$

$s_2 = \beta_1 v_1 + \beta_2 v_2 + \dots \beta_n v_n$

$s_1 + s_2 = (\alpha_1 + \beta_1)v_1 + (\alpha_2 + \beta_2)v_2 + \dots (\alpha_n + \beta_n)v_n \in \text{Span}(v_1, v_n).$

$S_1 + S_2 = \gamma_1 v_1 + \gamma_2 v_2 + \dots \gamma_n v_n$

3) let $S_1 \in \text{Span}(v_1 \dots v_n)$ and α is scalar.

$S_1 = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 \dots \alpha_n v_n.$

Now $\alpha S_1 = \alpha(\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 \dots \alpha_n v_n).$

$= \alpha\alpha_1 v_1 + \alpha\alpha_2 v_2 + \dots \alpha\alpha_n v_n$

$= \delta_1 v_1 + \delta_2 v_2 + \dots \delta_n v_n \in \text{Span}(v_1 \dots v_n)$ ✓

EX. is $p(x) = 3 + 4x - 5x^2$ a linear combination of $P_1(x) = 1 + x + x^2$, $P_2(x) = 1 - x$, $P_3(x) = 1 + x$.

* Solve $P(x) = 3 + 4x - 5x^2 = \alpha_1 P_1(x) + \alpha_2 P_2(x) + \alpha_3 P_3(x)$

$3 + 4x - 5x^2 = \alpha_1(1 + x + x^2) + \alpha_2(1 - x) + \alpha_3(1 + x)$

$3 = \alpha_1 + \alpha_2 + \alpha_3 \longrightarrow (1)$

$4 = \alpha_1 - \alpha_2 + \alpha_3 \longrightarrow (2)$

$-5 = \alpha_1 \longrightarrow (3)$

$8 = \alpha_2 + \alpha_3$

$+ \ 9 = -\alpha_2 + \alpha_3$

$\boxed{\frac{17}{2} = \alpha_3}$

$\boxed{\alpha_2 = -\frac{1}{2}}$

$\frac{16}{2} - \frac{17}{2}$

$-\frac{1}{2}$

$\therefore \ 3 + 4x - 5x^2 = -5(1 + x + x^2) + -\frac{1}{2}(1 - x) + \frac{17}{2}(1 + x).$

* the System should be Consistant to be linear Combination.

Def: let $v_1, v_2 \dots v_n \in V$, we say $v_1, v_2 \dots v_n$ is a spaning set for V, if $\text{span}(v_1 \dots v_n) = V$.

$\Leftrightarrow$ any $v \in V$ is a linear Combination of $v_1 \dots v_n$.

$\Leftrightarrow$ for any $v \in V$, there exists scalars $\alpha_1 \dots \alpha_n$, s.t $v = \alpha_1 v_1 + \alpha_2 v_2 \dots + \alpha_n v_n$

$\Leftrightarrow$ for any $v \in V$, the system $v = \alpha_1 v_1 + \alpha_2 v_2 + \dots \alpha_n v_n$ is Consistant.

check if $v_1 \dots v_n \in V$ is a spaning set?

EX. Is $\left\{ v_1 = \begin{pmatrix}1\\1\\1\end{pmatrix}, v_2 = \begin{pmatrix}1\\1\\0\end{pmatrix}, v_3 = \begin{pmatrix}1\\0\\1\end{pmatrix} \right\}$ a sp set of $\mathbb{R}^3$.

let $v = \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} \in \mathbb{R}^3 \rightarrow$ check if the system $\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = \begin{pmatrix}a\\b\\c\end{pmatrix}$ is Consistant for all a, b, c.

Solve :- $\alpha_1 \begin{bmatrix}1\\1\\1\end{bmatrix} + \alpha_2 \begin{bmatrix}1\\1\\0\end{bmatrix} + \alpha_3 \begin{bmatrix}1\\0\\1\end{bmatrix} = \begin{bmatrix}a\\b\\c\end{bmatrix}$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & a \\ 1 & 1 & 0 & b \\ 1 & 0 & 1 & c \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & a \\ 0 & 0 & -1 & b-a \\ 0 & -1 & 0 & c-a \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & a \\ 0 & -1 & 0 & c-a \\ 0 & 0 & -1 & b-a \end{array}\right]$$

$\therefore$ the system is Consistant, So v_1, v_2, v_3 is spaning set of $\mathbb{R}^3$.

EX. Is $\left\{ v_1 = \begin{pmatrix}1\\1\\1\end{pmatrix}, v_2 = \begin{pmatrix}1\\2\\3\end{pmatrix}, v_3 = \begin{pmatrix}1\\3\\5\end{pmatrix} \right\}$ a spaning set of $\mathbb{R}^3$?

$S_1 = \begin{pmatrix}a\\b\\c\end{pmatrix}$ — $\begin{bmatrix}a\\b\\c\end{bmatrix} = \alpha_1 \begin{bmatrix}1\\1\\1\end{bmatrix} + \alpha_2 \begin{bmatrix}1\\2\\3\end{bmatrix} + \alpha_3 \begin{bmatrix}1\\3\\5\end{bmatrix}$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & a \\ 1 & 2 & 3 & b \\ 1 & 3 & 5 & c \end{array}\right] \xrightarrow{-R_1+R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & a \\ 0 & +1 & 2 & b-a \\ 0 & 2 & 4 & c-a \end{array}\right] \xrightarrow{-2R_2+R_3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & a \\ 0 & 1 & 2 & b-a \\ 0 & 0 & 0 & -2b+c+a \end{array}\right]$$

Consistant if $-2b+c+a=0$

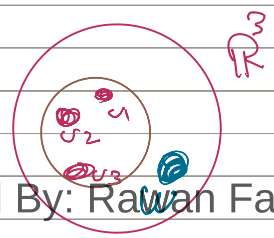
$\therefore$ the system is inConsistant, So, v_1, v_2, v_3 is not Spaning set for $\mathbb{R}^3$.

Find w s.t $w \notin \text{span}(v_1, v_2, v_3)$.

in Consistant.

$-2b+c+a \neq 0$

$w = \begin{pmatrix}3\\1\\1\end{pmatrix} \notin \text{span}(v_1, v_2, v_3)$.



EX. Is $\{P_1(x) = x^2+x+1,\ P_2(x) = x+3,\ P_3(x) = x^2-x+2\}$ a spaning set for P_3

$P_3 = a_1x^2 + a_2x + a_3$

$a_1x^2 + a_2x + a_3 = \alpha_1 P_1(x) + \alpha_2 P_2(x) + \alpha_3 P_3(x).$

$a_1x^2 + a_2x + a_3 = \alpha_1(x^2+x+1) + \alpha_2(x+3) + \alpha_3(x^2-x+2)$

$a_1 = \alpha_1 + \alpha_3$
$a_2 = \alpha_1 + \alpha_2 + -\alpha_3$
$a_3 = \alpha_1 + 3\alpha_2 + 2\alpha_3$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & a_1 \\ 1 & 1 & -1 & a_2 \\ 1 & 3 & 2 & a_3 \end{array}\right] \xrightarrow[-R_1+R_3]{-R_1+R_2} \left[\begin{array}{ccc|c} 1 & 0 & 1 & a_1 \\ 0 & 1 & -2 & a_2-a_1 \\ 0 & 3 & 1 & a_3-a_1 \end{array}\right] \xrightarrow{-2R_2+R_3} \left[\begin{array}{ccc|c} 1 & 0 & 1 & a_1 \\ 0 & 1 & -2 & a_2-a_1 \\ 0 & 0 & 7 & a_1-2a_2+a_3 \end{array}\right]$$

Consistant for any a, b, c.

$\{P_1(x) = x^2+x+1,\ P_2(x) = x+3,\ P_3(x) = x^2-x+2\}$ is a spaning set for P_3.

EX. Is $\left\{E_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, E_2 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, E_3 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}\right\}$ is spaning set for $\mathbb{R}^{2\times2}$.

let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbb{R}^2$.

Solve $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \alpha_1 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \alpha_2 \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + \alpha_3 \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$

$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \alpha_1+\alpha_2+\alpha_3 & \alpha_2 \\ \alpha_3 & \alpha_1 \end{bmatrix}$

$a = \alpha_1 + \alpha_2 + \alpha_3$
$b = \alpha_2$
$c = \alpha_3$
$d = \alpha_1$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \\ 1 & 0 & 0 & d \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \\ 0 & -1 & -1 & d-a \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \\ 0 & 0 & -1 & b+d-a \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & b+d-a+c \end{array}\right]$$

it's consistant if $b+d-a+c = 0$
otherwise it's inconsistant, So it's not spaning set for $\mathbb{R}^{2\times2}$

Example 8-

let $v_1 = \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}$, $v_2 = \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix}$, $v_3 = \begin{bmatrix} 8 \\ 1 \\ 3 \end{bmatrix}$, and $H = \text{Span}\{v_1, v_2, v_3\}$

Note $v_3 = 2v_2 - v_1$, show that $\text{span}\{v_1, v_2, v_3\} = \text{span}\{v_1, v_2\}$ And Find A Basis For the Subspace of H,

نعوض

Sol :

$$
\begin{aligned}
X &= C_1 v_1 + C_2 v_2 + C_3 v_3 \\
&= C_1 v_1 + C_2 v_2 + C_3 (2v_2 - v_1) \\
&= C_1 v_1 + C_2 v_2 + 2C_3 v_2 - C_3 v_1 \\
&= (C_1 - C_3) v_1 + (2C_3 + C_2) v_2
\end{aligned}
$$

So X Can be written as a linear Combination of v_1 and v_2.

Example 8-

let $v_1 = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$, $v_2 = \begin{bmatrix} 1 \\ -5 \\ -6 \end{bmatrix}$, $v_3 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$, and $H = \text{Span}\{v_1, v_2, v_3\}$, also $4v_1 + v_2 + 3v_3 = 0$. Find 3 distinct Bases for H.

$B_1 = \{v_2, v_3\}$

$B_2 = \{v_1, v_3\}$

$B_3 = \{v_2, v_1\}$

Remark :-

in R_3 :- $e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$e_i = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \end{bmatrix} \rightarrow$ ith row.

فقط تكون (1) عند i
وباقي أصفار.

$e_1 \in R^2 \rightarrow e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

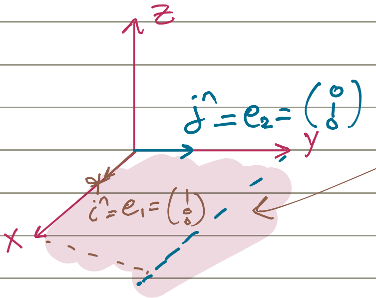
$e_2 \in R^2 \rightarrow e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

EX. Find the span (e_1, e_2) in R^3.

$$\text{Span}(e_1, e_2) = \{\alpha_1 e_1 + \alpha_2 e_2 ,\ \alpha_1, \alpha_2 \in R\}$$

$$\text{span}(e_1, e_2) = \left\{ \alpha_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ 0 \end{bmatrix},\ \alpha_1, \alpha_2 \in R \right\}$$

هندسياً :



موجودة في
x,y plane.

$$U = \alpha_1 \hat{i} + \alpha_2 \hat{j}$$

* $A_{m\times n}$, $AX=b$, $Ax=0$.

if x_0 is a solution to $Ax=b$, and x_1 is a solution to $Ax=0$, then x_1+x_0 is sol to $AX=b$.

$AX_0+AX_1 = 0+b = b$. #

* any sol to $AX=b$, is of the form $y=x_0+z$, x_0 is sol to $AX=b$

z is sol to $AX=0$.

Theorem: let A be $m\times n$ matrix. if x_0 is a particular solution to $Ax=b$, ($Ax=b$ is consistent), then y is a sol to $Ax=b$, iff $y=x_0+z$, $z\in \underset{\sim}{N(A)}$.

z is a sol to $Ax=0$.

Proof: if x_0 is a solution to $Ax=b$ and y is sol to $Ax=b$

$\rightarrow Ax_0=b$, $Ay=b$

$Ay - Ax_0 = 0$

$A(\underbrace{y-x_0}_{z})=0$

$Az=0$

z is solution to $AX=0$.

$z\in N(A)$.

$z=y-x_0 \rightarrow \boxed{y=z+x_0}$

EX. $\underline{A_{4\times 3}}$, $c=2a_1+a_2+a_3$, a_i columns of A.

$2 \to x_1$, $1 \to x_2$, $1 \to x_3 \rightarrow$ are the coefficient.

1) if $N(A)=\{0\}$, How many solutions does $Ax=c$ have?

$AX=C$ is consistant (c is linear combination of $A \Rightarrow Ax=c$ is consistant).

a solution to $Ax=c$ is $x_0=\begin{bmatrix}2\\1\\1\end{bmatrix}$

Are there other solution? No, why? $y=x_0+z$, $z\in N(A)$

$y=\begin{bmatrix}2\\1\\1\end{bmatrix}+z = \begin{bmatrix}2\\1\\1\end{bmatrix}$ # only one solution.

$\rightarrow N(A)=\{0\}$
معطى من السؤال

2) if $N(A)\neq\{0\} \rightarrow \{0\}$ هاي مش معناها إنو بدونه الـ
وإنما انو الـ null space يوجد فيه أرقام غير الصفر

$AX=C$, has infinite # of solutions.

$y=X+Z$

3.3 linear independence

$V_1 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$, $V_2 = \begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix}$, $V_3 = \begin{bmatrix} -1 \\ 3 \\ 8 \end{bmatrix}$

1. Is V_1 a linear Comb of V_2, V_3
2. Is V_2 〃 〃 〃 V_1, V_3
3. Is V_3 〃 〃 〃 V_1, V_2

Can we write one of them as a linear Combination of other 2 vectors?

Solve $\boxed{\alpha_1 V_1 + \alpha_2 V_2 + \alpha_3 V_3 = 0}$

Zero sol.
$\Downarrow$
$\{0\}$
$0V_1 + 0V_2 + 0V_3 = 0$
لـ وهون ما بقدر أكتب واحد بدلالة الثانية

* none of V_1, V_2, V_3 Can be written as a linear Solution of others.

non-Zero Sol.

$X = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$ is a Solution to $\alpha_1 V_1 + \alpha_2 V_2 + \alpha_3 V_3 = 0$

$1V_1 + 0V_2 + 2V_3 = 0$

$V_1 = 0V_2 - 2V_3$

$V_3 = -\frac{1}{2} V_1 + 0V_2$

$V_2 = X$

ما بقدر أكتبه لأنه الـ (coeffient) تاعه صفر

* one of them can be written as linear Combination of other.

* يعني إذا الـ (system) كان عنده بس الـ $\boxed{\text{Zero Solution}}$ ما بقدر أكتبهم بدلالة بعض

* بينما لو كان (non-Zero sol.) فهون بقدر أكتبهم بدلالة بعض.

Def let $V_1, V_2 \dots V_n \in V$, if the system $C_1V_1 + C_2V_2 + \dots + C_nV_n = 0$ has only the Zero sol. we say $V_1, \dots V_n$ are linearly independent. "مستقلين"

if $C_1V_1 + C_2V_2 \dots + C_nV_n$ has a non Zero Sol. so, we say they linearly dependent. "بيعتمدوا على بعض"

indep. ← only Zero Sol.

dep. ← non-Zero Sol.

$C_1V_1 + C_2V_2 \dots C_nV_n = 0$

لازم أحل

* يعني عشان أقرر حل هذا النظام dependent / independent

* وإذا كانه dependent ← مستحيل يكونه independent.

يعني فقط واحد من الاحتمالين.

$\mathbb{R}^n$

EX.

$v_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $v_2 = \begin{bmatrix} 4 \\ 8 \end{bmatrix}$ L.I or L.D.

solve $c_1v_1 + c_2v_2 = 0$

$c_1\begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2\begin{bmatrix} 4 \\ 8 \end{bmatrix} = 0$

$\left[\begin{array}{cc|c} 1 & 4 & 0 \\ 2 & 8 & 0 \end{array}\right] = \left[\begin{array}{cc|c} 1 & 4 & 0 \\ 0 & 0 & 0 \end{array}\right]$ ∅ has non zero solutions so, v_1 and v_2 are linearly dependent.

$\boxed{v_2 = 4v_1}$

* Remarks:-

if $v_1, v_2 \in V$, v_1, v_2 are L.D $\Leftrightarrow$ one of them can be written as a scalar multiple of the other.

EX. $P_1(x) = x^2+x$, $P_2(x) = x+1$. L.I, L.D ?

→ non of them can be written as a constant multiple of the other.

So. L.I.

method (2)

$c_1v_1 + c_2v_2 = 0$

$c_1(x^2+x) + c_2(x+1) = 0$

$c_1x^2 + c_1x + c_2x + c_2 = 0$

$\left.\begin{array}{l} c_1 = 0 \\ c_1 + c_2 = 0 \\ c_2 = 0 \end{array}\right\} \to$ L.I.

EX. $v_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$, $v_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $v_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ in $\mathbb{R}^3$ L.I, L.D ?

$c_1v_1 + c_2v_2 + c_3v_3 = 0$

$\begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & -1 \end{bmatrix} \to$ 3 leading ones so, only zero solution.

$c_3 = 0$

$c_2 + 0 = 0$

$c_1 + 0 + 0 = 0$

∴ L.I → non of them can be written as a linear combination of the other vectors.

Theorem:-

if $v_1, v_2, v_3 \dots v_n \in \mathbb{R}^n$, then the vectors $v_1, v_2 \dots v_n$ are independent iff $X = (v_1, v_2 \dots v_n)_{n\times n}$, is non singular $\Leftrightarrow$ $\det(X) \neq 0$.

∅ $v_1, v_2 \dots v_n$ are L.D $\Leftrightarrow$ X is Singular.

EX. $v_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 2 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 1 \end{bmatrix}, v_3 = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, v_4 = \begin{bmatrix} 2 \\ -2 \\ 4 \\ 6 \end{bmatrix} \in \mathbb{R}^4$

$X = \begin{bmatrix} 1 & 1 & -1 & 2 \\ -1 & 0 & 0 & -2 \\ 1 & 2 & 0 & 4 \\ 2 & 1 & 1 & 6 \end{bmatrix}$

singular → dependent, $|X| = 0$

nonsingular → independent, $|X| \neq 0$.

$|X| = \begin{vmatrix} 1 & 1 & -1 & 2 \\ -1 & 0 & 0 & -2 \\ 1 & 2 & 0 & 4 \\ 2 & 1 & 1 & 6 \end{vmatrix} = \begin{vmatrix} 1 & 1 & -1 & 2 \\ 0 & 1 & -2 & 0 \\ 0 & 1 & 1 & 3 \\ 0 & -1 & 3 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 1 & -1 & 2 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 2 & 2 \end{vmatrix} = 0 \rightarrow$ singular → dependent.

Pn 8-

EX:- $P_1(x) = x^2 - 2x + 3,\ P_2(x) = 2x^2 + x + 8,\ P_3(x) = x^2 + 8x + 7.$

$\rightarrow c_1 P_1(x) + c_2 P_2(x) + c_3 P_3(x) = 0$

$c_1(x^2 - 2x + 3) + c_2(2x^2 + x + 8) + c_3(x^2 + 8x + 7) = 0$

$c_1 + 2c_2 + c_3 = 0$
$-2c_1 + c_2 + 8c_3 = 0$
$3c_1 + 8c_2 + 7c_3 = 0$

$\begin{bmatrix} 1 & 2 & 1 \\ -2 & 1 & 8 \\ 3 & 8 & 7 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 1 \\ 0 & 5 & 10 \\ 0 & 2 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$

having one leading variable so, it has infinite number of sol.

wornskian test

let $f_1(x), f_2(x) \dots f_n(x) \in C^{n-1}[a,b]$. ← إذا عندي 3 فنكشن بيلزمني المشتقة 2
4 = = 3

we define wornskian of $f_1 \dots f_n$ as the function

$$W[f_1(x), f_2(x) \dots f_n(x)](x) = \begin{vmatrix} f_1(x) & f_2(x) & \dots & f_n(x) \\ f_1'(x) & f_2'(x) & \dots & f_n'(x) \\ f''(x) & & & \\ \vdots & & & \\ f^{(n-1)}(x) & \dots & \dots & \dots \end{vmatrix}_{n \times n}$$

if their exist at least one point $x \in [a,b]$ Such that $W[f_1, \dots f_n] \neq 0$, then $f_1(x) \dots f_n(x)$ are linearly independent.

EX. $f_1(x) = e^{x}$, $f_2(x) = e^{-x}$, on $(-\infty, \infty)$.

$$W[f_1(x), f_2(x)] = \begin{vmatrix} e^{x} & e^{-x} \\ e^{x} & -e^{-x} \end{vmatrix} = -1-1 = -2 \neq 0$$

, so they are linearly indep.

$W \neq 0 \longrightarrow \det(f) \neq 0 \longrightarrow$ nonsingular $\longrightarrow$ has only the zero sol $\longrightarrow$ Independent.

EX. $f_1(x) = x$, $f_2(x) = x\ln x$ on $(0, \infty)$.

$$W = \begin{vmatrix} x & x\ln x \\ 1 & 1+\ln x \end{vmatrix} = x + x\ln x - x\ln x = x \;, \quad (0, \infty)$$

هل يوجد على الأقل نقطة
واحدة تجعل ال W لا
تساوي صفر؟ إذا نعم معناها linearly indep

Remark :- if $W[f_1 \dots f_n](x) = 0$, $\forall x \in I$

→ ✱ Test fails
✱ يعني إذا (W=0) ما بستنتج ولا إشي).

EX. $f_1(x) = x^2$, $f_2(x) = x|x|$ on $(-1, 1)$.

$$x|x| = \begin{cases} -x^2, & -1 \le x < 0 \\ x^2, & 0 \le x \le 1 \end{cases}$$

Cont on $(-1,1)$.

$$(x|x|)' = \begin{cases} -2x, & -1 < x < 0 \\ 2x, & 0 < x < 1 \end{cases} = 2|x|$$

$$(x|x|)'(0)\Big|_{+} = 0 \quad , \quad (x|x|)'(0)\Big|_{-} = 0$$

$=$

so it's diffrentiable.

$$W = \begin{vmatrix} x^2 & x|x| \\ 2x & 2|x| \end{vmatrix} = 2x^2|x| - 2x^2|x| = 0$$

test fails

$\Downarrow$

so, back to definition.

$C_1 V_1 + C_2 V_2 = 0$

$C_1 x^2 + C_2 x|x| = 0$

$$\left.\begin{array}{l} x = 1 \longrightarrow C_1 + C_2 = 0 \\ x = -1 \longrightarrow C_1 - C_2 = 0 \end{array}\right\} \Longrightarrow C_1 = C_2 = 0 \longrightarrow \text{so } \exists \text{ only the zero Sol.}$$

$\Downarrow$

Independent.

EX. $f_1(x) = \sin(x)$, $f_2(x) = 2\sin x$ on $[-\pi/2, \pi/2]$

$$W = \begin{vmatrix} \sin x & 2\sin x \\ \cos x & 2\cos x \end{vmatrix} = 0$$

$f_1(x) = \sin x$, $f_2(x) = 2\sin(x)$

linearly dependent
since $f_2 = 2f_1$

* the vector space $C^{n-1}[a,b]$

* $C^3[a,b]$: all function with cont. third derivative on $[a,b]$

Def: let $f_1 - f_n \in C^{n-1}[a,b]$, define $W[f_1,\dots f_n]$ on $[a,b]$ by

$$W[f_1,\dots,f_n](x) = \begin{vmatrix} f_1(x) & \dots & f_n(x) \\ f_1'(x) & \dots & f_n'(x) \\ f^{(n-1)}(x) & \dots & \dots \end{vmatrix}$$

wornskion $f_1 \dots f_n$.

if $\exists$ x_0 in $[a,b]$ such that $W[f_1,\dots f_n](x_0) \neq 0$, then $f_1, \dots f_n$, then its independent.

ex. $1, x, x^2, x^3$

$$W[1,x,x^2,x^3](x) = \begin{vmatrix} 1 & x & x^2 & x^3 \\ 0 & 1 & 2x & 3x^2 \\ 0 & 0 & 2 & 6x \\ 0 & 0 & 0 & 6 \end{vmatrix} = 12 \neq 0$$

so, $1, x, x^2, x^3$ is indep.

ex2. e^x, e^{-x}

$$W[e^x, e^{-x}](x) = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix} = -1-1 \neq 0 \quad \text{Indep.}$$

* if $f_1 \dots f_n$ are linearly indep in $C^{n-1}[a,b]$ then they are also linearly independent in $C[a,b]$.

ex. $x^2, x|x|$ in $C(-1,1)$

$$W[x^2, x|x|] = \begin{vmatrix} x^2 & x|x| \\ 2x & 2|x| \end{vmatrix} = 0 \text{, No conclus.}$$

$$x|x| = \begin{cases} -x^2, & -1 \le x \le 0 \\ x^2, & 0 \le x \le 1 \end{cases}$$

$$\begin{cases} -2x, & -1 < x < 0 \\ 2x, & 0 < x < 1 \end{cases} \longrightarrow 2|x|$$

تاخد اي اشي اذا تغيرت

نرجع للتعريف

$c_1(x^2) + c_2(x|x|) = 0$

$x = \frac{1}{2}$, $\frac{1}{4}c_1 + \frac{1}{4}c_2 = 0$

$x = -\frac{1}{2}$, $\frac{1}{4}c_1 - \frac{1}{4}c_2 = 0$

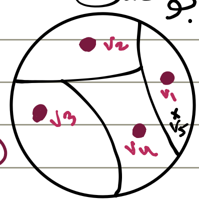
$c_1 = c_2 = 0$

linearly indep.

Remark 1 :-

1. if $v_1, \dots, v_n \in V$, and one of them (say v_n), can be written as a linear Combination of the other vectors $\{v_1 \dots v_n\}$ are linear dependent}, then $span(v_1, \dots, v_n) = span(v_1, \dots, v_{n-1})$

أبو قش



2. if $\{v_1 \dots v_n\}$ is a spanning set for V and $\{v_1 \dots v_n\}$ are LD and v_n can be written as linear Combination of the others, then $\{v_1, \dots v_{n-1}\}$ is a spanning set for V.

Remark 2 :-

- if $\{v_1 \dots v_n\}$ are L.D (say v_n can be written as linear Combination of the others). then $\{v_1 \dots v_n\}$ may or may not be L.I.

- let $v_1 \dots v_n \in V$. a vector $v \in Span\{v_1, \dots v_n\}$ can be written uniqly as a linear Combination of other vector if and only if $v_1 \dots v_n$ are linearly independent.

$v \in Span(v_1, \dots v_n) \Rightarrow v$ can be written as L.C of $\{v_1 \dots v_n\}$

One way $\longrightarrow$ linearly Independent
or many ways $\longrightarrow$ Linearly dependent.

- * $\{v_1, v_2 \dots v_n\}$ spaning for V, $\{v_1, v_2 \dots v_{n-1}\}$ spaning set. F
- if $\{v_1 \dots v_n\}$ is spanning set for V, $v_{n+1} \in V$, $\{v_1 \dots v_n, v_{n+1}\}$ Spanning set T.
- if $\{v_1 \dots v_n\}$ are linearly D. and $v_{n+1} \in V$, then $\{v_1, \dots v_n, v_{n+1}\}$ is L.D T
- if $\{v_1 \dots v_n\}$ are L.D then $\{v_1 \dots v_{n-1}\}$ L.D (F)
- if $\{v_1 \dots v_n\}$ are LD then $\{v_1 \dots v_n\}$ L.I (F)
- if $\{v_1 \dots v_n\}$ are L.I then $\{v_1, \dots v_{n-1}\}$ are L.I (T)
- if $\{v_1, \dots v_n\}$ are L.I then $v_{n+1} \in V$ $\{v_1 \dots v_{n+1}\}$ are L.I. F
- // // // // // // // // L.D F

* L.I. none of them can be written as linear Combination of the other.

* L.D at least one of them can be written as a linear Combination of others.

3.4 basis and dimision

* A set of vector $\{v_1 ... v_n\}$ is Called a basis for V if :-
(1) $\{v_1 ... v_n\}$ is a spaning set of V.
(2) $\{v_1 ... v_n\}$ are L.I.
} any vector can be written uniqly as a linear Combination

Ex. Is $\{e_1, e_2, e_3\}$ a basis for $\mathbb{R}^3$

① span $\{e_1, e_2, e_3\}$, let $v = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathbb{R}^3$

$$\alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3 = \alpha_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \alpha_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \end{array}\right] \rightarrow \text{Consistant for all } a, b, c$$

So $\{e_1, e_2, e_3\}$ is Spanning set for $\mathbb{R}^3$.

② $\{e_1, e_2, e_3\}$ independent.

$$\alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3 = 0$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \text{only has the zero sol.} \Rightarrow \text{nonSingular} \Rightarrow |X| = 1 \neq 0$$
So, its Independent.

So $\{e_1, e_2, e_3\}$ is linearly I.

So, $\{e_1, e_2, e_3\}$ are Basis for $\mathbb{R}^3$.

Ex. Is $\left\{ v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, v_3 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \right\}$ a basis for $\mathbb{R}^3$.

① spaning set for $\mathbb{R}^3$.

let $v = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \in \mathbb{R}^3$, $\alpha_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + \alpha_3 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & a \\ 1 & 1 & 0 & b \\ 1 & 1 & 1 & c \end{array}\right] \xrightarrow[-R_1+R_3]{-R_1+R_2} \left[\begin{array}{ccc|c} 1 & 0 & 2 & a \\ 0 & 1 & -2 & b-a \\ 0 & 1 & -1 & c-a \end{array}\right] \xrightarrow{-R_2+R_3} \left[\begin{array}{ccc|c} 1 & 0 & 2 & a \\ 0 & 1 & -2 & b-a \\ 0 & 0 & 1 & c-b \end{array}\right]$$

the system is Consistant, has only one Sol. , so it's span ✓.

② linear independent.

$$\alpha_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + \alpha_3 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = 0$$

$$\begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \xrightarrow{\;\cdots\cdots\;} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \longrightarrow \det(A) = 1 \to \text{nonsingular} \to \text{Independent.}$$

So yes, its' Basis.

EX. $V_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, V_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, V_3 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$, basis for $\mathbb{R}^3$

① spaning

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & a \\ 1 & 1 & 0 & b \\ 2 & 0 & 1 & c \end{array}\right] \xrightarrow[-2R_1+R_3]{-R_1+R_2} \left[\begin{array}{ccc|c} 1 & 1 & 2 & a \\ 0 & 0 & -2 & b-a \\ 0 & -2 & -3 & c-2a \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 2 & a \\ 0 & -2 & -3 & c-2a \\ 0 & 0 & -2 & c-a \end{array}\right]$$

Consistant, so yes its spaning set.

② L.I ?

$$\begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \xrightarrow{\;\cdots\;} \begin{bmatrix} 1 & 1 & 2 \\ 0 & -2 & -3 \\ 0 & 0 & -2 \end{bmatrix} \longrightarrow \det(A) \neq 0$$

So it L.I

So it Basis

EX. Is $\left\{ E_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, E_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, E_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, E_{21} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\}$ a Basis for $\mathbb{R}^{2\times2}$

① span let $v \in V$, $v = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$\alpha_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \alpha_3 \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + \alpha_4 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\left.\begin{aligned} \alpha_1 &= a \\ \alpha_2 &= b \\ \alpha_3 &= c \\ \alpha_4 &= d \end{aligned}\right\} \Rightarrow \text{Consistant for all } a, b, c, d$$

So, $\{E_{11}, E_{12}, E_{21}, E_{22}\}$ sp. for $\mathbb{R}^{2\times2}$.

② L.I ?

$$\alpha_1 E_{11} + \alpha_2 E_{12} + \alpha_3 E_{21} + \alpha_4 E_{22} = 0$$

$$\begin{bmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0$, So non Singular.

So its L.I.

So it's Basis.

EX. Is $\{P_1(x) = x^2+x+1,\ P_2(x) = x+1,\ P_3(x) = 2\}$ Basis for P_3.

① Span $P(x) = ax^2+bx+c$

$\alpha_1 P_1(x) + \alpha_2 P_2(x) + \alpha_3 P_3(x) = P(x)$

$\alpha_1(x^2+x+1) + \alpha_2(x+1) + \alpha_3(2) = ax^2+bx+c.$

$$\left.\begin{aligned} \alpha_1 &= a \\ \alpha_1+\alpha_2 &= b \\ \alpha_1+\alpha_2+2\alpha_3 &= c \end{aligned}\right\} \quad \begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} \underline{2} & 1 & 1 \\ 0 & \underline{1} & 1 \\ 0 & 0 & \underline{1} \end{bmatrix} \rightarrow$$ Consistant, So its' Span Set.

② L.I ?

$\alpha_1(x^2+x+1) + \alpha_2(x+1) + \alpha_3(2) = 0$

$$\left.\begin{aligned} \alpha_1 &= 0 \\ \alpha_1+\alpha_2 &= 0 \\ \alpha_1+\alpha_2+2\alpha_3 &= 0 \end{aligned}\right\} \quad \begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \xrightarrow{\ \cdots\ } \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \longrightarrow$$ nonsingular and only the zero Sol.

⇓

So. Indep.

So its Basis ——

Theorem 8-

① if $\{v_1 \dots v_n\}$ is a spanning set for V, then any Collection of more than n vectors in V are L.D.

↳ Collection $\{w_1, w_2 \dots w_k\}$, $k > n$ ⟶ $W \to$ L.D

EX. $\left\{ v_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, v_2 = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, v_3 = \begin{bmatrix} 4 \\ -5 \\ 2 \end{bmatrix}, v_4 = \begin{bmatrix} 3 \\ 4 \\ 7 \end{bmatrix} \right\}$ L.I or L.D in $\mathbb{R}^3$

we know $\{e_1, e_2, e_3\}$ are basis for $\mathbb{R}^3 \Rightarrow$ spaning set for $\mathbb{R}^3$

3 vectors

$v_1, v_2, v_3, v_4 \to$ 4 vectors > 3, So $\{v_1, v_2, v_3, v_4\}$ L.D.

EX. $\left\{ v_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, v_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, v_3 = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} \right\}$ L.D or L.I for $\mathbb{R}^3$?

3 vectors → may D. / → may I. } we have to check.

$\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = 0.$

$$\begin{bmatrix} 1 & 2 & -1 \\ 1 & 1 & 0 \\ 2 & 0 & 2 \end{bmatrix} \xrightarrow[-2R_1+R_3]{-R_1+R_2} \begin{bmatrix} 1 & 2 & -1 \\ 0 & -1 & 1 \\ 0 & -4 & 4 \end{bmatrix} \xrightarrow{-4R_2+R_3} \begin{bmatrix} 1 & 2 & -1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

→ the system is has non zero Sol. So, it's L.D.

Remark:-
if V has a basis $\{v_1, v_2 \dots v_n\}$ and $\{u_1, u_2 \dots u_k\}$ are L.I, then $k \leq n$

Theorem:-
if $\{u_1, u_2 \dots u_n\}$ and $\{w_1, w_2 \dots w_m\}$ are two bases for V, then $n = m$

Proof:- Assume $\{u_1, u_2 \dots u_n\}$ is Basis → Spaning and L.I
$\{w_1, w_2 \dots w_m\}$ is Basis → Spaning and L.I.

since $\{u_1, u_2, \dots u_n\}$ are spaning and $\{w_1, w_2, \dots w_m\}$ are L.I so $\boxed{m \leq n}$

and since $\{w_1, w_2 \dots w_m\}$ is span and $\{u_1, u_2 \dots u_n\}$ is L.I So, $\boxed{n \leq m}$

Since $m \leq n$ and $n \leq m$

So $n = m$ #

Def:- let V be a vector space, If V has a basis $\{v_1, v_2 \dots v_n\}$, we say V is finite diminsional and diminsion of $V = n$. $\dim(V) = n$

EX. $\mathbb{R}^3$, $\dim(\mathbb{R}^3) = 3$
since $\{e_1, e_2, e_3\}$ is a basis for $\mathbb{R}^3$.

* يعني باختصار: بجيب basis وبعد الـ elements اللي فيه.

EX. $\dim(P_3) = 3$
since $\{P_1(x) = x^2+x+1, P_2(x) = x+1, P_3(x) = 2\}$ is a basis for P_3

Remark:-
* If $V = \{0\}$, has no basis., we define $\dim(\{0\}) = 0$

* If $V \neq \{0\}$ has no basis, we say $\dim(V) = \infty$.

* Exampels for vector space that don't have Basis.

1. The space $C[a,b] \to$ Cont. function on closed interval.
2. The space $\underline{\underline{P}}$: all poly. of all deg. (any degree)

* $\dim(\mathbb{R}^{2\times2}) = 4$
* $\dim(\mathbb{R}^n) = n$, basis for $\mathbb{R}^n$ is $\{e_1, e_2, e_3, \dots e_n\} \to$ standard Basis for $\mathbb{R}^n$.
* $\dim(P_n) = n$, basis for P_n is $\{x^{n-1}, x^{n-2}, \dots, x, 1\}$ basis for P_n.
* $\dim(\mathbb{R}^{m\times n}) = \underline{\underline{m.n}}$
  حاصل ضرب
  m و n

dim هو عدد ال elements الي في basis

* $\dim(\mathbb{R}^{3\times4}) = 12 = 3\times4$

EX. let $A = \begin{bmatrix} 1 & 2 & -1 & 1 \\ -1 & -1 & 0 & 2 \\ 1 & 0 & 2 & 1 \end{bmatrix}$, Find a basis and dimension of $N(A)$.

$N(A) \to Ax = 0$

$$\begin{bmatrix} 1 & 2 & -1 & 1 \\ -1 & -1 & 0 & 2 \\ 1 & 0 & 2 & 1 \end{bmatrix} \xrightarrow[-R_1+R_3]{R_1+R_2} \begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & 1 & -1 & 3 \\ 0 & -2 & 3 & 0 \end{bmatrix} \xrightarrow{2R_2+R_3} \begin{bmatrix} \underline{\underline{1}} & 2 & -1 & 1 \\ 0 & \underline{\underline{1}} & -1 & 3 \\ 0 & 0 & \underline{\underline{1}} & 6 \end{bmatrix}$$

$x_4 = \alpha$
$x_3 = -6\alpha$
$x_2 = x_3 - 3x_4 = -6\alpha - 3\alpha = -9\alpha$
$x_1 = -2x_2 + x_3 - x_4 = 18\alpha - 6\alpha - \alpha = 11\alpha$

$N(A) = \left\{ \alpha \begin{pmatrix} 11 \\ -9 \\ -6 \\ 1 \end{pmatrix}, \alpha \in \mathbb{R} \right\}$

So $v_1 = \begin{pmatrix} 11 \\ -9 \\ -6 \\ 1 \end{pmatrix}$ is spanning set for $N(A)$.

α أي element لحاله بشرط ماكون الصفر بكونه L.I.

$\{v_1\}$ L.I $\to$ فيه element وحيد. , $\dim(N(A)) = 1$

EX. $S = \left\{ \begin{pmatrix} a+b \\ a-b \\ a+b+c \end{pmatrix} : a,b,c \in \mathbb{R} \right\}$ is a subspace of $\mathbb{R}^3$

Find Basis and dim(s).

$$S = \left\{ \overset{\text{scalor}}{a} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + b \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + c \begin{pmatrix} 0 \\ 0 \\ +1 \end{pmatrix} , a,b,c \in \mathbb{R} \right\}$$

spanning set for S is $\left\{ v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, v_2 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, v_3 = \begin{pmatrix} 0 \\ 0 \\ +1 \end{pmatrix} \right\}$

L.I or L.D ?

$x = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$, $|x| = -2$, so L.I.

$\therefore$ a basis for S is $\{v_1, v_2, v_3\}$.

$\dim(S) = \underline{\underline{3}}$.

Remark :-

1. let V be a vector space, $\dim(V) = n > 0$, $V \neq \{0\}$
   - 1.1. any set $\{v_1, v_2 \dots v_n\}$ are $\underline{\text{L.I}}$ then $\{v_1 \dots v_n\}$ is a spanning set for V. (and so they are a basis for V).
   - 1.2. if $\{u_1 \dots u_n\}$ form a sp. set for V, then $\{u_1, u_2 \dots u_n\}$ are linearly I so, they form a basis for V.

EX. Is $\left\{ v_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, v_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, v_3 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \right\}$ basis for $\mathbb{R}^3$?

$\dim(\mathbb{R}^3) = 3$

* 3 vectors.

$$\begin{bmatrix} 1 & -1 & 1 \\ 2 & 0 & -1 \\ 3 & 1 & 0 \end{bmatrix} \xrightarrow[-3R_1+R_3]{-2R_1+R_2} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & -3 \\ 0 & 4 & -3 \end{bmatrix} \xrightarrow{-2R_2+R_3} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & -3 \\ 0 & 0 & 3 \end{bmatrix} , \quad |x| \neq 0 \quad \text{so L.I} \checkmark$$

So it's spanning set and then $\{V_1, V_2, V_3\}$ are basis.

Theorem 8

1. if V is a vector space, $\dim(V) = n > 0$.

↳ 1. no set of fewer than n vectors can span V.

↳ if $\{v_1, v_2 \dots v_k\} \in V$, $k < n$, then $v_1, \dots, v_k$ is not span set for V.

2. if $\{u_1, u_2 \dots, u_s\}$ are L.I, $s < n$, then this set can be extended to a basis.

2. if $\{w_1, w_2 \dots w_r\}$ is a sp set for V, $r > n$, then $\{w_1 \dots w_n\}$ can be pared down to a basis

EX. $v_1 = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$, $v_2 = \begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix}$, $v_3 = \begin{pmatrix} 2 \\ 6 \\ 4 \end{pmatrix}$, $S = \mathrm{Span}(v_1, v_2, v_3)$. Find a basis for S.

S, supsbase of $\mathbb{R}^3$

$\dim(S) = 3$.