Key

Birzeit University
Mathematics Department
Math331-Section (1)
First Short Exam

Instructor: Dr. Ala Talahmeh
Time: 40 minutes
Name:...

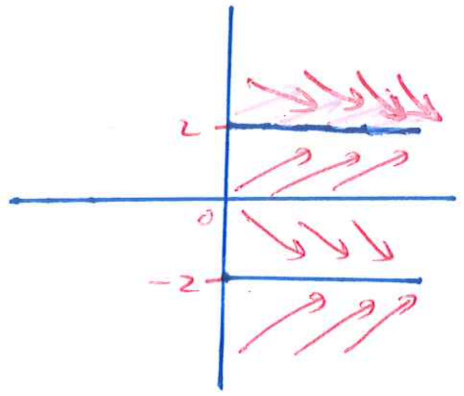
First Semester 2024/2025
Date: 30/10/2024
Number:...............

Question#1 [4 marks].

(a) Let $y(t)$ be the solution of the IVP

$$\frac{dy}{dt} + y^3 - 4y = 0, \quad y(0) = -1.$$

Find $\lim_{t\to\infty} y(t)$.



Sol. $\frac{dy}{dt} = 4y - y^3 = y(2-y)(2+y)$

1 $\frac{dy}{dt} = 0 \Rightarrow y = 0, 2, -2$

1 $\lim_{t\to\infty} y(t) = -2.$

(b) Find the **largest interval** in which the solution of the initial-value problem

$$(x-1)(x-4)\frac{dy}{dx} + \frac{\ln(x-1)}{x^2+1}y = 4x^2 + 1, \quad y(2) = -1$$

is certain to exist.

1 $p(x) = \frac{\ln(x-1)}{(x^2+1)(x-1)(x-4)}$, $q(x) = \frac{4x^2+1}{(x-1)(x-4)}$

p and q are continuous on $x > 1, x \neq 4$.

The largest open interval is $(1, 4)$. 1

Question#2 [6 marks].

(a) Consider the initial-value problem

$$\frac{dy}{dx} = \frac{\sqrt{y^2-1}}{\ln(2x-4)}, \; y(3) = -3.$$

Find the **largest open rectangle** in which the conditions of the existence and uniqueness theorem are satisfied .

1

$f(x,y) = \dfrac{\sqrt{y^2-1}}{\ln(2x-4)}$, $\dfrac{\partial f}{\partial y} = \dfrac{y}{\sqrt{y^2-1}\,\ln(2x-4)}$

f and $\frac{\partial f}{\partial y}$ are continuous on

1

$R = \{(x,y) : x > 2, x \neq \frac{5}{2}$ and $|y| > 1\}$. The largest open rectangle is $x > \frac{5}{2}$ and $y < -1$.

(b) Solve the initial-value problem:

$$\frac{dx}{dy} = \frac{2y\left(1+\sin^2 x\right)}{\left(1+y^2\right)\cos x}, \;\; y(0) = 0.$$

1

$$\int \frac{\cos x}{1+\sin^2 x}\,dx = \int \frac{2y}{1+y^2}\,dy$$

1 1

$$\tan^{-1}(\sin x) = \ln(1+y^2) + C$$

0.5

$y(0) = 0 \Rightarrow \tan^{-1}(0) = \ln 1 + C \Rightarrow C = 0.$

0.5

$\therefore$ the solution is $\tan^{-1}(\sin x) = \ln(1+y^2)$.

Question#3 [5 marks]. Find the general solution of the differential equation

$$\frac{dy}{dx} - \frac{y}{x} = \frac{1}{y}, \quad x > 0, \quad y > 0.$$

0.5 $\frac{dy}{dx} - \frac{1}{x} y = y^{-1}$ Bernoulli with $n = -1$.

$y \frac{dy}{dx} - \frac{1}{x} y^2 = 1$

1 $y^2 = v, \quad 2y \frac{dy}{dx} = \frac{dv}{dx}$

1 $\frac{1}{2} \frac{dv}{dx} - \frac{1}{x} v = 1$

$\frac{dv}{dx} - \frac{2}{x} v = 2$ linear in v.

0.5 $\mu(x) = e^{\int -\frac{2}{x} dx} = e^{-2 \ln|x|} = x^{-2}, \; x > 0.$

1 $v = x^2 \left[\int 2x^{-2} dx + c \right]$

1 $y^2 = x^2 \left(-\frac{2}{x} + c \right)$

$y^2 = -2x + cx^2$

Question#4 [**5 marks**]. A hot iron was left in a room with temperature equals to 20°C. After one minute the temperature of the rod is equal to 40°C. After two minutes it is equal to 30°C. What was the **initial temperature** of the rod.

1 $\frac{du}{dt} = -k(u-20)$, $u(1) = 40°c$, $u(2) = 30°c$.

$\int \frac{du}{u-20} = -k\int dt$, $u \neq 20$

$\ln|u-20| = -kt + c$

1 $u(t) = 20 + Ae^{-kt}$

1 $\begin{cases} 40 = u(1) = 20 + Ae^{-k} \Rightarrow Ae^{-k} = 20 \quad \text{--- (i)} \\ 30 = u(2) = 20 + Ae^{-2k} \Rightarrow Ae^{-2k} = 10 \quad \text{--- (ii)} \end{cases}$

(i) ÷ (ii) $\Rightarrow e^{k} = 2 \Rightarrow \boxed{k = \ln 2}$ 0.5

(i) $\Rightarrow \frac{1}{2}A = 20 \Rightarrow \boxed{A = 40}$ 0.5

$\therefore u(t) = 20 + 40e^{-(\ln 2)t} = 20 + 40(2^{-t})$.

1 $u(0) = 20 + 40(1) = 60°c$.

Good Luck