Exercises :

3.2.0 : True or False .

a. If $f(x) \to \infty$ as $x \to \infty$ and $g(x) > 0$ then $\frac{g(x)}{f(x)} \to 0$ as $x \to \infty$. False

If $f(x) = x^2 + 1 = g(x)$ then $f(x) \to \infty$ and $g(x) > 0$ $\forall x$.

But $\frac{g(x)}{f(x)} = 1$ Dose not converge to 0.

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b. If $f(x) \to 0$ as $x \to a^+$ and $g(x) \geq 1$ $\forall x \in \mathbb{R}$ then $\frac{g(x)}{f(x)} \to \infty$ as $x \to a^+$. False

If $f(x) = -x^2$ and $g(x) = 1$ then $f(x) \to 0$ as $x \to 0^+$ and $g(x) \geq 1$

But $\frac{g(x)}{f(x)} = -\frac{1}{x^2} \to -\infty$ as $x \to 0^+$.

C. If $f(x) \to \infty$ as $x \to \infty$ then $\frac{\sin(x^2+x+1)}{f(x)} \to 0$ as $x \to \infty$. True.

let $\varepsilon > 0$ and set $M = \frac{1}{\varepsilon}$

If $x > M$ then $\left| \frac{\sin(x^2+x+1)}{f(x)} \right|$ $\qquad -1 \leq \sin x \leq 1$

$f(x) > M$

$\frac{1}{f(x)} < \frac{1}{M}$ $\qquad \leq \frac{1}{f(x)} < \frac{1}{M} < \frac{1}{\frac{1}{\varepsilon}} = \varepsilon$

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d. If P and Q are polynomiels s.t the degree of P is less than or equal to the degree of Q, then there is an $L \in \mathbb{R}$ s.t $\lim_{x \to \infty} \frac{P(x)}{Q(x)} = \lim_{x \to -\infty} \frac{P(x)}{Q(x)} = L$. True

proof :

من حل الدكتور

9.2.1 : use defenitions to prove that the limit exists.

a. $\lim\limits_{x\to 0^-} \frac{\sqrt{x^2}}{x} = -1 \quad \Rightarrow \quad \lim\limits_{x\to 0^-} \frac{|x|}{x} = \lim\limits_{x\to 0^-} \frac{-x}{x} = -1$ $x<0$

let $\varepsilon > 0$ and set $\delta = 1$

If $0-\delta < x < 0$ then $\left| \frac{\sqrt{x^2}}{x} - (-1) \right|$

$-\delta < x < 0$

$= \left| \frac{-x}{x} + 1 \right|$ $\sqrt{x^2} = -x$

$= |-1+1|$

$= 0 < \varepsilon$ $\boxed{3}$

b. $\lim\limits_{x\to \infty} \frac{\sin x}{x^2} = 0$

let $\varepsilon > 0$, set $M = \frac{1}{\sqrt{\varepsilon}}$

If $x > M$ then $\left| \frac{\sin x}{x^2} - 0 \right| = \left| \frac{\sin x}{x^2} \right|$

$\left(\frac{1}{x} < \frac{1}{M} \right)^{\wedge 2}$

$\frac{1}{x^2} < \frac{1}{M^2}$ $\leq \frac{1}{x^2}$

$< \frac{1}{M^2}$

$< \frac{1}{\frac{1}{\sqrt{\varepsilon}}^2}$

$< \varepsilon$ $\boxed{3}$

d. $\lim_{x \to 1^+} \frac{x-3}{3-x-2x^2} = \frac{-2}{0} = \infty$

suppose $M > 0$ and set $\delta = \min\{1,$

If $1 < x < 1+\delta$ then $\frac{x-3}{-2x^2-x+3} = \frac{3-x}{2x^2+x-3} = \frac{3-x}{(x-1)(2x+3)}$

$< \frac{3-x}{\delta 7}$

since $3-x \geq 3-2 = 1$

$\frac{1}{7\delta}$

$1 \div 7 \frac{1}{7M}$

$\hookrightarrow$ If $1 < x < 1+\delta$ then $(1 < x < 2)$ since $\delta \leq 1$ δ [illegible]

$(2 < 2x < 4)$ $+3$

$5 < 2x+3 < 7$

and $0 < x-1 < \delta$

e. $\lim_{x \to -\infty} \frac{\cos(\tan x)}{x+1} = 0$

let $\varepsilon > 0$, set $M = -1 - \frac{1}{\varepsilon}$

if $x < M$ then $\left| \frac{\cos(\tan x)}{x+1} - 0 \right|$

$x+1 < M+1$

$-(x+1) > -(M+1)$

$\frac{-1}{x+1} < -\frac{1}{M+1}$

$\leq \left| \frac{1}{x+1} \right|$ $\cos x \leq 1 \quad \forall x$.

لأنو انا بتحرك عند $-\infty$

$< \frac{1}{-(x+1)}$ x : negative

$|x+1| = x+1 , x \geq -1$

$= -(x+1) , x < -1$

$< \frac{-1}{x+1}$

واحنا $-\infty$ عشان هيك السالب

$< \frac{-1}{M+1} = \varepsilon$ $\rightarrow (M+1)\varepsilon = -1$

$M+1 = -\frac{1}{\varepsilon} - 1$

$M = -1 - \frac{1}{\varepsilon}$.

$< -1 \div (\cancel{-1} - \frac{1}{\varepsilon} + \cancel{1})$

$< -1 * -\varepsilon$

$< \varepsilon$

3.2.2: evaluate the following limits when they exists:

a. $\lim_{x \to 2^-} \frac{x^3 - x^2 - 4}{x^2 - 4} = \lim_{x \to 2^-} \frac{\cancel{(x-2)}(x^2 + x + 2)}{\cancel{(x-2)}(x+2)}$

$= \lim_{x \to 2^-} \frac{x^2 + x + 2}{x + 2}$

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$\rightarrow \quad = 2$

b. $\lim_{x\to\infty} \frac{5x^2+3x-2}{3x^2-2x+1} = \lim_{x\to\infty} \frac{5+\frac{3}{x}-\frac{2}{x^2}}{3-\frac{2}{x}+\frac{1}{x^2}} = \frac{5+0-0}{3-0+0} = \frac{5}{3}$

c. $\lim_{x\to-\infty} e^{-\frac{1}{x^2}} = e^{\lim_{x\to-\infty} -\frac{1}{x^2}} = e^0 = 1$

d. $\lim_{x\to 0^+} \frac{e^{x^2+2x-1}}{\sin x} = \frac{e^{-1}}{\sin 0} = \frac{e^{-1}}{0^+} = \infty$.

e. $\lim_{x\to 0^-} \frac{\sin(x+\frac{\pi}{2})}{\sqrt[3]{\cos x - 1}} = \frac{\sin\frac{\pi}{2}}{\sqrt[3]{0^-}} = \frac{1}{0^-} = -\infty$

f. $\lim_{x\to 0^+} \frac{\sqrt{1-\cos x}}{\sin x} =$ since $\sin^2 x = 1-\cos^2 x = (1-\cos x)(1+\cos x)$

$$\frac{\sqrt{1-\cos x}}{\sin x} = \frac{1}{\sqrt{1+\cos x}}$$

So $\lim_{x\to 0} \frac{1}{\sqrt{1+\cos x}} = \frac{1}{\sqrt{2}}$

3.2.3 :

a. prove that $\lim_{x\to a} x^n = a^n$ for $n=0,1,\dots,$ and $a\in\mathbb{R}$. By Thm 3

$$\lim_{x\to x_0} x^n = \left(\lim_{x\to x_0} x\right)^n = x_0^n \quad \forall n\in\mathbb{N}.$$

b. prove that if P is a polynomial, then $\lim_{x\to a} P(x) = P(a)$, $\forall a\in\mathbb{R}$.

By Thm 3 and part a↑ :

$$\lim_{x\to x_0} P(x) = \lim_{x\to x_0} (a_n x^n + \dots + a_0) = a_n x_0^n + \dots + a_0$$
$$= P(x_0).$$

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3.2.4: Prove the following comparison theorems for real functions f and g and $a \in \mathbb{R}$:

a. If $f(x) \geq g(x)$ and $g(x) \to \infty$ as $x \to a$ then $f(x) \to \infty$ as $x \to a$.

If $g(x) \to \infty$ as $x \to a$ then

given $M \in \mathbb{R}$ and choose $\delta > 0$ s.t $0 < |x-a| < \delta$ implies $g(x) > M$.

since $f(x) > g(x) \implies f(x) > M$.

So By def: $g(x) \to \infty$ as $x \to a$ and $f(x) \to \infty$ as $x \to a$.

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b. If $f(x) \leq g(x) \leq h(x)$ and $L := \lim_{x \to \infty} f(x) = \lim_{x \to \infty} h(x)$ then $g(x) \to L$ as $x \to \infty$.

let $\varepsilon > 0$ and choose $M \in \mathbb{R}$ s.t $x > M$ implies:

$|f(x) - L| < \varepsilon$ and $|h(x) - L| < \varepsilon$. By hypotheses and def.

this means: $L - \varepsilon < f(x) \leq g(x) \leq h(x) < L + \varepsilon$.

Thus, $L - \varepsilon < g(x) < L + \varepsilon$

$\to |g(x) - L| < \varepsilon$. $\blacksquare$

3.2.5 : prove ..., suppose that $a \in \mathbb{R}$ and $f : [a, \infty) \to \mathbb{R}$ for some $a \in \mathbb{R}$. Then $f(x) \to L$ as $x \to \infty$ iff $f(x_n) \to L$ for any seq. $x_n \in (a, \infty)$ which converges to ∞ as $n \to \infty$.

suppose $f(x) \to L$ as $x \to \infty$.

let $\varepsilon > 0$ and choose $M \in \mathbb{R}$ s.t $x > M \Rightarrow |f(x) - L| < M$

If $x_n \to \infty$ as $n \to \infty$ choose $N \in \mathbb{N}$ s.t $n \geq N \Rightarrow x_n > M$

Then $|f(x_n) - L| < \varepsilon \quad \forall n \in \mathbb{N}$

i.e, $f(x_n) \to L$ as $n \to \infty$

Conversely, suppose $f(x)$ Does not converge to L as $x \to \infty$.

Then there is an $\varepsilon_0 > 0$ s.t given $n > 0$ there is an $x_n > n$ satisfying

$f(x_n) \geq L + \varepsilon_0$ OR $f(x_n) \leq L - \varepsilon_0$

i.e $|f(x_n) - L| \geq \varepsilon_0$, Thus $x_n \to \infty$ But $f(x_n)$ DN Converge to L as $n \to \infty$

$\boxtimes$

3.2.6 : suppose that $f : [0,1] \to \mathbb{R}$ and $f(a) = \lim_{x \to a} f(x) \quad \forall a \in [0,1]$. prove that $f(q) = 0 \quad \forall q \in \mathbb{Q} \cap [0,1]$ iff $f(x) = 0 \quad \forall x \in [0,1]$.

given $x_0 \in [0,1]$, choose $q_n \in \mathbb{Q} \cap [0,1]$ s.t $q_n \to x_0$ as $n \to \infty$

By Thm , $f(q_n) \to f(x_0)$.

If $f(q) = 0$ for all $q \in \mathbb{Q} \cap [0,1]$ it follows that $f(x_0) = 0$.

Thus, $f(x) = 0 \quad \forall x \in [0,1]$.

converse Trivial.

$\boxtimes$