

Insertion Sort

Parosh Aziz Abdulla

Uppsala University

July 24, 2012

Insertion Sort

INSERTION-SORT(A)

```
1  for  $j \leftarrow 2$  to  $\text{length}[A]$ 
2      do  $\text{key} \leftarrow A[j]$ 
3          ▷ Insert  $A[j]$  into  $A[1..j-1]$ .
4           $i \leftarrow j - 1$ 
5          while  $i > 0$  and  $A[i] > \text{key}$ 
6              do  $A[i+1] \leftarrow A[i]$ 
7               $i \leftarrow i - 1$ 
8           $A[i+1] \leftarrow \text{key}$ 
```

6	3	5	1	2	4
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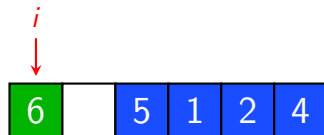


$j = 2$

Insertion Sort

INSERTION-SORT(A)

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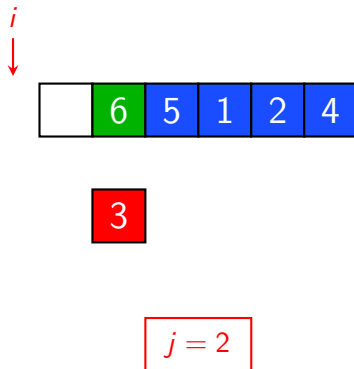


$j = 2$

Insertion Sort

INSERTION-SORT(A)

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1  for  $j \leftarrow 2$  to  $\text{length}[A]$ 
2      do  $\text{key} \leftarrow A[j]$ 
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$j = 2$

Insertion Sort

INSERTION-SORT(A)

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2      do  $\text{key} \leftarrow A[j]$ 
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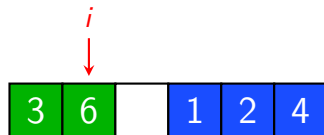


$j = 3$

Insertion Sort

INSERTION-SORT(A)

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2      do  $\text{key} \leftarrow A[j]$ 
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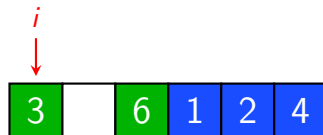
5

$j = 3$

Insertion Sort

INSERTION-SORT(A)

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5

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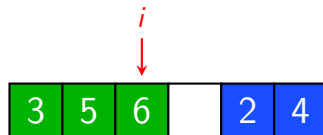


$j = 4$

Insertion Sort

INSERTION-SORT(A)

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1  for  $j \leftarrow 2$  to  $\text{length}[A]$ 
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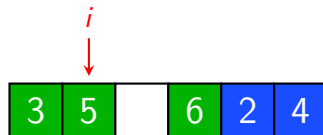
1

$j = 4$

Insertion Sort

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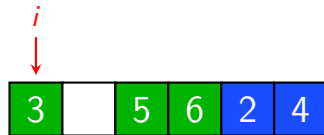
1

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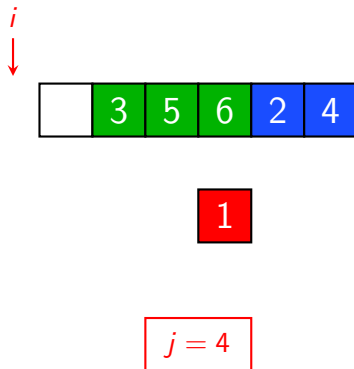
1

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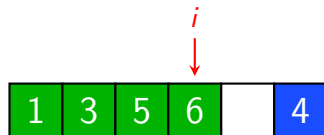


$j = 5$

Insertion Sort

INSERTION-SORT(A)

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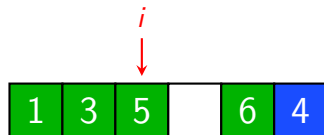
2

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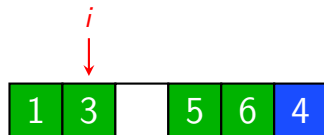
2

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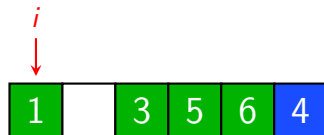
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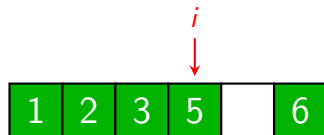
4

$j = 6$

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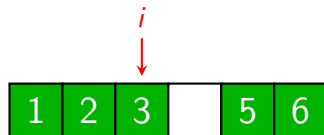
4

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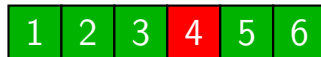
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$j = 6$

Insertion Sort - Cost Analysis

INSERTION-SORT(A)

cost

times

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3           $\triangleright$  Insert  $A[j]$  into  $A[1..j-1]$ .
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Insertion Sort - Cost Analysis

INSERTION-SORT(A)	cost	times
1 for $j \leftarrow 2$ to $\text{length}[A]$	C_1	
2 do $\text{key} \leftarrow A[j]$	C_2	
3 $\triangleright$ Insert $A[j]$ into $A[1..j-1]$	0	
4 $i \leftarrow j - 1$	C_4	
5 while $i > 0$ and $A[i] > \text{key}$	C_5	
6 do $A[i+1] \leftarrow A[i]$	C_6	
7 $i \leftarrow i - 1$	C_7	
8 $A[i+1] \leftarrow \text{key}$	C_8	

Insertion Sort - Cost Analysis

INSERTION-SORT(A)	cost	times
1 for $j \leftarrow 2$ to $\text{length}[A]$	C_1	n
2 do $\text{key} \leftarrow A[j]$	C_2	
3 $\triangleright$ Insert $A[j]$ into $A[1..j-1]$	0	
4 $i \leftarrow j - 1$	C_4	
5 while $i > 0$ and $A[i] > \text{key}$	C_5	
6 do $A[i+1] \leftarrow A[i]$	C_6	
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Insertion Sort - Cost Analysis

INSERTION-SORT(A)	cost	times
1 for $j \leftarrow 2$ to $\text{length}[A]$	C_1	n
2 do $\text{key} \leftarrow A[j]$	C_2	$n - 1$
3 $\triangleright$ Insert $A[j]$ into $A[1..j-1]$	0	
4 $i \leftarrow j - 1$	C_4	
5 while $i > 0$ and $A[i] > \text{key}$	C_5	
6 do $A[i+1] \leftarrow A[i]$	C_6	
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3 $\triangleright$ Insert $A[j]$ into $A[1..j-1]$	0	$n - 1$
4 $i \leftarrow j - 1$	C_4	
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3 $\triangleright$ Insert $A[j]$ into $A[1..j-1]$	0	$n - 1$
4 $i \leftarrow j - 1$	C_4	$n - 1$
5 while $i > 0$ and $A[i] > \text{key}$	C_5	
6 do $A[i+1] \leftarrow A[i]$	C_6	
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3 $\triangleright$ Insert $A[j]$ into $A[1..j-1]$	0	$n - 1$
4 $i \leftarrow j - 1$	C_4	$n - 1$
5 while $i > 0$ and $A[i] > \text{key}$	C_5	$\sum_{j=2}^n t_j$
6 do $A[i+1] \leftarrow A[i]$	C_6	
7 $i \leftarrow i - 1$	C_7	
8 $A[i+1] \leftarrow \text{key}$	C_8	

Insertion Sort - Cost Analysis

INSERTION-SORT(A)	cost	times
1 for $j \leftarrow 2$ to $\text{length}[A]$	C_1	n
2 do $\text{key} \leftarrow A[j]$	C_2	$n - 1$
3 $\triangleright$ Insert $A[j]$ into $A[1..j-1]$	0	$n - 1$
4 $i \leftarrow j - 1$	C_4	$n - 1$
5 while $i > 0$ and $A[i] > \text{key}$	C_5	$\sum_{j=2}^n t_j$
6 do $A[i+1] \leftarrow A[i]$	C_6	$\sum_{j=2}^n (t_j - 1)$
7 $i \leftarrow i - 1$	C_7	
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1 for $j \leftarrow 2$ to $\text{length}[A]$	c_1	n
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4 $i \leftarrow j - 1$	c_4	$n - 1$
5 while $i > 0$ and $A[i] > \text{key}$	c_5	$\sum_{j=2}^n t_j$
6 do $A[i+1] \leftarrow A[i]$	c_6	$\sum_{j=2}^n (t_j - 1)$
7 $i \leftarrow i - 1$	c_7	$\sum_{j=2}^n (t_j - 1)$
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4 $i \leftarrow j - 1$	c_4	$n - 1$
5 while $i > 0$ and $A[i] > \text{key}$	c_5	$\sum_{j=2}^n t_j$
6 do $A[i+1] \leftarrow A[i]$	c_6	$\sum_{j=2}^n (t_j - 1)$
7 $i \leftarrow i - 1$	c_7	$\sum_{j=2}^n (t_j - 1)$
8 $A[i+1] \leftarrow \text{key}$	c_8	$n - 1$

Insertion Sort - Best Case

INSERTION-SORT(A)

```
1  for  $j \leftarrow 2$  to  $\text{length}[A]$ 
2      do  $\text{key} \leftarrow A[j]$ 
3          ▷ Insert  $A[j]$  into  $A[1..j-1]$ .
4           $i \leftarrow j - 1$ 
5          while  $i > 0$  and  $A[i] > \text{key}$ 
6              do  $A[i+1] \leftarrow A[i]$ 
7                   $i \leftarrow i - 1$ 
8           $A[i+1] \leftarrow \text{key}$ 
```

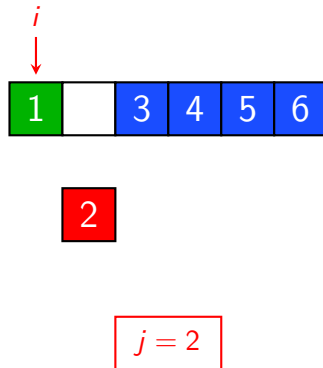


$j = 2$

Insertion Sort - Best Case

INSERTION-SORT(A)

```
1  for  $j \leftarrow 2$  to  $\text{length}[A]$ 
2      do  $\text{key} \leftarrow A[j]$ 
3          ▷ Insert  $A[j]$  into  $A[1..j-1]$ .
4           $i \leftarrow j - 1$ 
5          while  $i > 0$  and  $A[i] > \text{key}$ 
6              do  $A[i+1] \leftarrow A[i]$ 
7                   $i \leftarrow i - 1$ 
8           $A[i+1] \leftarrow \text{key}$ 
```



Insertion Sort - Best Case

INSERTION-SORT(A)

```
1  for  $j \leftarrow 2$  to  $\text{length}[A]$ 
2      do  $\text{key} \leftarrow A[j]$ 
3          ▷ Insert  $A[j]$  into  $A[1..j-1]$ .
4           $i \leftarrow j - 1$ 
5          while  $i > 0$  and  $A[i] > \text{key}$ 
6              do  $A[i+1] \leftarrow A[i]$ 
7                   $i \leftarrow i - 1$ 
8           $A[i+1] \leftarrow \text{key}$ 
```



$j = 2$

$t_j = 1$

Insertion Sort - Best Case

INSERTION-SORT(A)

```
1  for  $j \leftarrow 2$  to  $\text{length}[A]$ 
2      do  $\text{key} \leftarrow A[j]$ 
3          ▷ Insert  $A[j]$  into  $A[1..j-1]$ .
4           $i \leftarrow j - 1$ 
5          while  $i > 0$  and  $A[i] > \text{key}$ 
6              do  $A[i+1] \leftarrow A[i]$ 
7                   $i \leftarrow i - 1$ 
8           $A[i+1] \leftarrow \text{key}$ 
```

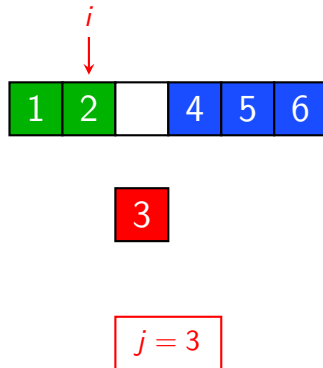


$j = 3$

Insertion Sort - Best Case

INSERTION-SORT(A)

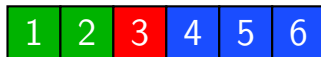
```
1  for  $j \leftarrow 2$  to  $\text{length}[A]$ 
2      do  $\text{key} \leftarrow A[j]$ 
3          ▷ Insert  $A[j]$  into  $A[1..j-1]$ .
4           $i \leftarrow j - 1$ 
5          while  $i > 0$  and  $A[i] > \text{key}$ 
6              do  $A[i+1] \leftarrow A[i]$ 
7                   $i \leftarrow i - 1$ 
8           $A[i+1] \leftarrow \text{key}$ 
```



Insertion Sort - Best Case

INSERTION-SORT(A)

```
1  for  $j \leftarrow 2$  to  $\text{length}[A]$ 
2      do  $\text{key} \leftarrow A[j]$ 
3          ▷ Insert  $A[j]$  into  $A[1..j-1]$ .
4           $i \leftarrow j - 1$ 
5          while  $i > 0$  and  $A[i] > \text{key}$ 
6              do  $A[i+1] \leftarrow A[i]$ 
7                   $i \leftarrow i - 1$ 
8           $A[i+1] \leftarrow \text{key}$ 
```



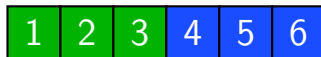
$j = 3$

$t_j = 1$

Insertion Sort - Best Case

INSERTION-SORT(A)

```
1  for  $j \leftarrow 2$  to  $\text{length}[A]$ 
2      do  $\text{key} \leftarrow A[j]$ 
3          ▷ Insert  $A[j]$  into  $A[1..j-1]$ .
4           $i \leftarrow j - 1$ 
5          while  $i > 0$  and  $A[i] > \text{key}$ 
6              do  $A[i+1] \leftarrow A[i]$ 
7                   $i \leftarrow i - 1$ 
8           $A[i+1] \leftarrow \text{key}$ 
```

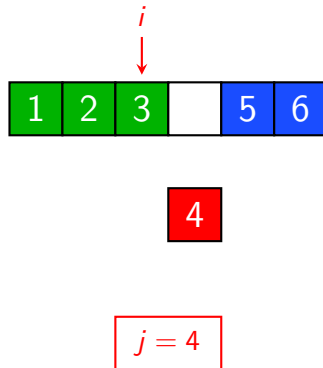


$j = 4$

Insertion Sort - Best Case

INSERTION-SORT(A)

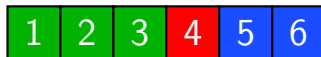
```
1  for  $j \leftarrow 2$  to  $\text{length}[A]$ 
2      do  $\text{key} \leftarrow A[j]$ 
3          ▷ Insert  $A[j]$  into  $A[1..j-1]$ .
4           $i \leftarrow j - 1$ 
5          while  $i > 0$  and  $A[i] > \text{key}$ 
6              do  $A[i+1] \leftarrow A[i]$ 
7                   $i \leftarrow i - 1$ 
8           $A[i+1] \leftarrow \text{key}$ 
```



Insertion Sort - Best Case

INSERTION-SORT(A)

```
1  for  $j \leftarrow 2$  to  $\text{length}[A]$ 
2      do  $\text{key} \leftarrow A[j]$ 
3          ▷ Insert  $A[j]$  into  $A[1..j-1]$ .
4           $i \leftarrow j - 1$ 
5          while  $i > 0$  and  $A[i] > \text{key}$ 
6              do  $A[i+1] \leftarrow A[i]$ 
7                   $i \leftarrow i - 1$ 
8           $A[i+1] \leftarrow \text{key}$ 
```



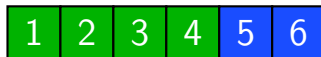
$j = 4$

$t_j = 1$

Insertion Sort - Best Case

INSERTION-SORT(A)

```
1  for  $j \leftarrow 2$  to  $\text{length}[A]$ 
2      do  $\text{key} \leftarrow A[j]$ 
3          ▷ Insert  $A[j]$  into  $A[1..j-1]$ .
4           $i \leftarrow j - 1$ 
5          while  $i > 0$  and  $A[i] > \text{key}$ 
6              do  $A[i+1] \leftarrow A[i]$ 
7                   $i \leftarrow i - 1$ 
8           $A[i+1] \leftarrow \text{key}$ 
```

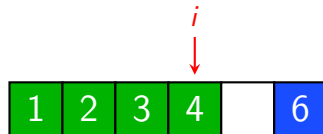


$j = 5$

Insertion Sort - Best Case

INSERTION-SORT(A)

```
1  for  $j \leftarrow 2$  to  $\text{length}[A]$ 
2      do  $\text{key} \leftarrow A[j]$ 
3          ▷ Insert  $A[j]$  into  $A[1..j-1]$ .
4           $i \leftarrow j - 1$ 
5          while  $i > 0$  and  $A[i] > \text{key}$ 
6              do  $A[i+1] \leftarrow A[i]$ 
7                   $i \leftarrow i - 1$ 
8           $A[i+1] \leftarrow \text{key}$ 
```



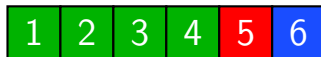
5

$j = 5$

Insertion Sort - Best Case

INSERTION-SORT(A)

```
1  for  $j \leftarrow 2$  to  $\text{length}[A]$ 
2      do  $\text{key} \leftarrow A[j]$ 
3          ▷ Insert  $A[j]$  into  $A[1..j-1]$ .
4           $i \leftarrow j - 1$ 
5          while  $i > 0$  and  $A[i] > \text{key}$ 
6              do  $A[i+1] \leftarrow A[i]$ 
7                   $i \leftarrow i - 1$ 
8           $A[i+1] \leftarrow \text{key}$ 
```



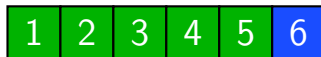
$j = 5$

$t_j = 1$

Insertion Sort - Best Case

INSERTION-SORT(A)

```
1  for  $j \leftarrow 2$  to  $\text{length}[A]$ 
2      do  $\text{key} \leftarrow A[j]$ 
3          ▷ Insert  $A[j]$  into  $A[1..j-1]$ .
4           $i \leftarrow j - 1$ 
5          while  $i > 0$  and  $A[i] > \text{key}$ 
6              do  $A[i+1] \leftarrow A[i]$ 
7                   $i \leftarrow i - 1$ 
8           $A[i+1] \leftarrow \text{key}$ 
```

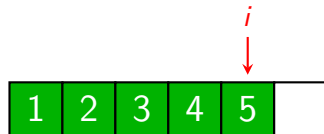


$j = 6$

Insertion Sort - Best Case

INSERTION-SORT(A)

```
1  for  $j \leftarrow 2$  to  $\text{length}[A]$ 
2      do  $\text{key} \leftarrow A[j]$ 
3          ▷ Insert  $A[j]$  into  $A[1..j-1]$ .
4           $i \leftarrow j - 1$ 
5          while  $i > 0$  and  $A[i] > \text{key}$ 
6              do  $A[i+1] \leftarrow A[i]$ 
7                   $i \leftarrow i - 1$ 
8           $A[i+1] \leftarrow \text{key}$ 
```



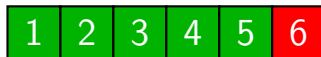
6

$j = 6$

Insertion Sort - Best Case

INSERTION-SORT(A)

```
1  for  $j \leftarrow 2$  to  $\text{length}[A]$ 
2      do  $\text{key} \leftarrow A[j]$ 
3          ▷ Insert  $A[j]$  into  $A[1..j-1]$ .
4           $i \leftarrow j - 1$ 
5          while  $i > 0$  and  $A[i] > \text{key}$ 
6              do  $A[i+1] \leftarrow A[i]$ 
7                   $i \leftarrow i - 1$ 
8           $A[i+1] \leftarrow \text{key}$ 
```



$j = 6$

$t_j = 1$

Insertion Sort - Worst Case

INSERTION-SORT(A)

```
1  for  $j \leftarrow 2$  to  $\text{length}[A]$ 
2      do  $\text{key} \leftarrow A[j]$ 
3          ▷ Insert  $A[j]$  into  $A[1..j-1]$ .
4           $i \leftarrow j - 1$ 
5          while  $i > 0$  and  $A[i] > \text{key}$ 
6              do  $A[i+1] \leftarrow A[i]$ 
7                   $i \leftarrow i - 1$ 
8           $A[i+1] \leftarrow \text{key}$ 
```

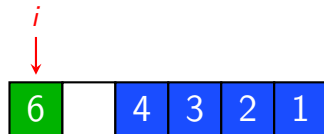


$j = 2$

Insertion Sort - Worst Case

INSERTION-SORT(A)

```
1  for  $j \leftarrow 2$  to  $\text{length}[A]$ 
2      do  $\text{key} \leftarrow A[j]$ 
3          ▷ Insert  $A[j]$  into  $A[1..j-1]$ .
4           $i \leftarrow j - 1$ 
5          while  $i > 0$  and  $A[i] > \text{key}$ 
6              do  $A[i+1] \leftarrow A[i]$ 
7                   $i \leftarrow i - 1$ 
8           $A[i+1] \leftarrow \text{key}$ 
```

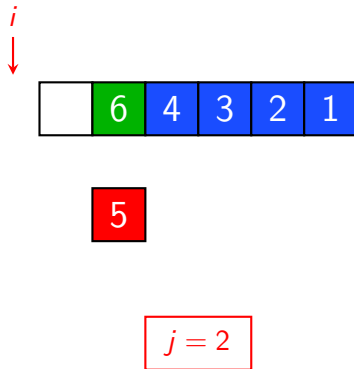


$j = 2$

Insertion Sort - Worst Case

INSERTION-SORT(A)

```
1  for  $j \leftarrow 2$  to  $\text{length}[A]$ 
2      do  $\text{key} \leftarrow A[j]$ 
3          ▷ Insert  $A[j]$  into  $A[1..j-1]$ .
4           $i \leftarrow j - 1$ 
5          while  $i > 0$  and  $A[i] > \text{key}$ 
6              do  $A[i+1] \leftarrow A[i]$ 
7                   $i \leftarrow i - 1$ 
8           $A[i+1] \leftarrow \text{key}$ 
```



Insertion Sort - Worst Case

INSERTION-SORT(A)

```
1  for  $j \leftarrow 2$  to  $\text{length}[A]$ 
2      do  $\text{key} \leftarrow A[j]$ 
3          ▷ Insert  $A[j]$  into  $A[1..j-1]$ .
4           $i \leftarrow j - 1$ 
5          while  $i > 0$  and  $A[i] > \text{key}$ 
6              do  $A[i+1] \leftarrow A[i]$ 
7                   $i \leftarrow i - 1$ 
8           $A[i+1] \leftarrow \text{key}$ 
```



$j = 2$

$t_j = 2$

Insertion Sort - Worst Case

INSERTION-SORT(A)

```
1  for  $j \leftarrow 2$  to  $\text{length}[A]$ 
2      do  $\text{key} \leftarrow A[j]$ 
3          ▷ Insert  $A[j]$  into  $A[1..j-1]$ .
4           $i \leftarrow j - 1$ 
5          while  $i > 0$  and  $A[i] > \text{key}$ 
6              do  $A[i+1] \leftarrow A[i]$ 
7                   $i \leftarrow i - 1$ 
8           $A[i+1] \leftarrow \text{key}$ 
```

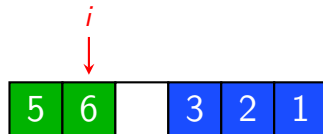


$j = 3$

Insertion Sort - Worst Case

INSERTION-SORT(A)

```
1  for  $j \leftarrow 2$  to  $\text{length}[A]$ 
2      do  $\text{key} \leftarrow A[j]$ 
3          ▷ Insert  $A[j]$  into  $A[1..j-1]$ .
4           $i \leftarrow j - 1$ 
5          while  $i > 0$  and  $A[i] > \text{key}$ 
6              do  $A[i+1] \leftarrow A[i]$ 
7                   $i \leftarrow i - 1$ 
8           $A[i+1] \leftarrow \text{key}$ 
```



4

$j = 3$

Insertion Sort - Worst Case

INSERTION-SORT(A)

```
1  for  $j \leftarrow 2$  to  $\text{length}[A]$ 
2      do  $\text{key} \leftarrow A[j]$ 
3          ▷ Insert  $A[j]$  into  $A[1..j-1]$ .
4           $i \leftarrow j - 1$ 
5          while  $i > 0$  and  $A[i] > \text{key}$ 
6              do  $A[i+1] \leftarrow A[i]$ 
7                   $i \leftarrow i - 1$ 
8           $A[i+1] \leftarrow \text{key}$ 
```



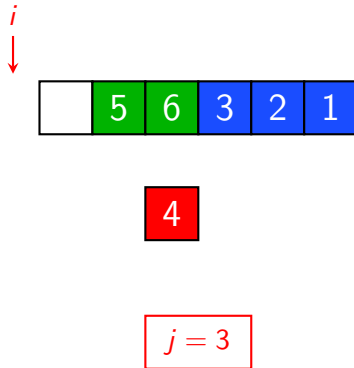
4

$j = 3$

Insertion Sort - Worst Case

INSERTION-SORT(A)

```
1  for  $j \leftarrow 2$  to  $\text{length}[A]$ 
2      do  $\text{key} \leftarrow A[j]$ 
3          ▷ Insert  $A[j]$  into  $A[1..j-1]$ .
4           $i \leftarrow j - 1$ 
5          while  $i > 0$  and  $A[i] > \text{key}$ 
6              do  $A[i+1] \leftarrow A[i]$ 
7                   $i \leftarrow i - 1$ 
8           $A[i+1] \leftarrow \text{key}$ 
```



Insertion Sort - Worst Case

INSERTION-SORT(A)

```
1  for  $j \leftarrow 2$  to  $\text{length}[A]$ 
2      do  $\text{key} \leftarrow A[j]$ 
3          ▷ Insert  $A[j]$  into  $A[1..j-1]$ .
4           $i \leftarrow j - 1$ 
5          while  $i > 0$  and  $A[i] > \text{key}$ 
6              do  $A[i+1] \leftarrow A[i]$ 
7                   $i \leftarrow i - 1$ 
8           $A[i+1] \leftarrow \text{key}$ 
```



$j = 3$

$t_j = 3$

Insertion Sort - Worst Case

INSERTION-SORT(A)

```
1  for  $j \leftarrow 2$  to  $\text{length}[A]$ 
2      do  $\text{key} \leftarrow A[j]$ 
3          ▷ Insert  $A[j]$  into  $A[1..j-1]$ .
4           $i \leftarrow j - 1$ 
5          while  $i > 0$  and  $A[i] > \text{key}$ 
6              do  $A[i+1] \leftarrow A[i]$ 
7                   $i \leftarrow i - 1$ 
8           $A[i+1] \leftarrow \text{key}$ 
```

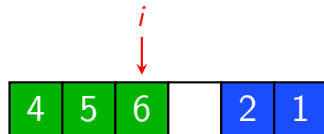


$j = 4$

Insertion Sort - Worst Case

INSERTION-SORT(A)

```
1  for  $j \leftarrow 2$  to  $\text{length}[A]$ 
2      do  $\text{key} \leftarrow A[j]$ 
3          ▷ Insert  $A[j]$  into  $A[1..j-1]$ .
4           $i \leftarrow j - 1$ 
5          while  $i > 0$  and  $A[i] > \text{key}$ 
6              do  $A[i+1] \leftarrow A[i]$ 
7                   $i \leftarrow i - 1$ 
8           $A[i+1] \leftarrow \text{key}$ 
```



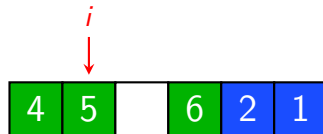
3

$j = 4$

Insertion Sort - Worst Case

INSERTION-SORT(A)

```
1  for  $j \leftarrow 2$  to  $\text{length}[A]$ 
2      do  $\text{key} \leftarrow A[j]$ 
3          ▷ Insert  $A[j]$  into  $A[1..j-1]$ .
4           $i \leftarrow j - 1$ 
5          while  $i > 0$  and  $A[i] > \text{key}$ 
6              do  $A[i+1] \leftarrow A[i]$ 
7                   $i \leftarrow i - 1$ 
8           $A[i+1] \leftarrow \text{key}$ 
```



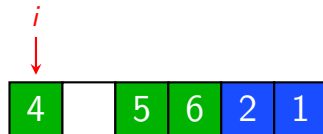
3

$j = 4$

Insertion Sort - Worst Case

INSERTION-SORT(A)

```
1  for  $j \leftarrow 2$  to  $\text{length}[A]$ 
2      do  $\text{key} \leftarrow A[j]$ 
3          ▷ Insert  $A[j]$  into  $A[1..j-1]$ .
4           $i \leftarrow j - 1$ 
5          while  $i > 0$  and  $A[i] > \text{key}$ 
6              do  $A[i+1] \leftarrow A[i]$ 
7                   $i \leftarrow i - 1$ 
8           $A[i+1] \leftarrow \text{key}$ 
```



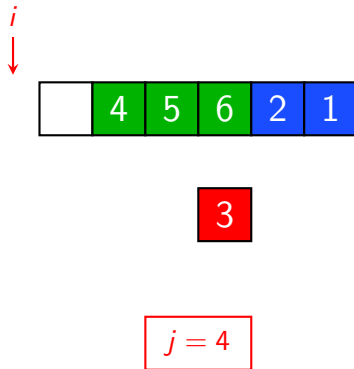
3

$j = 4$

Insertion Sort - Worst Case

INSERTION-SORT(A)

```
1  for  $j \leftarrow 2$  to  $\text{length}[A]$ 
2      do  $\text{key} \leftarrow A[j]$ 
3          ▷ Insert  $A[j]$  into  $A[1..j-1]$ .
4           $i \leftarrow j - 1$ 
5          while  $i > 0$  and  $A[i] > \text{key}$ 
6              do  $A[i+1] \leftarrow A[i]$ 
7                   $i \leftarrow i - 1$ 
8           $A[i+1] \leftarrow \text{key}$ 
```



Insertion Sort - Worst Case

INSERTION-SORT(A)

```
1  for  $j \leftarrow 2$  to  $\text{length}[A]$ 
2      do  $\text{key} \leftarrow A[j]$ 
3          ▷ Insert  $A[j]$  into  $A[1..j-1]$ .
4           $i \leftarrow j - 1$ 
5          while  $i > 0$  and  $A[i] > \text{key}$ 
6              do  $A[i+1] \leftarrow A[i]$ 
7                   $i \leftarrow i - 1$ 
8           $A[i+1] \leftarrow \text{key}$ 
```



$j = 4$

$t_j = 4$

Insertion Sort - Worst Case

INSERTION-SORT(A)

```
1  for  $j \leftarrow 2$  to  $\text{length}[A]$ 
2      do  $\text{key} \leftarrow A[j]$ 
3          ▷ Insert  $A[j]$  into  $A[1..j-1]$ .
4           $i \leftarrow j - 1$ 
5          while  $i > 0$  and  $A[i] > \text{key}$ 
6              do  $A[i+1] \leftarrow A[i]$ 
7                   $i \leftarrow i - 1$ 
8           $A[i+1] \leftarrow \text{key}$ 
```

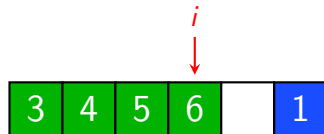


$j = 5$

Insertion Sort - Worst Case

INSERTION-SORT(A)

```
1  for  $j \leftarrow 2$  to  $\text{length}[A]$ 
2      do  $\text{key} \leftarrow A[j]$ 
3          ▷ Insert  $A[j]$  into  $A[1..j-1]$ .
4           $i \leftarrow j - 1$ 
5          while  $i > 0$  and  $A[i] > \text{key}$ 
6              do  $A[i+1] \leftarrow A[i]$ 
7                   $i \leftarrow i - 1$ 
8           $A[i+1] \leftarrow \text{key}$ 
```



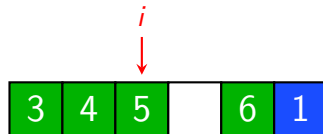
2

$j = 5$

Insertion Sort - Worst Case

INSERTION-SORT(A)

```
1  for  $j \leftarrow 2$  to  $\text{length}[A]$ 
2      do  $\text{key} \leftarrow A[j]$ 
3          ▷ Insert  $A[j]$  into  $A[1..j-1]$ .
4           $i \leftarrow j - 1$ 
5          while  $i > 0$  and  $A[i] > \text{key}$ 
6              do  $A[i+1] \leftarrow A[i]$ 
7                   $i \leftarrow i - 1$ 
8           $A[i+1] \leftarrow \text{key}$ 
```



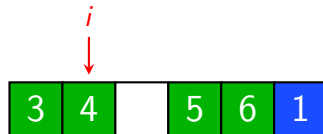
2

$j = 5$

Insertion Sort - Worst Case

INSERTION-SORT(A)

```
1  for  $j \leftarrow 2$  to  $\text{length}[A]$ 
2      do  $\text{key} \leftarrow A[j]$ 
3          ▷ Insert  $A[j]$  into  $A[1..j-1]$ .
4           $i \leftarrow j - 1$ 
5          while  $i > 0$  and  $A[i] > \text{key}$ 
6              do  $A[i+1] \leftarrow A[i]$ 
7                   $i \leftarrow i - 1$ 
8           $A[i+1] \leftarrow \text{key}$ 
```



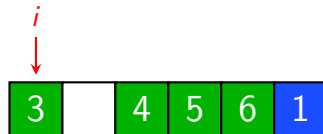
2

$j = 5$

Insertion Sort - Worst Case

INSERTION-SORT(A)

```
1  for  $j \leftarrow 2$  to  $\text{length}[A]$ 
2      do  $\text{key} \leftarrow A[j]$ 
3          ▷ Insert  $A[j]$  into  $A[1..j-1]$ .
4           $i \leftarrow j - 1$ 
5          while  $i > 0$  and  $A[i] > \text{key}$ 
6              do  $A[i+1] \leftarrow A[i]$ 
7                   $i \leftarrow i - 1$ 
8           $A[i+1] \leftarrow \text{key}$ 
```



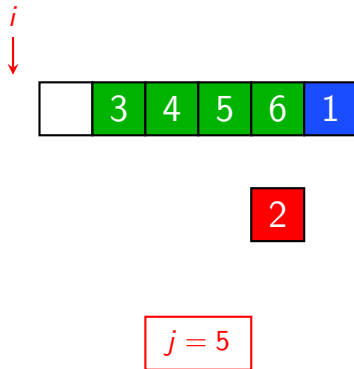
2

$j = 5$

Insertion Sort - Worst Case

INSERTION-SORT(A)

```
1  for  $j \leftarrow 2$  to  $\text{length}[A]$ 
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Insertion Sort - Worst Case

INSERTION-SORT(A)

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6              do  $A[i + 1] \leftarrow A[i]$ 
7                   $i \leftarrow i - 1$ 
8           $A[i + 1] \leftarrow \text{key}$ 
```



$j = 5$

$t_j = 5$

Insertion Sort - Worst Case

INSERTION-SORT(A)

```
1  for  $j \leftarrow 2$  to  $\text{length}[A]$ 
2      do  $\text{key} \leftarrow A[j]$ 
3          ▷ Insert  $A[j]$  into  $A[1..j-1]$ .
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5          while  $i > 0$  and  $A[i] > \text{key}$ 
6              do  $A[i+1] \leftarrow A[i]$ 
7                   $i \leftarrow i - 1$ 
8           $A[i+1] \leftarrow \text{key}$ 
```



$j = 6$

Insertion Sort - Worst Case

INSERTION-SORT(A)

```
1  for  $j \leftarrow 2$  to  $\text{length}[A]$ 
2      do  $\text{key} \leftarrow A[j]$ 
3          ▷ Insert  $A[j]$  into  $A[1..j-1]$ .
4           $i \leftarrow j - 1$ 
5          while  $i > 0$  and  $A[i] > \text{key}$ 
6              do  $A[i+1] \leftarrow A[i]$ 
7                   $i \leftarrow i - 1$ 
8           $A[i+1] \leftarrow \text{key}$ 
```



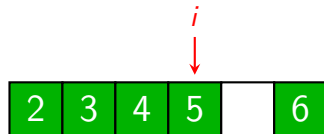
1

$j = 6$

Insertion Sort - Worst Case

INSERTION-SORT(A)

```
1  for  $j \leftarrow 2$  to  $\text{length}[A]$ 
2      do  $\text{key} \leftarrow A[j]$ 
3          ▷ Insert  $A[j]$  into  $A[1..j-1]$ .
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5          while  $i > 0$  and  $A[i] > \text{key}$ 
6              do  $A[i+1] \leftarrow A[i]$ 
7                   $i \leftarrow i - 1$ 
8           $A[i+1] \leftarrow \text{key}$ 
```



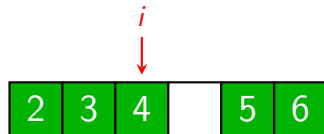
1

$j = 6$

Insertion Sort - Worst Case

INSERTION-SORT(A)

```
1  for  $j \leftarrow 2$  to  $\text{length}[A]$ 
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5          while  $i > 0$  and  $A[i] > \text{key}$ 
6              do  $A[i+1] \leftarrow A[i]$ 
7                   $i \leftarrow i - 1$ 
8           $A[i+1] \leftarrow \text{key}$ 
```



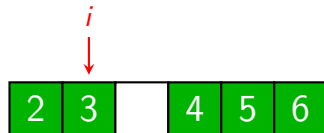
1

$j = 6$

Insertion Sort - Worst Case

INSERTION-SORT(A)

```
1  for  $j \leftarrow 2$  to  $\text{length}[A]$ 
2      do  $\text{key} \leftarrow A[j]$ 
3          ▷ Insert  $A[j]$  into  $A[1..j-1]$ .
4           $i \leftarrow j - 1$ 
5          while  $i > 0$  and  $A[i] > \text{key}$ 
6              do  $A[i+1] \leftarrow A[i]$ 
7                   $i \leftarrow i - 1$ 
8           $A[i+1] \leftarrow \text{key}$ 
```



1

$j = 6$

Insertion Sort - Worst Case

INSERTION-SORT(A)

```
1  for  $j \leftarrow 2$  to  $\text{length}[A]$ 
2      do  $\text{key} \leftarrow A[j]$ 
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4           $i \leftarrow j - 1$ 
5          while  $i > 0$  and  $A[i] > \text{key}$ 
6              do  $A[i+1] \leftarrow A[i]$ 
7                   $i \leftarrow i - 1$ 
8           $A[i+1] \leftarrow \text{key}$ 
```



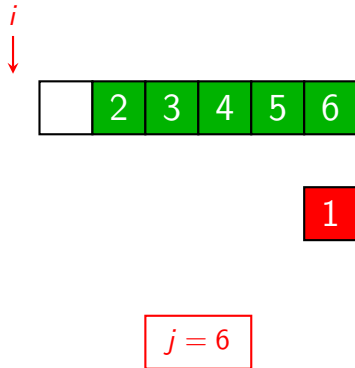
1

$j = 6$

Insertion Sort - Worst Case

INSERTION-SORT(A)

```
1  for  $j \leftarrow 2$  to  $\text{length}[A]$ 
2      do  $\text{key} \leftarrow A[j]$ 
3          ▷ Insert  $A[j]$  into  $A[1..j-1]$ .
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5          while  $i > 0$  and  $A[i] > \text{key}$ 
6              do  $A[i+1] \leftarrow A[i]$ 
7                   $i \leftarrow i - 1$ 
8           $A[i+1] \leftarrow \text{key}$ 
```



Insertion Sort - Worst Case

INSERTION-SORT(A)

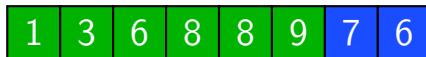
```
1  for  $j \leftarrow 2$  to  $\text{length}[A]$ 
2      do  $\text{key} \leftarrow A[j]$ 
3          ▷ Insert  $A[j]$  into  $A[1..j-1]$ .
4           $i \leftarrow j - 1$ 
5          while  $i > 0$  and  $A[i] > \text{key}$ 
6              do  $A[i+1] \leftarrow A[i]$ 
7                   $i \leftarrow i - 1$ 
8           $A[i+1] \leftarrow \text{key}$ 
```



$j = 6$

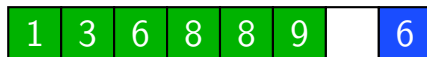
$t_j = 6$

Insertion Sort - Average Case



$j = 7$

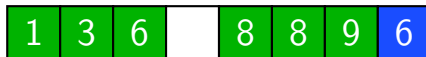
Insertion Sort - Average Case



7

$j = 7$

Insertion Sort - Average Case



7

$j = 7$

Insertion Sort - Average Case



$$j = 7$$

$$t_j \approx \frac{j}{2}$$