11.2 Calculus with parametric Curves

Tangents and Areas:

If $x = f(t)$ and $y = g(t)$ are differentiable fun's of t, then by Chain rule we have that

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt} ,$$

so if $\frac{dx}{dt} \neq 0$, we get that

$$\boxed{\frac{dy}{dx} = \frac{dy/dt}{dx/dt}}$$

Similarly, we can prove that

$$\frac{d^2y}{dx^2} = \frac{(dy'/dt)}{(dx/dt)}$$

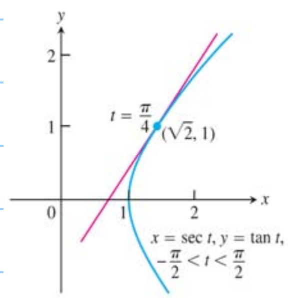
لاحظ هنا أن $y' = \frac{dy}{dx}$ وتكون دالة في t .

Examples: 1) a) Find the tangent to the curve

$x = \sec t$, $y = \tan t$ $-\pi/2 < t < \pi/2$

at the point $(\sqrt{2}, 1)$ when $t = \pi/4$.

sol: $\frac{dy}{dx} = \frac{(dy/dt)}{(dx/dt)} = \frac{\sec^2 t}{\sec t \tan t} = \frac{\sec t}{\tan t}$



$= \csc t$

$\therefore$ at $t = \pi/4$, $\frac{dy}{dx} = \sqrt{2}$

Eqn of the Tangent: $m_T = \sqrt{2}$, $P(\sqrt{2}, 1)$, so

$$y = y_0 + m_T(x - x_0) = 1 + \sqrt{2}(x - \sqrt{2}) = \boxed{\sqrt{2}x - 1}$$

b) Find $\frac{d^2y}{dx^2}$ at $t = \pi/4$.

sol: $y'(t) = \frac{dy}{dx} = \csc t \implies$

$$\frac{d^2y}{dx^2} = \frac{(dy'/dt)}{(dx/dt)} = \frac{-\csc t \cot t}{\sec t \tan t} = -\cot^3 t$$

at $t = \pi/4$, $\frac{d^2y}{dx^2} = -\cot^3 \pi/4 = \boxed{-1}$

2) Find d^2y/dx^2 as a fun of t if

$$x = t - t^2, \quad y = t - t^3, \quad -\infty < t < \infty$$

sol:

$$y'(t) = \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{1-3t^2}{1-2t}$$

Now

$$\frac{dy'}{dt} = \frac{(1-2t)(-6t) - (1-3t^2)\cdot(-2)}{(1-2t)^2} = \frac{2-6t+6t^2}{(1-2t)^2}$$

$$\therefore \frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt} = \frac{(2-6t+6t^2)/(1-2t)^2}{(1-2t)} = \frac{2-6t+6t^2}{(1-2t)^3}.$$

3) Find the normal to the curve

$$x = 2t^2 + 3, \quad y = t^4 \quad \text{at } t = -1$$

sol: The slope of the tangent to the curve at $t=-1$ is

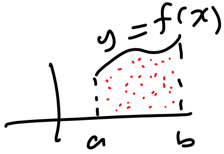
$$m_T = \left.\frac{dy}{dx}\right|_{t=-1} = \left.\frac{dy/dt}{dx/dt}\right|_{t=-1} = \left.\frac{4t^3}{4t}\right|_{t=-1} = 1$$

So the slope of the normal line is $m_\perp = \frac{-1}{m_T} = -1$

Moreover, at $t=-1$, $(x,y) = (5,1)$

$\therefore$ Normal Line:

$$y = 1 + (-1)(x-5) = \boxed{6-x}$$



Area: تذكر أنه إذا كانت $(y = f(x) \geq 0 \,,\, a \leq x \leq b)$ فإن المساحة تحت المنحنى تساوي التكامل

$$Area = \int_a^b y \, dx$$

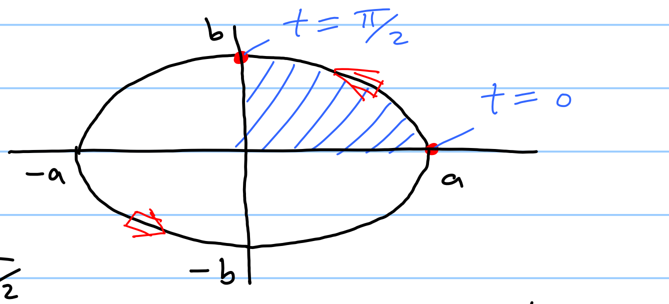
فإذا كانت لدينا معادلات وسيطية ، فإنه يراعى التحويل بدلالة t كالتالي:

Example: Find the area enclosed by the ellipse

$$x = a\cos t, \quad y = b\sin t, \qquad 0 \leq t \leq 2\pi$$

<u>Sol:</u> Clearly,

$$Area = 4\int_0^a y \, dx$$



but $x = 0$ when $y = b$ at $t = \pi/2$ and $x = a$ at $t = 0$. Moreover, $y = b\sin t$ and

$$dx = a \cdot -\sin t \, dt = -a\sin t \, dt \implies$$

$$Area = 4 \cdot \int_{\pi/2}^{0} b\sin t \cdot (-a\sin t) \, dt = 4ab\int_0^{\pi/2} \sin^2 t \, dt$$

$$= \frac{4ab}{2}\int_0^{\pi/2} 1 - \cos 2t \, dt = 2ab\left[t - \frac{\sin 2t}{2}\right]_0^{\pi/2}$$

$$= 2ab\left[\left(\frac{\pi}{2} - 0\right) - (0 - 0)\right] = \boxed{ab\pi}$$

Length of a Parametrically Defined Curve:

DEFINITION If a curve C is defined parametrically by $x = f(t)$ and $y = g(t)$, $a \leq t \leq b$, where f' and g' are continuous and not simultaneously zero on $[a, b]$, and C is traversed exactly once as t increases from $t = a$ to $t = b$, then **the length of C** is the definite integral

$$L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2} \, dt.$$

Examples: 1) Prove that the length of the circle of radius r is $2\pi r$.

Sol: Using the parametrization

$$x = r\sin t \quad , \quad y = r\cos t \quad , \qquad 0 \leq t \leq 2\pi$$

or any other parametrization

we get that (هناك عدد لانهائي منهم، جرب معادلة أخرى لتحصل على نفس النتيجة)

$$\frac{dx}{dt} = r\cos t \;, \qquad \frac{dy}{dt} = -r\sin t$$ which are continuous

and not simultaneously zero, so

$$L = \int_0^{2\pi} \sqrt{r^2\cos^2 t + r^2\sin^2 t}\; dt = \int_0^{2\pi} r\, dt = \boxed{2\pi r}$$

2) Find the length of the curve

$x = 8\cos t + 8t\sin t \;, \quad y = 8\sin t - 8t\cos t \;, \quad 0 \leq t \leq \pi/2$

Sol: $$\frac{dx}{dt} = -8\sin t + 8\sin t + 8t\cos t = 8t\cos t$$

$$\frac{dy}{dt} = 8\cos t - 8\cos t + 8t\sin t = 8t\sin t$$

$$\therefore L = \int_0^{\pi/2} \sqrt{64t^2\cos^2 t + 64t^2\sin^2 t} = \int_0^{\pi/2} 8t\, dt = 4t^2\Big]_0^{\pi/2}$$

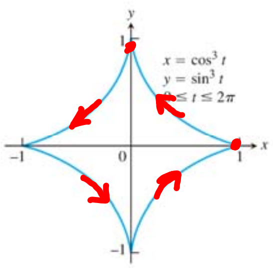
$$= 4\left[\frac{\pi^2}{4} - 0\right] = \boxed{\pi^2}$$

EXAMPLE 5 Find the length of the astroid (Figure 11.13)

$$x = \cos^3 t, \qquad y = \sin^3 t, \qquad 0 \leq t \leq 2\pi.$$

Sol: $$\frac{dx}{dt} = -3\cos^2 t\,\sin t$$

$$\frac{dy}{dt} = 3\sin^2 t\,\cos t$$



لاحظ أنه حدود التكامل هي في اتجاه تزايد t وليس في اتجاه تزايد x (كما في المساحات)

$$\therefore L = 4\int_0^{\pi/2} \sqrt{9\cos^4 t \sin^2 t + 9 \sin^4 t \cos^2 t}\, dt$$

$$= 4\int_0^{\pi/2} \sqrt{9 \sin^2 t \cos^2 t}\, dt = 4\int_0^{\pi/2} 3\,|\sin t \cos t|\, dt$$

$$= \frac{12}{2}\int_0^{\pi/2} \sin 2t \, dt \qquad (\sin 2t = 2\sin t \cos t \text{ , and } \sin 2t > 0 \text{ in the interval } [0, \tfrac{\pi}{2}])$$

$$= 6 \cdot -\frac{\cos 2t}{2}\Big]_0^{\pi/2} = -3\,(-1-1) = \boxed{6}$$

ملحوظة: سابقاً تم دراسة طول المنحنى وكان قانونه باستخدام المعادلات العادية

$$L = \int_a^b \sqrt{1 + \left(\frac{df}{dx}\right)^2}\, dx$$

لاحظ أن هذا القانون هو حالة خاصة من القانون الجديد تكون فيه المعادلات البارامترية

$$x = t, \qquad y = f(t), \qquad a < t < b$$

$$\Rightarrow \frac{dx}{dt} = 1,\ dx = dt, \qquad \frac{dy}{dt} = f'(t) \quad \text{and} \quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{dy}{dt}$$

$$\Rightarrow L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

Area of Surfaces of Revolution

Area of Surface of Revolution for Parametrized Curves

If a smooth curve $x = f(t), y = g(t), a \le t \le b$, is traversed exactly once as t increases from a to b, then the areas of the surfaces generated by revolving the curve about the coordinate axes are as follows.

1. Revolution about the x-axis ($y \ge 0$):

$$S = \int_a^b 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \qquad (5)$$

2. Revolution about the y-axis ($x \ge 0$):

$$S = \int_a^b 2\pi x \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \qquad (6)$$

Example: Find the area of the surface generated by revolving the circle

$$x = \cos t \,, \quad y = 1 + \sin t \,, \qquad 0 \le t \le 2\pi$$

about x-axis.

(نلاحظ أن المعادلة البارامترية هي معادلة دائرة مركزها (0,1) ونصف قطرها 1)

Sol: $\left(\frac{dx}{dt}\right)^2 = (-\sin t)^2 = \sin^2 t$

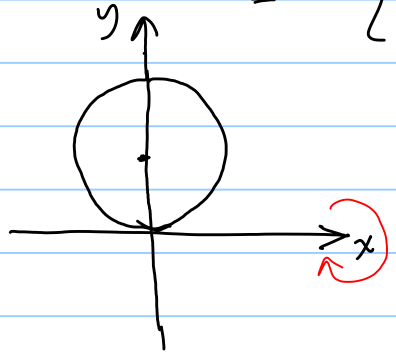
$(dy/dt)^2 = \cos^2 t$

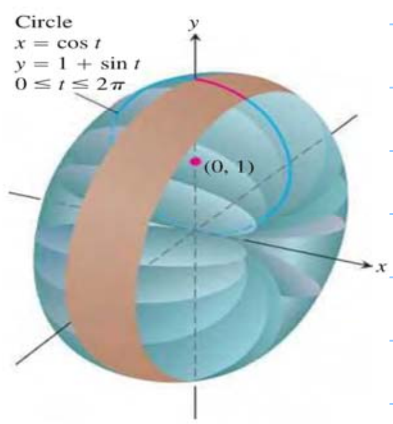
$$\therefore S = \int 2\pi y \sqrt{(dx/dt)^2 + (dy/dt)^2}\, dt$$

$$2\pi \int_0^{2\pi} (1 + \sin t) \sqrt{\sin^2 t + \cos^2 t}\; dt$$

$$= 2\pi \int_0^{2\pi} 1 + \sin t \, dt = 2\pi \left[t - \cos t \right]_0^{2\pi}$$

$$= 2\pi \left[(2\pi - 1) - (0 - 1) \right] = \boxed{4\pi^2}$$





ملاحظه

1) Find the tangent to the curve

$$x = \frac{1}{t+1}, \quad y = \frac{t}{t-1}, \quad \text{at } t_0 = 2.$$

<u>Sol:</u> At $t_0 = 2$, $(x_0, y_0) = (1/3, 2)$.

Now, $m_T = \left.\frac{dy}{dx}\right|_{t=2} = \frac{dy/dt}{dx/dt} = \frac{(t-1-t)/(t-1)^2}{-1/(t+1)^2} = \left.\frac{(t+1)^2}{(t-1)^2}\right|_{t=2} = 9$

<u>Tangent line:</u> $y = y_0 + m_T(x - x_0)$

$$= 2 + 9(x - 1/3) \Rightarrow \boxed{y = 9x - 1}$$

Note that

$$\frac{d^2y}{dx^2} = \frac{\left(\frac{dy'}{dt}\right)}{\left(dx/dt\right)} = \frac{\left[(t-1)^2 \cdot 2(t+1) - (t+1)^2 \cdot 2(t-1)\right]/(t-1)^4}{-1/(t+1)^2}$$

$$= \frac{4(t+1)^3}{(t-1)^3}$$

2) Find the area of the surface generated by revolving the curve:

$x = \ln(\sec t + \tan t) - \sin t$, $y = \cos t$, $0 \le t \le \pi/3$

about x-axis.

<u>Sol:</u> $\frac{dx}{dt} = \frac{\sin^2 t}{\cos t}$ (Do it) and $\frac{dy}{dt} = -\sin t$

بعد الحسابات

$$\therefore S = 2\pi \int_0^{\pi/3} \cos t \sqrt{\frac{\sin^4 t}{\cos^2 t} + \sin^2 t}\, dt$$

$$= 2\pi \int_0^{\pi/3} \cos t \sqrt{\frac{\sin^4 t + \sin^2 t \cos^2 t}{\cos^2 t}}\, dt = 2\pi \int_0^{\pi/3} \cancel{\cos t}\, \frac{\sin t}{\cancel{\cos t}}\, dt$$

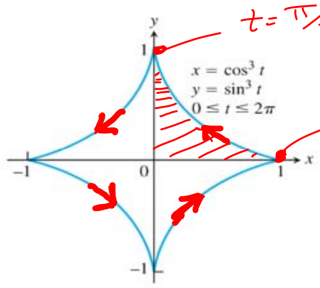
$$= -2\pi \cos t \Big]_0^{\pi/3} = -2\pi\left(\tfrac{1}{2} - 1\right) = \pi$$

in book

EXAMPLE 3 Find the area enclosed by the astroid (Figure 11.13)

$$x = \cos^3 t, \qquad y = \sin^3 t, \qquad 0 \le t \le 2\pi.$$

Solution By symmetry, the enclosed area is 4 times the area beneath the curve in the first quadrant where $0 \le t \le \pi/2$. We can apply the definite integral formula for area studied in Chapter 5, using substitution to express the curve and differential dx in terms of the parameter t. So,



t = π/2

t = 0

$$A = 4\int_0^1 y\,dx = 4\int_{\pi/2}^{0} \sin^3 t \cdot (-3\cos^2 t \sin t)\,dt$$

$$= 4\int_0^{\pi/2} \sin^3 t \cdot 3\cos^2 t \sin t\,dt$$

$$= 12\int_0^{\pi/2} \left(\frac{1-\cos 2t}{2}\right)^2 \left(\frac{1+\cos 2t}{2}\right) dt \qquad \sin^4 t = \left(\frac{1-\cos 2t}{2}\right)^2$$

$$= \frac{3}{2}\int_0^{\pi/2} (1 - 2\cos 2t + \cos^2 2t)(1 + \cos 2t)\,dt \qquad \text{Expand square term.}$$

$$= \frac{3}{2}\int_0^{\pi/2} (1 - \cos 2t - \cos^2 2t + \cos^3 2t)\,dt \qquad \text{Multiply terms.}$$

$$= \frac{3}{2}\left[\int_0^{\pi/2} (1 - \cos 2t)\,dt - \int_0^{\pi/2} \cos^2 2t\,dt + \int_0^{\pi/2} \cos^3 2t\,dt\right]$$

$$= \frac{3}{2}\left[\left(t - \frac{1}{2}\sin 2t\right) - \frac{1}{2}\left(t + \frac{1}{4}\sin 2t\right) + \frac{1}{2}\left(\sin 2t - \frac{1}{3}\sin^3 2t\right)\right]_0^{\pi/2} \qquad \text{Section 8.2, Example 3}$$

$$= \frac{3}{2}\left[\left(\frac{\pi}{2} - 0 - 0 - 0\right) - \frac{1}{2}\left(\frac{\pi}{2} + 0 - 0 - 0\right) + \frac{1}{2}(0 - 0 - 0 + 0)\right] \qquad \text{Evaluate.}$$

$$= \frac{3\pi}{8}.$$

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