9.1.3. Suppose that $\mathbf{x}_k \to \mathbf{0}$ in $\mathbf{R}^n$ as $k \to \infty$ and that $\mathbf{y}_k$ is bounded in $\mathbf{R}^n$.

$\|x_k - 0\| < \frac{\varepsilon}{2M}$

$\|y_k\| \le M, \ M > 0$

a) Prove that $\mathbf{x}_k \cdot \mathbf{y}_k \to 0$ as $k \to \infty$.

b) If $n = 3$, prove that $\mathbf{x}_k \times \mathbf{y}_k \to 0$ as $k \to \infty$.

Pf. (a) $\|x_k \cdot y_k\| \le \|x_k\| \, \|y_k\|$

$\le M \|x_k\| < M \cdot \frac{\varepsilon}{2M} < \varepsilon.$

(b) $\|x_k \times y_k\| \le \|x_k\| \, \|y_k\| \le M \|x_k\| \to 0$ as $k \to \infty$

9.1.6. Let E be a nonempty subset of $\mathbf{R}^n$.

a) Show that a sequence $\mathbf{x}_k \in E$ converges to some point $\mathbf{a} \in E$ if and only if for every set U, which is relatively open in E and contains $\mathbf{a}$, there is an $N \in \mathbf{N}$ such that $\mathbf{x}_k \in U$ for $k \ge N$.

b) Prove that a set $C \subseteq E$ is relatively closed in E if and only if the limit of every sequence $\mathbf{x}_k \in C$ which converges to a point in E satisfies $\lim_{k\to\infty} \mathbf{x}_k \in C$.

C is closed

(b) ($\Rightarrow$) Let C be a relv. closed in E,

i.e, $C = E \cap B$, B is closed.

$x_k \longrightarrow a$, $x_k \in C \subseteq B$ "closed"

$\Rightarrow$ $a \in B$. and $a \in E$ "given".

$\Rightarrow$ $a \in B \cap E = C$.

($\Leftarrow$) $\overset{\text{closed}}{C} \cap E = C$

$\Rightarrow$ C is rel. closed in E.

9.1.6. Let E be a nonempty subset of $\mathbf{R}^n$.

a) Show that a sequence $\mathbf{x}_k \in E$ converges to some point $\mathbf{a} \in E$ if and only if for every set U, which is relatively open in E and contains $\mathbf{a}$, there is an $N \in \mathbf{N}$ such that $\mathbf{x}_k \in U$ for $k \geq N$.

Pf: ($\Rightarrow$) $x_k \in E$, $x_k \to a \in E$.

U rel. open in E, i.e, $U = E \cap V^{\text{open}}$

$a \in U \Rightarrow a \in V \Rightarrow x_k \in V$, for large k

$\Rightarrow x_k \in U$ for large k.

($\Leftarrow$) $x_k \in U$ "relv. open in E", $a \in U$

Since $U = E \cap B_\varepsilon(a)$ is rel. open in E

and $a \in U$, then $x_k \in U \subseteq B_\varepsilon(a)$, for large k, i.e, $\|x_k - a\| < \varepsilon$

$x_k \to a$ as $k \to \infty$. □

9.1.1. Using Definition 9.1i, prove that the following limits exist.

a) $$\mathbf{x}_k = \left(\frac{1}{k}, 1 - \frac{1}{k^2}\right)$$

b) $$\mathbf{x}_k = \left(\frac{k}{k+1}, \frac{\sin k^3}{k}\right)$$

c) $$\mathbf{x}_k = \left(\log(k+1) - \log k, 2^{-k}\right)$$

(c) $x_k = \left(\log\left(\frac{k+1}{k}\right), 2^{-k} \right) \to (0, 0)$

as $k \to \infty$.

Let $\varepsilon > 0$.

$$\|x_k - 0\|^2 = \left\| \left(\log \frac{k+1}{k}, 2^{-k} \right) \right\|^2$$

$$= \log^2\left(\frac{k+1}{k}\right) + 2^{-2k}$$

$$< \left(\frac{\varepsilon}{\sqrt{2}}\right)^2 + \left(\frac{\varepsilon}{\sqrt{2}}\right)^2 = \varepsilon^2;$$

$k > N := \max\{N_1, N_2\}$.

Thus,

$$\left\| \left(\log\left(\frac{k+1}{k}\right), 2^{-k} \right) - (0,0) \right\| < \varepsilon, \quad \forall k > N$$

$x_k \to 0$

$k \geq N \Rightarrow \|x_k\| < \varepsilon$

$\log\left(\frac{k+1}{k}\right) \to 0$

$\left\| \log \frac{k+1}{k} \right\| < \frac{\varepsilon}{\sqrt{2}}$

$k \geq N_1$ ✓

$2^{-2k} \to 0$,

$\| 2^{-2k} \| < \frac{\varepsilon}{\sqrt{2}}$

$k \geq N_2$

9.2.7. Define the distance between two nonempty subsets A and B of $\mathbf{R}^n$ by

$$\operatorname{dist}(A, B) := \inf\{\|\mathbf{x} - \mathbf{y}\| : \mathbf{x} \in A \quad \text{and} \quad \mathbf{y} \in B\}.$$

a) Prove that if A and B are compact sets which satisfy $A \cap B = \emptyset$, then $\operatorname{dist}(A, B) > 0$.

b) Show that there exist nonempty, closed sets A, B in $\mathbf{R}^2$ such that $A \cap B = \emptyset$ but $\operatorname{dist}(A, B) = 0$.

(b) $A = \{(x,y) : y = 0\} = x\text{-axis}$.

$B = \{(x,y) : y = \frac{1}{x}\}$ closed.

(a) Recall, $\inf A = \beta < \infty \iff \exists$ a seq. $x_n \in A$ s.t. $x_k \longrightarrow \beta = \inf A$.

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Since $\|x-y\| \geq 0$ and both sets are nonempty ($\|x-y\|$ is bdd below by 0), then

dist(A,B) exists and finite. By Approximation Property for Infima, then $\exists\ x_k \in A$, $y_k \in B$ s.t.

$$\|x_k - y_k\| \longrightarrow \text{dist}(A,B) \text{ as } k \to \infty.$$

($x_k \to x_0$, $y_k \to y_0$)

Since A and B are Compact (closed & bdd)

$\Rightarrow$ x_k and y_k are bdd seq.

by the Bolzano-Weierstrass thm,

$\exists$ Subsequences:

$x_{k_j} \to x_0 \in A$ (closed) and $y_{k_j} \to y_0 \in B$ (closed).

Since $A \cap B = \phi$, $x_0 \neq y_0$.

$\text{dist}(A,B) = \|x_0 - y_0\| > 0$ since $x_0 \neq y_0$.

9.3.7. Suppose that $g : \mathbf{R} \to \mathbf{R}$ is differentiable and that $g'(x) > 1$ for all $x \in \mathbf{R}$. Prove that if $g(1) = 0$ and $f(x, y) = (x-1)^2(y+1)/(yg(x))$, then there is an $L \in \mathbf{R}$ such that $f(x, y) \to L$ as $(x, y) \to (1, b)$ for all $b \in \mathbf{R} \setminus \{0\}$.

$$f(x,y) = \frac{(x-1)^2(y+1)}{y\,g(x)} \to L \in \mathbb{R}$$

?? as $(x,y) \to (1,b)$, $b \neq 0$.

By the MVT, g on $[1, x]$.

$$g(x) = g(x) - g(1) = g'(c)(x-1), \quad c \text{ is between } x \text{ and } 1.$$

$$|g(x)| = |g'(c)|\,|x-1| > 1 \cdot |x-1|$$

$$\frac{1}{|g(x)|} < \frac{1}{|x-1|}, \quad x \neq 1.$$

$$|f(x,y)| = \left| \frac{(x-1)^2 (y+1)}{y\, g(x)} \right|$$

$$< \frac{|x-1|^2 \, |y+1|}{|y| \, |x-1|}, \quad x \neq 1.$$

$$= \frac{|x-1| \, |y+1|}{|y|} \longrightarrow 0$$

as $(x,y) \longrightarrow (1,b)$, $b \neq 0$.

$$\therefore \lim_{(x,y) \to (1,b)} f(x,y) = 0, \quad b \neq 0.$$

ex. $\lim_{(x,y)\to(1,1)} (x^3+y^3) = 2$.

$\forall \varepsilon>0, \exists \delta>0$
$\|(x,y)-(1,1)\|<\delta \Rightarrow |(x^3+y^3)-2|<\varepsilon$.

$\sqrt{(x-1)^2+(y-1)^2} < \delta$

$|x-1| < \delta$ and $|y-1| < \delta$

$|x^3+y^3-2| = |x^3-1+y^3-1|$

$\leq |x^3-1| + |y^3-1|$

$\leq |x-1|\,|x^2+x+1| + |y-1|\,|y^2+y+1|$

$\delta = 1$ $\quad |x-1| < 1 \Rightarrow 0 < x < 2$

$|y-1| < 1 \Rightarrow 0 < y < 2$

$\leq \delta\,|x^2+x+1| + \delta\,|y^2+y+1|$

$< 7\delta + 7\delta = 14\delta < \varepsilon$

Take $\delta = \min\left\{1, \frac{\varepsilon}{14}\right\}$. Then

If $\|(x,y) - (1,1)\| < \delta$, then

$$|f(x,y) - 2| = |x^3 + y^3 - 2|$$

$\delta = 1$ $\delta = 1$

$$\leq |x-1|\,|x^2+x+1| + |y-1|\,|y^2+y+1|$$

$$< 7\delta + 7\delta$$

$$= 14\delta < 14 \cdot \frac{\varepsilon}{14} = \varepsilon.$$

9.4.4. Suppose that A is closed in $\mathbf{R}^n$ and $\mathbf{f} : A \to \mathbf{R}^m$. Prove that $\mathbf{f}$ is continuous on A if and only if $\mathbf{f}^{-1}(E)$ is closed in $\mathbf{R}^n$ for every closed subset E of $\mathbf{R}^m$.

Pf. f is cont. on A and E closed in $\mathbb{R}^m$

we need to show $f^{-1}(E)$ is closed in $\mathbb{R}^n$.

Let $x_k \in f^{-1}(E)$, $x_k \longrightarrow a$. then $f(x_k) \in E$

and $f(x_k) \longrightarrow f(a)$ since f is cont.

$\Longrightarrow f(a) \in E$, since E is closed

i.e, $a \in f^{-1}(E)$.

($\Leftarrow$) Spse that $f^{-1}(E)$ is closed in $\mathbb{R}^n$, for every E closed in $\mathbb{R}^m$.

Spse f is not cont. at $a \in A$

$\exists\ x_k \in A \quad x_k \to a$ but $f(x_k) \not\to f(a)$.

$\exists\ \varepsilon_0 > 0$ and k_j s.t

$$\| f(x_{k_j}) - f(a) \| \geq \varepsilon_0 .$$

then $f(x_{k_j}) \in B^c_{\varepsilon_0}(f(a))$

$$x_{k_j} \in f^{-1}\left(B^c_{\varepsilon_0}(f(a)) \right) \equiv \text{closed set}.$$

(closed)

$$\Rightarrow \quad a \in f^{-1}\left(B^c_{\varepsilon_0}(f(a)) \right).$$

$\Rightarrow \quad f(a) \in B_{\varepsilon_0}^c(f(a))$, a contradiction.

$\therefore$ f is cont. on A.

9.4.5. Suppose that $E \subseteq \mathbf{R}^n$ and that $\mathbf{f} : E \to \mathbf{R}^m$.

a) Prove that $\mathbf{f}$ is continuous on E if and only if $\mathbf{f}^{-1}(B)$ is relatively closed in E for every closed subset B of $\mathbf{R}^m$.

b) Suppose that $\mathbf{f}$ is continuous on E. Prove that if V is relatively open in $\mathbf{f}(E)$, then $\mathbf{f}^{-1}(V)$ is relatively open in E, and if B is relatively closed in $\mathbf{f}(E)$, then $\mathbf{f}^{-1}(B)$ is relatively closed in E.

Pf. (a) ($\Rightarrow$) spse f is cont. on E and B is closed set. we need to show $f^{-1}(B)$ is rel. closed in E ($f^{-1}(B) = E \cap C$, C closed).

Since, B^c is open and f is cont. on E,

$f^{-1}(B^c)$ is rel. open in E. That is,

$f^{-1}(B^c) = V \cap E$, V is open.

$E \cap f^{-1}(B^c) = V \cap E$. ✓

$(f^{-1}(B))^c = f^{-1}(B^c)$

$f^{-1}(B) = E \setminus (E \setminus f^{-1}(B))$ [$E \setminus f^{-1}(B) = V \cap E$]

$f^{-1}(B) = (V \cap E)^c \cap E$.

$= (V^c \cup E^c) \cap E$

$\Leftrightarrow x \in f^{-1}(B^c)$

$\Leftrightarrow f(x) \in B^c$

$\Leftrightarrow f(x) \notin B$

$\Leftrightarrow x \notin f^{-1}(B)$

$\Leftrightarrow x \in (f^{-1}(B))^c$

$f^{-1}(B) = V^c \cap E$

(V^c closed)

$\therefore f^{-1}(B)$ is rel. closed in E.

($\Leftarrow$) $f^{-1}(B)$ is rel. closed in E for every B closed in $\mathbb{R}^n$,

show f is cont. on E.

B is closed $\Rightarrow$ B^c is open.

✓ We need $f^{-1}(B^c)$ is rel. open in E.:

b) Suppose that $\mathbf{f}$ is continuous on E. Prove that if V is relatively open in $\mathbf{f}(E)$, then $\mathbf{f}^{-1}(V)$ is relatively open in E, and if B is relatively closed in $\mathbf{f}(E)$, then $\mathbf{f}^{-1}(B)$ is relatively closed in E.

Pf: V rel. open in $f(E)$, i.e.,

$V = f(E) \cap U$, U open. $\Rightarrow$ f cont.

$f^{-1}(U)$ rel. open in E

Now, $f^{-1}(V) \cap E = f^{-1}(V \cap f(E))$ → $V = f(E) \cap U$

$= f^{-1}(U \cap f(E))$

$= f^{-1}(U) \cap E.$

Hence, $f^{-1}(V)$ is rel. open in E.