		11	.11 /	, <u>~</u>
	Suppose that $\mathbf{x}_k \to 0$	11 Xr	3// =	2 M
9.1.3.	Suppose that $\mathbf{x}_k \to 0$	0 in $\mathbf{R}^n$ as	$s k \rightarrow$	$\infty$ and

- **9.1.3.** Suppose that  $\mathbf{x}_k \to \mathbf{0}$  in  $\mathbf{R}^n$  as  $k \to \infty$  and that  $\mathbf{y}_k$  is bounded in  $\mathbf{R}^n$ .
  - a) Prove that  $\mathbf{x}_k \cdot \mathbf{y}_k \to 0$  as  $k \to \infty$ .
  - b) If n = 3, prove that  $\mathbf{x}_k \times \mathbf{y}_k \to 0$  as  $k \to \infty$ .

$$Pf. (a) 11 \times_{k} \cdot y_{k} 11 \leq 11 \times_{k} 11 11 y_{k} 11 \leq M \cdot \frac{2}{2M} \leq 2.$$

- **9.1.6.** Let E be a nonempty subset of  $\mathbb{R}^n$ .
  - a) Show that a sequence  $\mathbf{x}_k \in E$  converges to some point  $\mathbf{a} \in E$  if and only if for every set U, which is relatively open in E and contains  $\mathbf{a}$ , there is an  $N \in \mathbb{N}$  such that  $\mathbf{x}_k \in U$  for  $k \geq N$ .
  - b) Prove that a set  $C \subseteq E$  is relatively closed in E if and only if the limit of every sequence  $\mathbf{x}_k \in C$  which converges to a point in E satisfies  $\lim_{k\to\infty} \mathbf{x}_k \in C$ .

119, 11 < M, M>0

(b) (=) Let C be a relv. closed in E i.e, C=ENB, Bis closed. X<sub>L</sub> -> a, X<sub>L</sub> ∈ C C B" closed" aEB. and aEf" giver" a & Bof = C. =) Cis rel. clased in E.

9.1.6.	Let	E	be	a	nonempty	subset	of $\mathbf{R}^n$
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a) Show that a sequence  $\mathbf{x}_k \in E$  converges to some point  $\mathbf{a} \in E$  if and only if for every set U, which is relatively open in E and contains  $\mathbf{a}$ , there is an  $N \in \mathbb{N}$  such that  $\mathbf{x}_k \in U$  for  $k \geq N$ .

(Xe et/, Xe -> a e f. U rel. open in f, i.e, U=En Vopen aeU =) aeV =) (XneV) for lurge k => xke U for lorge k. (E) X1 e l'relv. epen in É', a E l since U=EnBs(a) is reliapenint and a ∈ U, then Xn ∈ U ⊆ B, (a),

## hr Corge k, c.e, 11/4-all < s

x<sub>k</sub> → a as k → ∞.

**9.1.1.** Using Definition 9.1i, prove that the following limits exist.

$$\mathbf{x}_k = \left(\frac{1}{k}, 1 - \frac{1}{k^2}\right)$$

$$\mathbf{x}_k = \left(\frac{k}{k+1}, \frac{\sin k^3}{k}\right)$$

$$\mathbf{x}_k = \left(\log(k+1) - \log k, 2^{-k}\right)$$

$$log(\frac{k+1}{k}), \frac{-k}{2}$$

as e-jas

let 2>0.

$$=$$
  $\int_{-\infty}^{\infty} (k+1) + 2^{-2k}$ 

$$\leq \left(\frac{2}{\sqrt{2}}\right)^2 + \left(\frac{2}{\sqrt{2}}\right)^2 = 2^2,$$

-2k

11 2 2 k 11 < 52

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<b>9.2.7.</b> Define the distance between two nonempt	ty subsets A	and B of $\mathbb{R}^n$ by
---	--------------	----------------------------

$$dist(A, B) := \inf\{\|\mathbf{x} - \mathbf{y}\| : \mathbf{x} \in A \text{ and } \mathbf{y} \in B\}.$$

- a) Prove that if A and B are compact sets which satisfy  $A \cap B = \emptyset$ , then dist(A, B) > 0.
- b) Show that there exist nonempty, closed sets A, B in  $\mathbb{R}^2$  such that  $A \cap B = \emptyset$  but  $\operatorname{dist}(A, B) = 0$ .

(b) 
$$A = \{ (x,y) : y = 0 \} = x - \alpha x is .$$

$$B = \{(x,y): y = \frac{1}{x}\}$$
 closed

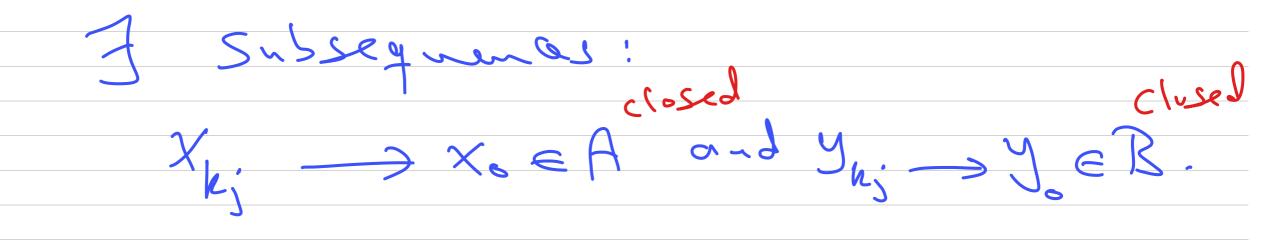
(a) 
$$\mathbb{R}e(all, \inf A = \beta < \infty \Leftrightarrow \exists aseq.)$$
  
Sis  $\mathbb{X}_{h} \in A$  s.t  $\mathbb{X}_{h} \longrightarrow \beta = \inf A$ .

Sine 1/x-y11>0 and Softh Sets are honeupty (11x-311 istald below by o), then dist (A,B) exists and finite. By Approximation Property for Infima, plen J XxEA, Y, EB S.t. 11 Xn-yn11 - > dist (A,B) ask-sal.

Since A and B are Compact (closed + 6dd) => Xh and ye are Gdd Seg. by the Bolzano-Weierstrass Mm,

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**9.3.7.** Suppose that  $g : \mathbf{R} \to \mathbf{R}$  is differentiable and that g'(x) > 1 for all  $x \in \mathbf{R}$ . Prove that if g(1) = 0 and  $f(x, y) = (x - 1)^2(y + 1)/(yg(x))$ , then there is an  $L \in \mathbf{R}$  such that  $f(x, y) \to L$  as  $(x, y) \to (1, b)$  for all  $b \in \mathbf{R} \setminus \{0\}$ .

$$f(x,y) = (x-1)^{2}(y+1) \longrightarrow 1 \in \mathbb{R}$$

$$y(g(x)) = (x,y) \longrightarrow (1,b), b+$$

By the MVT, 
$$g = g(c)(x)$$
.

$$g(x) = g(x) - g(1) = g'(c)(x-1), c is$$
Letween  $x$ 
and  $1$ .

$$|g(x)| = |g'(c)||x-1|| > |.|x-1|$$

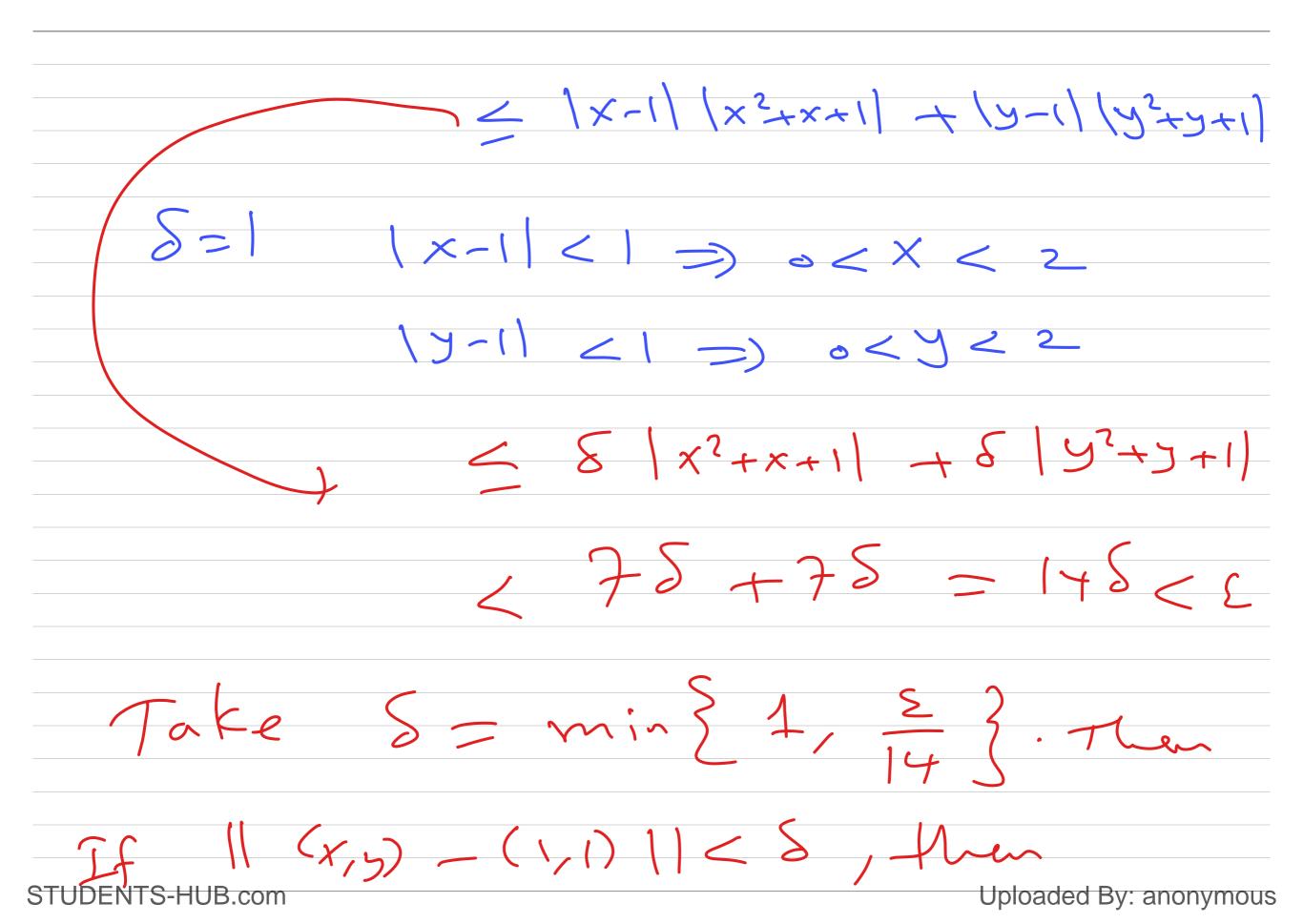
$$|g(x)| = |g'(c)||x-1|| > |.|x-1|$$

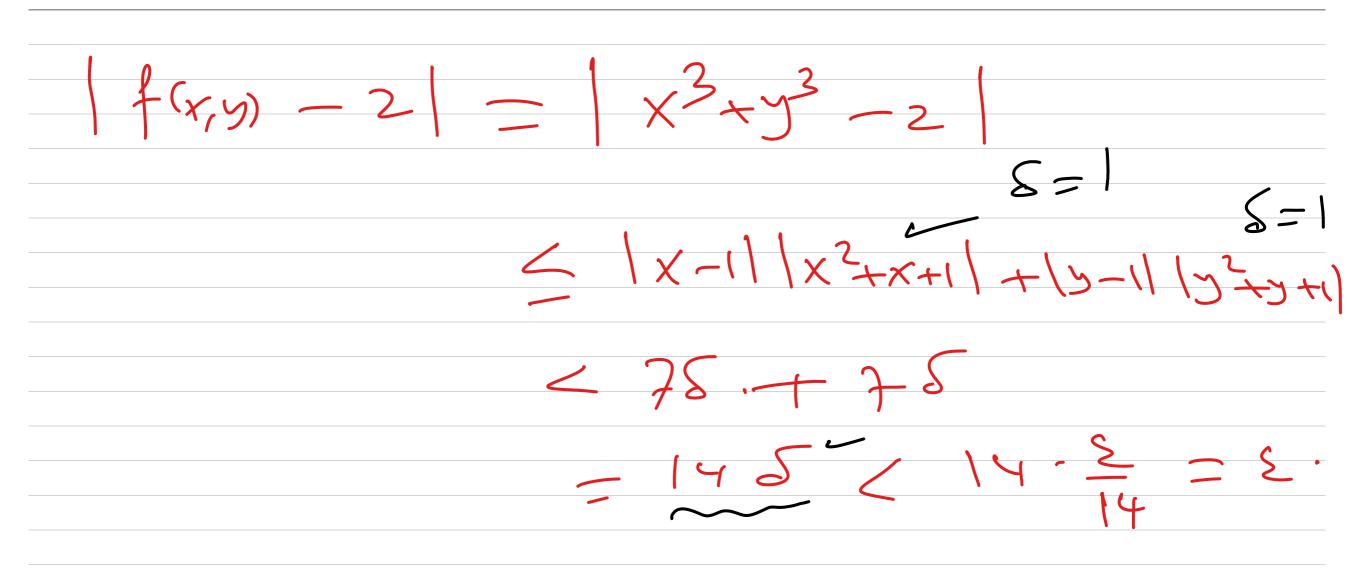
$$|f(x_1)| = \frac{(x-1)^2(y+1)}{yg(x)}$$

ex. 
$$\lim_{(x,y) \to (y,1)} (x^3 + y^3) = 2$$
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 $\lim_{(x,y) \to (y,1)} (x^3 + y^3) = 2$ .  
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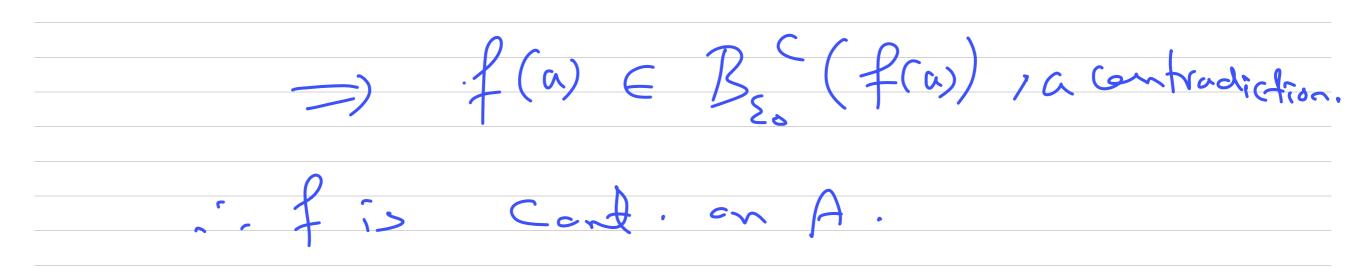




**9.4.4.** Suppose that A is closed in  $\mathbb{R}^n$  and  $\mathbf{f}: A \to \mathbb{R}^m$ . Prove that  $\mathbf{f}$  is continuous on A if and only if  $\mathbf{f}^{-1}(E)$  is closed in  $\mathbb{R}^n$  for every closed subset E of  $\mathbb{R}^m$ .

If. fis cont. on A and E closed in R. we need to show f'(E) is closed in R.

Let Yhe f (E), (Xh ) a. Then f(Xh)ef and  $f(x_n) \rightarrow f(a)$  since f is cont. =) f(a) ef, since E is closed i.e, aef-(6). (C) Sporthat folk) is closed in Pr for every E closed in Rm. Spse fis not cont. at a EA



- **9.4.5.** Suppose that  $E \subseteq \mathbb{R}^n$  and that  $\mathbf{f}: E \to \mathbb{R}^m$ .
  - a) Prove that **f** is continuous on E if and only if  $\mathbf{f}^{-1}(B)$  is relatively closed in E for every closed subset B of  $\mathbf{R}^m$ .
  - b) Suppose that **f** is continuous on E. Prove that if V is relatively open in  $\mathbf{f}(E)$ , then  $\mathbf{f}^{-1}(V)$  is relatively open in E, and if B is relatively closed in  $\mathbf{f}(E)$ , then  $\mathbf{f}^{-1}(B)$  is relatively closed in E.

ging BC is open and fis cond. on 6, f-1(RC) is rel. open in E. Most is f-(BC) = V (BC), V is open. Enf(Bc) = VAE. = (Nenes) UE Wyses

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( X & F (B) f-1(B) = (10sed C) X e(f, (b)) -i. f (B) is rel. closed in E. (E) f'(B) is rel. Closed in E for every B closed in Rr, Show fis cont. on E. Bisclosed = BC is open We need f (BC) is rel. ogen in E. STUDENTS-HUB.com Uploaded By: anonymous

b) Suppose that **f** is continuous on E. Prove that if V is relatively open in  $(\mathbf{f}(E))$ , then  $(\mathbf{f}^{-1}(V))$  is relatively open in (E) and if (E) is relatively closed in (E), then (E), then (E) is relatively closed in (E).

	$=f^{-1}(U)\cap E$		
Hence	f-(V) is rel. open in E.		