the series div. by Ratio Test since $\rho = 4 > 1$.

(c) $\sum_{n=1}^{\infty} \frac{4^n n! n!}{(2n)!}$. Let $a_n = \frac{4^n n! n!}{(2n)!}$, then

$$\frac{a_{n+1}}{a_n} = \frac{4^{n+1}(n+1)!(n+1)!}{(2n+2)!} \cdot \frac{(2n)!}{4^n n! n!}$$

$$= \frac{\cancel{4^n} \cdot 4(n+1)\cancel{n!}(n+1)\cancel{n!} \quad \cancel{(2n)!}}{(2n+2)(2n+1)\cancel{(2n)!} \quad \cancel{4^n}\cancel{n!}\cancel{n!}}$$

$$= \frac{4(n+1)^2}{2(n+1)(2n+1)} = \frac{2n+2}{2n+1}.$$

Since $\rho = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$. We cannot decide

anything from the Ratio Test. But we

notice that $\frac{a_{n+1}}{a_n} = \frac{2n+2}{2n+1} > 1$ (أتزايدية)

Therefore, $a_{n+1} > a_n \quad \forall n$.

$\Rightarrow$ $a_2 > a_1$ and $a_3 > a_2 > a_1$ - - - - -

$\Rightarrow$ $a_n > a_1 = 2$, for all n.

$\Rightarrow$ $\lim_{n\to\infty} a_n \neq 0$

$\Rightarrow$ the series diverges by using nth term test for divergence

(d) $a_1 = 1$, $a_{n+1} = \frac{1+\tan^{-1} n}{n} a_n$

$$\rho = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\left(\frac{1+\tan^{-1} n}{n}\right) \cancel{a_n}}{\cancel{a_n}}$$

$$= \lim_{n\to\infty} \frac{1+\tan^{-1} n}{n} = 0 < 1$$

$\therefore$ the series Conv. by Ratio test since $\rho = 0 < 1$.

Thm [the Root Test] Let $\sum_{n=1}^{\infty} a_n$ be a series with $a_n \geq 0$, $\forall n \geq N$. Suppose that $\boxed{\lim_{n\to\infty} \sqrt[n]{a_n} = \rho.}$

then (a) the series conv. if $\rho < 1$.

(b) " " div. if $\rho > 1$ or infinite.

(c) the test is inconclusive if $\rho = 1$.

Examples which of the following series conv. and which diverge?

① $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$ Conv. because

$$\rho = \lim_{n\to\infty} \sqrt[n]{\frac{n^2}{2^n}} = \lim_{n\to\infty} \frac{(\sqrt[n]{n})^2}{2} = \frac{1^2}{2} = \frac{1}{2} <$$

② $\sum_{n=1}^{\infty} \left(\frac{1}{1+n}\right)^n$ Conv. by Root test since

$$\rho = \lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{\left(\frac{1}{1+n}\right)^n}$$

$$= \lim_{n\to\infty} \frac{1}{1+n} = 0 < 1.$$

③ $\sum_{n=1}^{\infty} \frac{n^{2n}}{(1+2n^2)^n}$ Conv. by Root test since

$$\rho = \lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \left(\frac{n^{2n}}{(1+2n^2)^n}\right)^{\frac{1}{n}}$$

$$= \lim_{n\to\infty} \frac{n^2}{1+2n^2} = \frac{1}{2} < 1 .$$

(4) Exercise 15 $\displaystyle\sum_{n=1}^{\infty} \left(1 - \frac{1}{n}\right)^{n^2}$

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{\left(1-\frac{1}{n}\right)^{n^2}}$$

$$= \lim_{n\to\infty} \left(1-\frac{1}{n}\right)^{n} = e^{-1} = \frac{1}{e} < 1$$

$\Rightarrow$ series conv. by Root test.

(5) Exercise 11 $\displaystyle\sum_{n=1}^{\infty} \left(\frac{4n+3}{3n-5}\right)^{n}$

$a_n = \left(\frac{4n+3}{3n-5}\right)^n \geq 0 \ , \forall \ n \geq 2 ;$

$$\rho = \lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{\left(\frac{4n+3}{3n-5}\right)^n}$$

$$= \lim_{n\to\infty} \frac{4n+3}{3n-5} = \frac{4}{3} > 1$$

$\Rightarrow$ Series div. by Root test $\left(\rho = \frac{4}{3} > 1\right)$

Lecture problems

(6) $\sum_{n=2}^{\infty} \frac{3^{n+2}}{\ln n}$. Let $a_n = \frac{3^{n+2}}{\ln n} > 0, \forall n \geq 2$

$$\rho = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{3^{n+3}}{\ln(n+1)} \cdot \frac{\ln n}{3^{n+2}}$$

$$= \lim_{n\to\infty} \frac{3\ln n}{\ln(n+1)} = \lim_{n\to\infty} \frac{\frac{3}{n}}{\frac{1}{n+1}}$$

$$= 3 > 1$$

$\Rightarrow$ series div. by Ratio Test ($\rho = 3 > 1$).

---

(15) done see p.6.

(20) $\sum_{n=1}^{\infty} \frac{n!}{10^n}$ diverges by Ratio test since

$$\rho = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)!}{10^{n+1}} \cdot \frac{10^n}{n!}$$

$$= \lim_{n\to\infty} \frac{(n+1)\cancel{n!}}{\cancel{10^n}\cdot 10} \frac{\cancel{10^n}}{\cancel{n!}} = \infty$$

(30) $\sum_{n=1}^{\infty} \overbrace{\left(\frac{1}{n} - \frac{1}{n^2}\right)^n}^{a_n}$ Conv. by Root test since

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{\left(\frac{1}{n} - \frac{1}{n^2}\right)^n} = \lim_{n\to\infty} \left(\frac{1}{n} - \frac{1}{n^2}\right)$$

$$= 0 < 1.$$

10.6 Alternating Series, Absolute and Conditional Convergence

Df. A series in which the term are alternately positive and negative is an alternating series

ex. ① $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}- \cdots + \frac{(-1)^{n+1}}{n}+\cdots = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}$

② $\sum_{n=1}^{\infty}\frac{4(-1)^n}{2^n} = -2+1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\cdots$

③ $\sum_{n=1}^{\infty}(-1)^{n+1}n = 1-2+3-4+5-\cdots$

Thm ① [Alternating series Test] [A.S.T]

the series $\sum_{n=1}^{\infty}(-1)^{n+1}u_n = u_1-u_2+u_3-\cdots$

converges if all the following conditions satisfy

(1) u_n's are positive for all n.

(2) u_n eventually nonincreasing for all $n \geq N$ (N positive).

(3) $\lim_{n\to\infty} u_n = 0$.

ex. $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n}$, 1) $U_n = \frac{1}{n} > 0$ for all n.

2) $U_{n+1} = \frac{1}{n+1} < \frac{1}{n} = U_n \Rightarrow U_{n+1} < U_n , \forall n \geq 1$

$\therefore U_n$ is nonincreasing $\left[\text{or } f(x) = \frac{1}{x} \Rightarrow f' = \frac{-1}{x^2} < 0 \right]$

3) $\lim_{n\to\infty} U_n = \lim_{n\to\infty} \frac{1}{n} = 0$.

$\Rightarrow$ series Converges by A.S.T.

ex. $\sum_{n=1}^{\infty} (-1)^n \left(1+\frac{1}{n}\right)^n$

$U_n = \left(1+\frac{1}{n}\right)^n$

$\lim_{n\to\infty} U_n = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e^1 = e \neq 0$

$\Rightarrow$ series div. by Divergence test

ex. $\sum_{n=1}^{\infty} (-1)^n$, $\boxed{U_n = 1}$, $\lim_{n\to\infty} U_n = \lim_{n\to\infty} 1 = 1 \neq 0$

$\therefore$ series div. by Divergence Test

ex. $\sum_{n=1}^{\infty} \frac{10n}{n^2+16}$, $U_n = \frac{10n}{n^2+16}$

clearly, $U_n > 0$, for all $n \geq 1$ and $\lim_{n\to\infty} \frac{10n}{n^2+16} = \lim_{n\to\infty} \frac{10}{2n}$

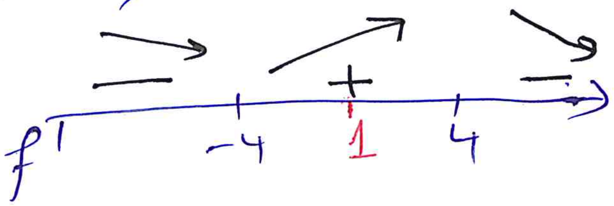
$= 0$

Define $f(x) = \dfrac{10x}{x^2+16}, \; x \geq 1$

$$f'(x) = \frac{(x^2+16)(10) - 10x(2x)}{(x^2+16)^2}$$

$$= \frac{10(x^2+16-2x^2)}{(x^2+16)^2} = \frac{10(16-x^2)}{(x^2+16)^2}$$

$\Rightarrow f'(x) \leq 0$, for all $x \geq 4$



$\therefore f$ is eventually nonincreasing $(N=4)$.

thus, $\{u_n\}$ is nonincreasing for $n \geq 4$.

$\Rightarrow$ the series converges by A.S.T.

Thm ②. [the Alternating series Estimation thm]

If the alternating series $\sum_{n=1}^{\infty} (-1)^{n+1} u_n$ converges. (satisfies the three condition of thm1)

then, for $n \geq N$,

$$S_n = u_1 - u_2 + u_3 - \cdots + (-1)^{n+1} u_n$$

approximates the sum L of the series with an error E such that $|E| < u_{n+1}$ (the absolute value of the first unused term.

Also, sum L lies between any two successive partial sums S_n and S_{n+1}

and the remainder, $L - S_n$, has the same sign as the first unused term.

ex: Use the 4^{th} partial sum S_4 to estimate the sum $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{2^{n-1}} = 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \cdots$

Sol: $S_4 = 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} = \frac{5}{8} = 0.625$

$S_5 = 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} = \frac{5}{8} + \frac{1}{16} = \frac{11}{16} = 0.6875$

exact sum $L = \frac{a}{1-r} = \frac{1}{1+\frac{1}{2}} = \frac{2}{3} = 0.\overline{6}$

Notice that $L = 0.66\ldots$ lies between S_4 & S_5

$|E| = L - S_4 = \frac{2}{3} - \frac{5}{8} \cong 0.0417$

By last thm (2), $|E| < u_5 \Rightarrow |E| < \frac{1}{16} = 0.0625$

$\Rightarrow -\frac{1}{16} < E < \frac{1}{16}$

Now, $E > 0$ (the same sign as the first unused term (a_5) نفس إشارة الحد الخامس)

So, we have $-\frac{1}{16} < E < \frac{1}{16}$ and $E > 0$

$$\Rightarrow \boxed{0 \leq E < \frac{1}{16}}$$

Q49) Estimate $|E|$ using S_3.

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} = \underbrace{1 - \frac{1}{2} + \frac{1}{3}}_{S_3} - \frac{1}{4} + \frac{1}{5} - \cdots$$

Sol. $U_n = \frac{1}{n}$, $|E| < U_4 = \frac{1}{4}$

إشارة E

$E < 0$

$$\Rightarrow -\frac{1}{4} < E < \frac{1}{4}$$

$$\Rightarrow -\frac{1}{4} < E \leq 0.$$

Q53) How many terms should be used to estimate the sum $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n^2+3}$ with an error less than 10^{-3} ?

u_n

Sol. $|E| < u_{n+1} < \frac{1}{10^3}$

$$\Rightarrow \frac{1}{(n+1)^2+3} < 10^{-3} \Rightarrow (n+1)^2+3 > 10^3$$

$$(n+1)^2 > 997$$

$\Rightarrow \quad n+1 > \sqrt{997}$

$\Rightarrow \quad n > -1 + \sqrt{997} \simeq 30.5753$

$\Rightarrow \quad n \geq 31.$

ex: How many terms should be used to estimate the sum $\sum_{n=0}^{\infty} (-1)^n \frac{1}{n!}$ with an error $|E| < 5 \times 10^{-6}$?

Sol. $\sum_{n=0}^{\infty} (-1)^n \frac{1}{n!} = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{(n-1)!} \quad u_n$

$|E| < u_{n+1} < 5 \times 10^{-6}$

$\Rightarrow \quad \frac{1}{n!} < \frac{5}{10^6} \Rightarrow n! > \frac{10^6}{5} = 2 \times 10^5$

$\Rightarrow \quad \boxed{n \geq 9}$

Absolute and Conditional Convergence

Df. (1) A series $\sum_{n=1}^{\infty} a_n$ converges absolutely (is absolutely convergent) if the series $\sum_{n=1}^{\infty} |a_n|$ converges.

(2) A series $\sum_{n=1}^{\infty} a_n$ that converges but doesnot converge absolutely converges conditionally

Thm (the absolute Convergence Test)
If $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ converges

Rmk. The Converse of this thm is not true in general

ex. $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n}$ conv. by A.S.T since

$u_n = \frac{1}{n} > 0$, decreasing, $\lim_{n\to\infty} \frac{1}{n} = 0$

but $\sum_{n=1}^{\infty} |(-1)^n \frac{1}{n}| = \sum_{n=1}^{\infty} \frac{1}{n}$ div. (p-test).

Examples which of the series converge absolutely, which converge, which converge conditionally, and which diverge?

① $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n^2}$ conv. by A.S.T
$\left(u_n = \frac{1}{n^2} \to 0, \ +ve, \ decr. \right)$.

$\sum_{n=1}^{\infty} \left| (-1)^n \frac{1}{n^2} \right| = \sum_{n=1}^{\infty} \frac{1}{n^2}$ conv. by p-test.

$\Rightarrow$ the orginal series Converges absolutely.

② $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n}$ conv. by A.S.T
$\left[u_n = \frac{1}{n} > 0, \ +ve, \ \lim_{n\to\infty} \frac{1}{n} = 0 \right]$

$\sum_{n=1}^{\infty} \left| (-1)^n \frac{1}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n}$ div. (p-test).

$\therefore \sum_{n=1}^{\infty} (-1)^n \frac{1}{n}$ Converges conditionally

In general, $\sum_{n=1}^{\infty} \frac{1}{n^p} = \begin{cases} \text{Conv. Conditionally if } 0 < p \le 1 \\ \text{conv. absolutely if } p > 1. \end{cases}$

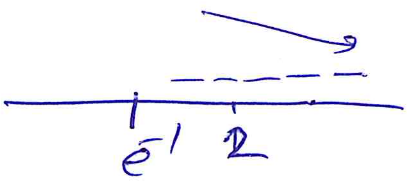
③ $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^{3-e}}$ conv. conditionally since $p = 3-e < 1$.

④ Exercise 28 $\sum_{n=2}^{\infty} (-1)^{n+1} \frac{1}{n \ln n}$

Converges by A.S.T $\Big[\ u_n = \frac{1}{n \ln n} > 0$

and if $f(x) = \frac{1}{x \ln x} \Rightarrow f' = \frac{-(x \cdot \frac{1}{x} + \ln x)}{(x \ln x)^2} < 0$

if $1 + \ln x > 0$ or $x > e^{-1}$



$\Rightarrow u_n = \frac{1}{n \ln n}$ is decreasing on $x > 2$.

$\lim_{n\to\infty} \frac{1}{n \ln n} = 0\ \Big]$

• $\sum_{n=2}^{\infty} \left| (-1)^{n+1} \frac{1}{n \ln n} \right| = \sum_{n=2}^{\infty} \frac{1}{n \ln n}$ div.

by integral test since

$$\int_2^{\infty} \frac{1}{x \ln x}\,dx = \lim_{A\to\infty} \int_2^{A} \frac{1}{x \ln x}\,dx$$

$$= \lim_{A\to\infty} \ln(\ln x)\Big|_2^{A} = \lim_{A\to\infty} \ln(\ln A) - \ln(\ln 2)$$

$= \infty$ div.

$$\Rightarrow \sum_{n=2}^{\infty} (-1)^{n+1} \frac{1}{n \ln n} \text{ conv. conditionally.}$$

Lecture problems 14, 19, 29

(14) $$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{3\sqrt{n+1}}{\sqrt{n}+1}.$$

let $$U_n = \frac{3\sqrt{n+1}}{\sqrt{n}+1} \Rightarrow \lim_{n\to\infty} \frac{3\sqrt{n+1}}{\sqrt{n}+1} \quad \frac{\infty}{\infty}.$$

$$= \lim_{n\to\infty} \frac{\frac{3}{\cancel{2}\sqrt{n+1}}}{\frac{1}{\cancel{2}\sqrt{n}}}$$

$$= \lim_{n\to\infty} \frac{3\sqrt{n}}{\sqrt{n+1}} = 3\sqrt{\lim_{n\to\infty} \frac{n}{n+1}}$$

$$= 3(1) = 3 \neq 0$$

$\therefore$ The series div. by the nth term test for divergence.

(19) $$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{n^3+1}.$$

$$\sum_{n=1}^{\infty} \left| (-1)^{n+1} \frac{n}{n^3+1} \right| = \sum_{n=1}^{\infty} \frac{n}{n^3+1}$$

$$\frac{n}{n^3+1} < \frac{n}{n^3} = \frac{1}{n^2}$$

Since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ conv. (p-test), then

$\sum_{n=1}^{\infty} \frac{n}{n^3+1}$ conv. by D.C.T.

$\Rightarrow \sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{n^3+1}$ Converges absolutely.

(29) $\sum_{n=1}^{\infty} (-1)^n \frac{\tan^{-1} n}{n^2+1}$ Converges absolutely since

$$\sum_{n=1}^{\infty} \left| (-1)^n \frac{\tan^{-1} n}{n^2+1} \right| = \sum_{n=1}^{\infty} \frac{\tan^{-1} n}{n^2+1}$$

$$\int_1^{\infty} \frac{\tan^{-1} x}{x^2+1} dx = \lim_{A\to\infty} \int_1^{A} \frac{\tan^{-1} x}{x^2+1} dx$$

$$= \lim_{A\to\infty} \left. \frac{(\tan^{-1} x)^2}{2} \right|_1^{A}$$

Let $u = \tan^{-1} x$

$du = \frac{1}{1+x^2} dx$

$$= \lim_{A\to\infty} \frac{(\tan^{-1} A)^2}{2} - \frac{(\tan^{-1} 1)^2}{2} = \frac{\left(\frac{\pi}{2}\right)^2}{2} - \frac{\left(\frac{\pi}{4}\right)^2}{2}$$

conv.

$\Rightarrow$ the orginal series conv. absolutely.

10.7 Power Series

. Power Series and Convergence

Df. A power series about $x=0$ is a series of the form $\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n + \cdots$

A power series about $x=a$ is a series of the form $\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$
$\cdots + c_n (x-a)^n + \cdots$

a is called the center.

$c_0, c_1, c_2, \ldots, c_n, \ldots$ are constants.

Ex. $\sum_{n=0}^{\infty} \left(-\frac{1}{2}\right)^n (x-2)^n = 1 - \frac{1}{2}(x-2) + \frac{1}{4}(x-2)^2 + \cdots$

is a power series with $c_n = \left(-\frac{1}{2}\right)^n$ coefficients

and center $= 2$.

Also, this series is a geometric series with $a=1$, $r = -\frac{1}{2}(x-2)$. The series

Converges if $|r|<1$, that is,

$$\left| -\tfrac{1}{2}(x-2) \right| < 1$$

$\Rightarrow \quad \frac{1}{2}|x-2| < 1$

$\Rightarrow \quad |x-2| < 2$

$\Rightarrow \quad -2 < x-2 < 2 \Rightarrow \boxed{0 < x < 4.}$

$$\text{Sum} = \frac{a}{1-r} = \frac{1}{1+\frac{1}{2}(x-2)} = \frac{2}{x}, \quad 0<x<4.$$

Ex. a) Find the series radius, center and interval of convergence.

b) For what values of x does the series converges absolutely?

c) For what values of x does the series converges conditionally?

$\boxed{1}$ $$\sum_{n=1}^{\infty} \frac{(3x-2)^n}{n}$$

Ans. we use Ratio or Root tests.

We use Ratio Test.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(3x-2)^{n+1}}{n+1}\cdot\frac{n}{(3x-2)^n}\right|$$

$$= \lim_{n\to\infty} |3x-2| \frac{n}{n+1}$$

$$= |3x-2| \lim_{n\to\infty} \frac{n}{n+1}$$

$$= |3x-2|\cdot 1 = |3x-2| < 1$$

$$\Rightarrow -1 < 3x-2 < 1$$

$$1 < 3x < 3$$

$$\Rightarrow \frac{1}{3} < x < 1.$$

at $x = \frac{1}{3}$, $\sum_{n=1}^{\infty} \frac{(3x-2)^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ conv. by A.S.T.

at $x=1$, $\sum_{n=1}^{\infty} \frac{(3x-2)^n}{n} = \sum_{n=1}^{\infty} \frac{1}{n}$ div. (p-test).

$\Rightarrow$ Interval of convergence is $[\frac{1}{3}, 1)$

the series conv. absolutely on $(\frac{1}{3}, 1)$

conv. conditionally at $x = \frac{1}{3}$

Center $= \frac{2}{3}$, radius $= R = \frac{1}{3}$.

Q14) $\sum_{n=1}^{\infty} \frac{(x-1)^n}{n^3 \cdot 3^n}$

We use root test

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \left(\frac{|x-1|^n}{n^3 \cdot 3^n} \right)^{\frac{1}{n}}$$

$$= \lim_{n\to\infty} \frac{|x-1|}{(\sqrt[n]{n})^3 \cdot 3}$$

$$= \frac{|x-1|}{(1)^3 \cdot 3} = \frac{|x-1|}{3} < 1$$

$$\Rightarrow |x-1| < 3 \Rightarrow -3 < x-1 < 3$$

$$\boxed{-2 < x < 4}$$

at $\boxed{x=-2}$, series $= \sum_{n=1}^{\infty} \frac{(-2-1)^n}{n^3 \cdot 3^n} = \sum_{n=1}^{\infty} (-1)^n \frac{1}{n^3}$

abs. Conv. ~~by~~ Since $\sum_{n=1}^{\infty} \left| (-1)^n \frac{1}{n^3} \right| = \sum_{n=1}^{\infty} \frac{1}{n^3}$ Conv.

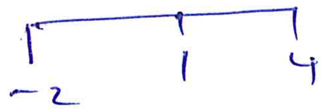
$\boxed{x=4}$, $\sum_{n=1}^{\infty} \frac{(4-1)^n}{n^3 \cdot 3^n} = \sum_{n=1}^{\infty} \frac{1}{n^3}$ abs. Conv.

Hence, the series Conv. abs. on $[-2, 4]$

" " " conditionally nowhere

$\therefore$ conv. on $[-2, 4]$

center $= 1$, radius $= 3$.



Q12) $\displaystyle\sum_{n=1}^{\infty} \frac{3^n x^n}{n!}$

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{3^{n+1} x^{n+1}}{(n+1)!} \cdot \frac{n!}{3^n x^n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{\cancel{3^n} \cdot 3 \cdot \cancel{x^n} \cdot x \cdot \cancel{n!}}{(n+1)\cancel{n!}\ \cancel{3^n}\ \cancel{x^n}} \right|$$

$$= |3x| \lim_{n\to\infty} \frac{1}{n+1}$$

$$= |3x| \cdot 0 < 1 \quad \text{for all } x.$$

$\therefore$ Interval of convergence $= (-\infty, \infty)$.

" " abs. conv. $= (-\infty, \infty)$.

The series conv. conditionally nowhere.

center $= 0$, radius $= \infty$.

ex. 4 $\sum_{n=0}^{\infty} n!\,(x-2)^n$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!\,(x-2)^{n+1}}{n!\,(x-2)^n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(n+1)\,\cancel{n!}\,\cancel{(x-2)^n}(x-2)}{\cancel{n!}\,\cancel{(x-2)^n}}\right|$$

$$= |x-2| \lim_{n\to\infty}(n+1)$$

$$= |x-2|\cdot\infty < 1 \text{ if } \boxed{x=2}$$

The series Conv. at $x=2$

Center $=2$, radius $R=0$.

<u>Summary</u> <u>Thm</u> for a given power series $\sum_{n=0}^{\infty} c_n (x-a)^n$ there are only three possibilities

1) There is a $R>0$ such that the series div. for x with $|x-a|>R$ but Conv. absolutely for x with $|x-a|<R$.

The series may or may not conv. at either of the endpoint $x = a - R$ and $x = a + R$.

2) The series conv. absolutely for every $x \in (-\infty, \infty)$ $(R = \infty)$.

3) The series conv. at $x = a$ and div. elsewhere $(R = 0)$.

Operation on Power Series

Thm (the series multiplication thm for power series). If

$$A(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots$$

$$B(x) = \sum_{n=0}^{\infty} b_n x^n = b_0 + b_1 x + b_2 x^2 + \cdots$$

Conv. absolutely for $|x| < R$, and

$$c_n = a_0 b_n + a_1 b_{n-1} + a_2 b_{n-2} + \cdots + a_{n-1} b_1 + a_n b_0$$

$$= \sum_{k=0}^{n} a_k b_{n-k}, \text{ then}$$

$\sum_{n=0}^{\infty} c_n x^n$ Conv. abs. to $A(x)B(x)$ for $|x| < R$:

$$\left(\sum_{n=0}^{\infty} a_n x^n\right) \cdot \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

$$= (a_0 + a_1 x + \cdots + \cdots)(b_0 + b_1 x + b_2 x^2 + \cdots)$$

$$= \sum_{n=0}^{\infty} c_n x^n .$$

Ex. Find the first four nonzero terms of

$$\left(\sum_{n=0}^{\infty} x^n\right) \cdot \left(\sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1}\right)$$

$$= \left(1 + x + x^2 + \cdots\right)\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots\right)$$

$$= \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots\right) + \left(x^2 - \frac{x^3}{2} + \frac{x^4}{3} - \cdots\right)$$

$$+ \left(x^3 - \frac{x^4}{2} + \frac{x^5}{3} - \cdots\right) + \cdots$$

$$= x + \frac{1}{2}x^2 + \frac{5}{6}x^3 - \frac{1}{6}x^4 + \cdots$$

Thm If $\sum_{n=0}^{\infty} a_n x^n$ conv. absolutely for $|x| < R$, then $\sum_{n=0}^{\infty} a_n (f(x))^n$ conv. abs. for any continuous fun f on $|f(x)| < R$.

ex: $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ conv. abs. for $|x| < 1$

Then $\frac{1}{1-4x^2} = \sum_{n=0}^{\infty} (4x^2)^n$ conv.

abs. for $|4x^2| < 1$ or $|x| < \frac{1}{2}$

Also, $\frac{1}{1-\ln x} = \sum_{n=0}^{\infty} (\ln x)^n$ conv.

abs. for $|\ln x| < 1$, i.e.,

$-1 < \ln x < 1 \Rightarrow \frac{1}{e} < x < e.$

• <u>Term by Term Differentiation</u>

<u>ex.</u> find series for $f'(x)$ and $f''(x)$ if

$f(x) = \frac{1}{1-x} = 1 + x + x^2 + \cdots = \sum_{n=0}^{\infty} x^n$,

$-1 < x < 1$

<u>Sol.</u> $f'(x) = \frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \cdots$

$= \sum_{n=1}^{\infty} n x^{n-1}, \quad -1 < x < 1$

$$f''(x) = \frac{2}{(1-x)^3} = 2+6x+12x^2+\cdots$$

$$= \sum_{n=2}^{\infty} n(n-1)\, x^{n-2}$$

Thm. (The term by term Differentiation Thm)

If $\sum_{n=0}^{\infty} c_n (x-a)^n$ has radius of Convergence

$R>0$, $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$ on $|x-a|<R$,

$$\Rightarrow f'(x) = \sum_{n=1}^{\infty} n c_n (x-a)^{n-1}$$

$$f''(x) = \sum_{n=2}^{\infty} n(n-1) c_n (x-a)^{n-2}, \cdots$$

Each of these series Conv. on $|x-a|<R$.

ex. Find the Sum $\sum_{n=1}^{\infty} \frac{n}{2^{n-1}}$

Sol. $\sum_{n=1}^{\infty} \frac{n}{2^{n-1}} = \sum_{n=1}^{\infty} n\left(\frac{1}{2}\right)^{n-1}$

Using $\sum_{n=1}^{\infty} n x^{n-1} = \frac{1}{(1-x)^2}$, $|x|<1$

put $x = \frac{1}{2}$, $\sum_{n=1}^{\infty} n\left(\frac{1}{2}\right)^{n-1} = \frac{1}{\left(1-\frac{1}{2}\right)^2} = 4.$

Term by Term integration thm

Spse that $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$

conv. for $a-R < x < a+R$ $(R>0)$.

Then $\sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}$ conv. for

$a-R < x < a+R$ and

$$\int f(x)\,dx = \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1} + C,$$

for $a-R < x < a+R$.

ex. The series $\frac{1}{1+t} = 1 - t + t^2 - \cdots$

Conv. on $-1 < t < 1$, therefore,

$$\ln(1+x) = \int_0^x \frac{1}{1+t}\,dt = t - \frac{t^2}{2} + \frac{t^3}{3} + \cdots \Big|_0^x$$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$$

or $\boxed{\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}, \quad -1 < x \le 1}$

ex. Find a power series of $y = \tan^{-1} x$.

sol. $\tan^{-1} x = \int \frac{1}{1+x^2} \, dx$

$$= \int \sum_{n=0}^{\infty} (-x^2)^n \, dx, \quad |-x^2| < 1$$

$$= \int \sum_{n=0}^{\infty} (-1)^n x^{2n}, \quad |x| < 1$$

$$\tan^{-1} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} + C, \quad |x| < 1$$

put $x = 0$, $\tan^{-1} 0 = 0 + C \Rightarrow \boxed{C = 0}$

$$\boxed{\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}, \quad |x| \le 1}$$

b.o.p

ex. Find the sum $\sum_{n=0}^{\infty} (-1)^n \frac{\left(\frac{\sqrt{3}}{2}\right)^{2n+1}}{2n+1}$

sol. Sum $= \tan^{-1}\left(\frac{\sqrt{3}}{2}\right)$.

ex. Find the sum $\sum_{n=0}^{\infty}(-1)^n \frac{1}{2n+1}$

Ans. Sum $= \tan^{-1} 1 = \pi/4$.

Ex. Represent the following function as a power series.

① $f(x) = \frac{x^2}{1+x}$

Sol. $f(x) = x^2 \cdot \frac{1}{1+x} = x^2 \sum_{n=0}^{\infty}(-1)^n x^n, \quad |x|<1$

$= \sum_{n=0}^{\infty}(-1)^n x^{n+2}, \quad |x|<1.$

② $g(x) = \frac{x}{(1-x)^2}$

we know $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \quad |x|<1$

$\Rightarrow \left(\frac{1}{1-x}\right)' = \left(\sum_{n=0}^{\infty} x^n\right)'$

$\frac{+1}{(1-x)^2} = \sum_{n=1}^{\infty} n x^{n-1}$

multiply by x : $\dfrac{x}{(1-x)^2} = \sum_{n=1}^{\infty} n x^n,$

$|x| < 1.$

(3) $h(x) = \dfrac{1}{x}$

Ans. $\dfrac{1}{x} = \dfrac{1}{1-(1-x)}$

$= \sum_{n=0}^{\infty} (1-x)^n, \quad |1-x| < 1$

$= \sum_{n=0}^{\infty} (1-x)^n, \quad -1 < 1-x < 1$

$-2 < -x < 0$

$0 < x < 2$

$\therefore \dfrac{1}{x} = \sum_{n=0}^{\infty} (1-x)^n, \quad 0 < x < 2.$

10.8 Taylor and Maclaurin Series

· Series Representation

Question: If a function $f(x)$ has derivatives of all orders on an interval I, Can it be expressed as a power series on I? And if it Can, what will its coefficients be?

Ans: Assume that $f(x) = \sum_{n=0}^{\infty} a_n (x-a)^n$

$$= a_0 + a_1(x-a) + a_2(x-a)^2 + \cdots$$

$$f'(x) = a_1 + 2a_2(x-a) + 3a_3(x-a)^2 + \cdots$$

$$f''(x) = 2a_2 + 6a_3(x-a) + 12a_4(x-a)^2 + \cdots$$

Notice that $f(a) = a_0$, $f'(a) = a_1$, $f''(a) = 2a_2 = 2!\,a_2$

$$f'''(a) = 6a_3 = 3!\,a_3$$

$$\vdots$$

$$f^{(n)}(a) = n!\,a_n \Rightarrow \boxed{a_n = \frac{f^{(n)}(a)}{n!}}$$

$$\therefore f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots$$

$$+ \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots$$

Taylor and Maclaurin series

Df's: let f be a function with derivatives of all order throughout some interval containing a. Then the Taylor series generated by f at $x=a$ is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

the Maclaurin series generated by f is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + f'(0)x + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots$$

(the Taylor series generated by f at $x=0$).

Ex. find the Taylor series generated by $f(x) = \frac{1}{x}$ at $a=2$. Does the series converge to $\frac{1}{x}$?

Ans: we need to find $f(2)$, $f'(2)$, $f''(2)$, ...

$f(x) = x^{-1}$, $f'(x) = -x^{-2}$, $f''(x) = 2x^{-3}$, ...

$f(2) = \frac{1}{2}$, $f'(2) = \frac{-1}{4} = -\frac{1}{2^2}$, $f''(2) = \frac{2}{2^3} = \frac{1}{4}$

$c_0 = f(2) = \frac{1}{2}$, $c_1 = f'(2) = -\frac{1}{2^2}$

$c_2 = \frac{f''(2)}{2!} = \frac{1}{2^3}, \ldots, \boxed{c_n = \frac{f^{(n)}(2)}{n!} = (-1)^n \frac{1}{2^{n+1}}}$

the Taylor series is $\sum_{n=0}^{\infty} \frac{f^{(n)}(2)}{n!} (x-2)^n$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{(x-2)^n}{2^{n+1}}.$$

$$= \frac{1}{2} - \frac{(x-2)}{2^2} + \frac{(x-2)^2}{2^3} - \cdots$$

this is a geometric series with first term $\frac{1}{2}$ and ratio $r = -\frac{(x-2)}{2}$. It converges absolutely for $|x-2| < 2$ or $0 < x < 4$.

and it sum is $\frac{1}{1 + \frac{x-2}{2}} = \frac{1}{\frac{2+x-2}{2}} = \frac{2}{x}$.

$\therefore$ the Taylor series generated by $f(x) = \frac{1}{x}$ at $a = 2$ converges to $\frac{1}{x}$ for $0 < x < 4$.

Ex. Find the Maclaurin series generated by $f(x) = e^x$.

sol. $f(0) = e^0 = 1$, $f'(x) = e^x$, $f''(x) = e^x, \ldots$

$f^{(n)}(0) = 1$, for every $n = 0, 1, 2, \ldots$

$\therefore$ the Maclaurin series for $f(x) = e^x$ is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{1}{n!} x^n = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots$$

Taylor Polynomials

Df: Let f be a function with derivatives of order k for $k = 1, 2, \ldots, N$ in some interval containing a. Then, for any integer n from 0 through N, the Taylor polynomial of order n generated by f at $x = a$ is the polynomial

$$P_n(x) = f(a) + f'(a)(x-a) + \cdots + \frac{f^{(k)}(a)}{k!}(x-a)^k + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

Rmk: We speak of a Taylor polynomial of order n rather than degree n because $f^{(n)}(a)$ may be zero.

For example: $f(x) = \cos x$, $a = 0$.

$f(0) = 1$, $f'(x) = -\sin x$, $f''(x) = -\cos x$

$f'(0) = 0$, $f''(0) = -1$.

$P_0(x) = f(0) = 1$, $P_1(x) = f(0) + f'(0)x$
$= 1 + 0 \cdot x = 1$

$P_1(x) = 1$ is the first order Taylor poly. ~~of~~ generated by $f(x) = \cos x$ has degree 0 not 1.

Ex. Find the Taylor polynomial generated by $f(x) = e^x$ at $x = 0$.

Sol. $P_n(x) = f(0) + f'(0)x + \cdots + \frac{f^{(n)}(0)}{n!}x^n$

$$= 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}$$

$P_0(x) = 1$, $P_1(x) = 1 + x$, $P_2(x) = 1 + x + \frac{x^2}{2}$.

Ex. Find the Taylor series and Taylor polynomials generated by $f(x) = \cos x$ at $x = 0$.

Sol. $f(x) = \cos x$, $f'(x) = -\sin x$

$f''(x) = -\cos x$ $\quad$ $f'''(x) = \sin x$

$f^{(4)}(x) = \cos x$ $\quad$ $f^{(5)}(x) = -\sin x$

$f^{(2n)}(x) = (-1)^n \cos x$, $f^{(2n+1)}(x) = (-1)^{n+1} \sin x$.

At $x=0$, $f^{(2n)}(0) = (-1)^n \cos 0 = (-1)^n$.

$f^{(2n+1)}(0) = 0$.

The Taylor series generated by f at 0 (or the Maclaurin series) is

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots + \frac{f^{(n)}(0)x^n}{n!} + \cdots$$

$$= 1 + 0\cdot x - \frac{x^2}{2!} + 0\cdot x^3 + \frac{x^4}{4!} + \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots$$

$$= \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}$$

The Taylor polynomials of order $2n$ and $2n+1$ are identical

$$P_{2n}(x) = P_{2n+1}(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^n \frac{x^{2n}}{(2n)!}$$

ex. $P_2(x) = P_3(x) = 1 - \frac{x^2}{2!}$

In the next section we will see that the series $\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$ converges to $\cos x$ for every x. Also, $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges to e^x for every x. But this is not the case ingeneral.

Ex. $f(x) = \begin{cases} 0 & x = 0 \\ e^{-\frac{1}{x^2}} & , x \neq 0 \end{cases}$.

You can show that $f^{(n)}(0) = 0$, for all n.

this means the Taylor series generated by f at $x=0$ is $f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots$

$$= 0 + 0 \cdot x + 0 \cdot x^2 + \cdots + 0 \cdot x^n + \cdots$$

$$= 0 + 0 + 0 + \cdots + 0 + \cdots$$

The series conv. for every x (its sum is 0).

but converges to $f(x)$ only at $x=0$.

That is, the Taylor series generated by $f(x)$ in this example is not equal to $f(x)$ itself. the question still remain for what values of x can we normally expect a Taylor series to converge to its generating function?

see sec. 10.9

10.9 Convergence of Taylor series

Taylor's Formula

If f has derivatives of all orders in an open interval I containing a, then for each positive integer n and $\forall x \in I$,

$$f(x) = f(a) + f'(a)(x-a) + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n(x),$$

where $R_n(x) = \dfrac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$ for some c between a and x

$$\therefore f(x) = \underbrace{P_n(x)}_{\text{Taylor Polynomial of order } n} + \underbrace{R_n(x)}_{\text{Remainder of order } n.}, \quad \forall x \in I$$

If $R_n(x) \longrightarrow 0$ as $n \to \infty$, $\forall x \in I$, we say the Taylor series generated by f at $x=a$ converges to f on I, & we write

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k.$$

Ex. Show that the Taylor series for $f(x) = e^x$ at $x=0$ converges to e^x, $\forall x$.

Solution. $e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + R_n(x)$, $x \in \mathbb{R}$

where $R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!} x^{n+1}$, c is between 0 and

$$= \frac{e^c x^{n+1}}{(n+1)!}$$

$$\Rightarrow |R_n(x)| = \frac{e^c |x|^{n+1}}{(n+1)!} \leq \frac{1 \cdot |x|^{n+1}}{(n+1)!}, \text{ if } x \leq 0 \quad (\text{since } e^c < 1)$$

$$|R_n(x)| = \frac{e^c |x|^{n+1}}{(n+1)!} < e^x \frac{x^{n+1}}{(n+1)!}, \text{ if } x > 0 \quad (\text{since } e^c < e^x)$$

because, $\lim_{n \to \infty} \frac{x^{n+1}}{(n+1)!} = 0$, for every x,

then $\lim_{n \to \infty} R_n(x) = 0$, and the series conv. to e^x,

and therefore $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^k}{k!} + \cdots$

ex. Find $\sum_{n=0}^{\infty} \frac{1}{n!} = e^1 = e$.

Estimating the Remainder

Thm If there is an $M > 0$ such that $|f^{(n+1)}(t)| \le M$ for all t between x and a, then

$$|R_n(x)| \le M \frac{|x-a|^{n+1}}{(n+1)!} \quad (*)$$

If $(*)$ holds for every n and other condition of Taylor's thm are satisfied by f, then the series conv. to $f(x)$.

Ex. show that the Taylor series for $f(x) = \sin x$ at $x=0$ converges for all x.

Sol. $f(x) = \sin x$, $f'(x) = \cos x$

$f''(x) = -\sin x$, $f'''(x) = -\cos x$

$\vdots \qquad \vdots$

 $f^{(2k)}(x) = (-1)^k \sin x$ $\qquad f^{(2k+1)}(x) = (-1)^k \cos x$

So, $f^{(2k)}(0) = 0$, $f^{(2k+1)}(0) = (-1)^k$.

$$\therefore \sin x = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + R_n(x).$$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + \frac{(-1)^k x^{2k+1}}{(2k+1)!} + R_{2k+1}(x),$$

$$R_{2k+1}(x) = \frac{f^{(2k+2)}(0)}{(2k+2)!} x^{2k+2}$$

$$\therefore \; |R_{2k+1}(x)| \le \frac{1 \cdot |x|^{2k+2}}{(2k+2)!} \longrightarrow 0 \text{ as } k \to \infty$$

So, $R_{2k+1}(x) \longrightarrow 0$ as $k \to \infty$.

thus, $\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$

for $-\infty < x < \infty$

Ex. Show that the Taylor series for $\cos x$ at $x=0$ converges to $\cos x$ for every x.

Sol. $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^k \frac{x^{2k}}{(2k)!} + R_{2k}(x),$

$$|R_{2k}(x)| \le \frac{1 \cdot |x|^{2k+1}}{(2k+1)!} \longrightarrow 0 \text{ as } k \to \infty$$

$$\Rightarrow \lim_{k \to \infty} R_{2k}(x) = 0 .$$

$$\therefore \cos x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

ex. Find the first few terms of the Taylor series for

ⓐ $f(x) = \frac{1}{3}(2x + x\cos x)$ ⓑ $g(x) = e^x \cos x$.

Sol. ⓐ $f(x) = \frac{2}{3}x + \frac{1}{3}x\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right)$

$= \frac{2}{3}x + \frac{1}{3}x - \frac{1}{6}x^3 + \frac{x^5}{3 \cdot 4!}$

$= x - \frac{1}{6}x^3 + \frac{x^5}{72} - \cdots$

ⓑ $g(x) = e^x \cos x$

$= \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right)$

$= \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\right)$

$- \left(\frac{x^2}{2!} + \frac{x^3}{2!} + \frac{x^4}{2!\,2!} + \frac{x^5}{2!\,3!} + \cdots\right)$

$+ \left(\frac{x^4}{4!} + \frac{x^5}{4!} + \frac{x^6}{2!\,4!} + \cdots\right) + \cdots$

$= 1 + x + \left(\frac{1}{6} - \frac{1}{2}\right)x^3 + \left(\frac{1}{4!} - \frac{1}{2!\,2!} + \frac{1}{4!}\right)x^4 + \cdots$

$= 1 + x - \frac{1}{3}x^3 - \frac{1}{6}x^4 + \cdots$

ex. Find the maclaurin series for $f(x) = x^3 e^{x^2}$.

Sol. $f(x) = x^3 e^{x^2} = x^3 \sum_{n=0}^{\infty} \frac{x^n}{n!}$, $-\infty < x < \infty$

$= \sum_{n=0}^{\infty} \frac{x^{n+3}}{n!}$, $-\infty < x < \infty$.

ex. Find the sum $S = \sum_{n=0}^{\infty} (-1)^n \frac{\pi^{2n}}{2 \cdot n!}$

Sol. $S = \frac{1}{2} \sum_{n=0}^{\infty} (-1)^n \frac{(\pi^2)^n}{n!} = \frac{1}{2} \sum_{n=0}^{\infty} \frac{(-\pi^2)^n}{n!}$

$= \frac{1}{2} e^{-\pi^2}$.

ex. $\sum_{n=0}^{\infty} (-1)^n \frac{\pi^{2n}}{2n!} = \cos\pi = -1$.

ex. Let $e^x \sim 1 + x + \frac{x^2}{2!}$, and $|x| < 0.1$

Estimate the error

Sol. $R_2(x) = \frac{f'''(c)}{3!} x^3$, c is between 0 & x.

$= \frac{e^c x^3}{3!}$

$|R_2(x)| \leq \frac{e^c |x|^3}{3!} < e^{0.1} \frac{(0.1)^3}{3!}$

Ex. Estimate the error if $\cos x \sim 1 - \frac{x^2}{2!}$, $|x| < 0.1$

$|E| < 1 \cdot \frac{|x|^4}{4!} < \frac{(0.1)^4}{4!}$, $E > 0$

$\Rightarrow 0 \leq E < \frac{1}{24} \times 10^{-4}$.

ex. For what values of x can we replace $\sin x$ by $x - \frac{x^3}{3!}$ with an error $|E| < 3 \times 10^{-4}$?

Sol. According to the ASET (Alternating Series Estimating thm sec. 10.6)

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

$$|E| < \frac{|x|^5}{5!} < 3 \times 10^{-4} \quad \text{or} \quad |x| < \sqrt[5]{360 \times 10^{-4}}$$

$$\approx 0.514.$$

ex. Estimate $\tan^{-1}(0.1)$ using $P_3(x)$, then estimate the error

sol: $$\tan^{-1}(0.1) = (0.1) - \frac{(0.1)^3}{3} = \frac{1}{10} - \frac{1}{3000}$$

$$= \frac{299}{3000}$$

$$|E| < \left|\frac{x^5}{5}\right| = \frac{|x|^5}{5} = \frac{(0.1)^5}{5} = 2 \times 10^{-6}$$

Rmk. $$\tan^{-1} x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{2k+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

10.10 The Binomial series and Applications of Taylor series

. The Binomial series

For $-1 < x < 1$, $(1+x)^m = 1 + \sum_{k=1}^{\infty} \binom{m}{k} x^k$,

where $\binom{m}{1} = m$, $\binom{m}{2} = \frac{m(m-1)}{2!}$,

and $\binom{m}{k} = \frac{m(m-1) \cdots (m-(k-1))}{k!}$, $k \geq 3$.

Ex. $\binom{-1}{1} = -1$, $\binom{-1}{2} = \frac{-1(-2)}{2!} = 1$

$$\binom{-1}{k} = \frac{-1(-2)(-3) \cdots (-1-(k-1))}{k!}$$

$$= \frac{(-1)(-2)(-3) \cdots (-k)}{k!} = \frac{(-1)^k k!}{k!}$$

$$= (-1)^k .$$

ex. $(1+x)^{-1} = 1 + \sum_{k=1}^{\infty} \binom{-1}{k} x^k$

$$= 1 + \sum_{k=1}^{\infty} (-1)^k x^k = \sum_{k=0}^{\infty} (-1)^k x^k$$

Ex. Find the first four nonzero terms for the Binomial series.

① $f(x) = \dfrac{1}{\sqrt[3]{1+x}} = (1+x)^{-\frac{1}{3}}$

$$= 1 + \sum_{k=1}^{\infty} \binom{-\frac{1}{3}}{k} x^k$$

$$= 1 + \binom{-\frac{1}{3}}{1} x + \binom{-\frac{1}{3}}{2} x^2 + \binom{-\frac{1}{3}}{3} x^3 + \cdots$$

$$= 1 - \frac{1}{3}x + \frac{-\frac{1}{3}\left(-\frac{1}{3}-1\right)}{2!} x^2 + \frac{\left(-\frac{1}{3}\right)\left(-\frac{1}{3}-1\right)\left(-\frac{1}{3}-2\right)}{3!} x^3 + \cdots$$

$$= 1 - \frac{1}{3}x + \frac{\left(-\frac{1}{3}\right)\left(-\frac{4}{3}\right)}{2} x^2 + \frac{\left(-\frac{1}{3}\right)\left(-\frac{4}{3}\right)\left(-\frac{7}{3}\right)}{6} x^3 + \cdots$$

$$= 1 - \frac{1}{3}x + \frac{2}{9}x^2 - \frac{14}{81}x^3 + \cdots$$

② $g(x) = \dfrac{x^3}{\sqrt[3]{1+x}}$

$$= x^3(1+x)^{-\frac{1}{3}}$$

part ① $= x^3\left[1 - \frac{1}{3}x + \frac{2}{9}x^2 - \frac{14}{81}x^3 + \cdots\right.$

$$= x^3 - \frac{1}{3}x^4 + \frac{2}{9}x^5 - \frac{14}{81}x^6 + \cdots$$

• Evaluating Nonelementary integrals.

ex. find $I = \int \sin(x^2)\, dx$.

Sol. $I = \int \left(x^2 - \frac{(x^2)^3}{3!} + \frac{(x^2)^5}{5!} - \cdots \right) dx$

$$= C + \frac{x^3}{3} - \frac{x^7}{7 \cdot 3!} + \frac{x^{11}}{11 \cdot 5!} - \frac{x^{15}}{15 \cdot 7!} + \cdots$$

ex. Estimate $\int_0^1 \tan^{-1} x\, dx$ with $|E| < 0.02$.

Sol. $\int_0^1 \tan^{-1} x\, dx = \int_0^1 \left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \right) dx$

$$= \frac{x^2}{2} - \frac{x^4}{12} + \frac{x^6}{30} - \frac{x^8}{56} + \cdots \Big|_0^1$$

$$= \frac{1}{2} - \frac{1}{12} + \frac{1}{30} - \frac{1}{56} + \cdots \quad (A.S)$$

$$\approx \frac{1}{2} - \frac{1}{12} + \frac{1}{30} \quad \left(\text{since } \frac{1}{56} < 0.02 \right)$$

$$= \frac{27}{60}.$$

• Evaluating Indeterminate Forms

$\left(\frac{0}{0}\right), \left(\frac{\infty}{\infty}\right), (\infty - \infty), (0 \cdot \infty)$

Ex. Evaluate $\lim_{x \to 1} \frac{\ln x}{x-1}$ $\left(\frac{0}{0}\right)$

Recall, $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{n-1}\frac{x^n}{n} + \cdots$,

$-1 < x \leq 1.$

Replace x by $x-1$:

$$\ln x = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \cdots + (-1)^{n-1}\frac{(x-1)^n}{n} + \cdots$$

$$\therefore \lim_{x \to 1} \frac{\ln x}{x-1} = \lim_{x \to 1}\left(1 - \frac{(x-1)}{2} + \frac{(x-1)^2}{3} - \cdots\right)$$

$$= 1.$$

Ex. Evaluate $\lim_{x \to 0}\left(\frac{1}{\sin x} - \frac{1}{x}\right)$

Sol. $= \lim_{x \to 0}\left(\frac{x - \sin x}{x \sin x}\right)$ $\left(\frac{0}{0}\right)$

$$= \lim_{x \to 0} \frac{x - \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)}{x\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)}$$

$$= \lim_{x \to 0} \frac{x^{\cancel{3}}\left(\frac{1}{3!} - \frac{x^2}{5!} + \cdots\right)}{\cancel{x^2}\left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \cdots\right)} = 0$$

Euler's Identity $\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$

ex: $e^{2+\frac{\pi}{3}i}$

$$= e^2 \, e^{\frac{\pi}{3}i}$$

$$= e^2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$$

$$= e^2\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)$$

$$= \frac{e^2}{2} + \frac{\sqrt{3}\,e^2}{2}i \, .$$

Proof (Euler's Identity)

$$e^{i\theta} = \sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!}$$

$$= 1 + (i\theta) + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \cdots$$

$$= 1 + i\theta - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + i\frac{\theta^5}{5!} + \cdots$$

$$= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!}\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right)$$

$$= \cos\theta + i\sin\theta$$

Please see Table 10.1 Frequently used Taylor series. page 602