Birzeit University

Mathematics Department

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Title: Mathematical Analysis II

CHAPTER 7

Infinite Series of Functions

7.1 UNIFORM CONVERGENCE OF SEQUENCES

set = E ✓ $a_n = \frac{1}{n}$ ✓

$f_n(x) = \frac{x}{n}$

$\{f_n(x)\}_{n=1}^{\infty}$ $f_1(x) = x$

$\{x, \frac{x}{2}, \frac{x}{3}, \ldots\}$ $f_2(x) = \frac{x}{2}$

7.1 Definition. $\phi \neq E \subseteq \mathbb{R}$.

Let E be a nonempty subset of $\mathbf{R}$. A sequence of functions $f_n : E \to \mathbf{R}$ is said to *converge pointwise* on E (notation: $f_n \to f$ pointwise on E as $n \to \infty$) if and only if $f(x) = \lim_{n\to\infty} f_n(x)$ exists for each $x \in E$.

$f_n(x) \longrightarrow f(x), \forall x \in E$ $\{f_n(x)\}$ conv.

7.2 Remark. *Let E be a nonempty subset of $\mathbf{R}$. Then a sequence of functions f_n converges pointwise on E, as $n \to \infty$, if and only if for every $\varepsilon > 0$ and $x \in E$ there is an $N \in \mathbf{N}$ (which may depend on x as well as ε) such that*

$$\boxed{n \geq N \quad \text{implies} \quad |f_n(x) - f(x)| < \varepsilon.}$$

$\{f_n(x)\}$

Proof. (·

$f_n \longrightarrow f$ pointwise on E.

تذكر 2.1

$\Longleftrightarrow$ by Def'n 7.1, $f_n(x) \longrightarrow f(x), \forall x \in E$.

$\Longleftrightarrow$ $\{f_n(x)\}$ converges to $f(x)$, $\forall x \in E$.

$\Longleftrightarrow$ by def 2.1, $\forall \varepsilon > 0$ and $x \in E$, $\exists N \in \mathbb{N}$ s.t $n \geq N \Longrightarrow |f_n(x) - f(x)| < \varepsilon$

7.3 Remark. *The pointwise limit of continuous (respectively, differentiable) functions is not necessarily continuous (respectively, differentiable).*

proof. let $f_n(x) = x^n$, $E = [0, 1]$

$$\lim_{n\to\infty} f_n(x) = \begin{cases} 0, & 0 \leq x < 1 \\ 1, & x = 1 \end{cases} = f(x).$$

Notice $f_n \longrightarrow f$ pointwise on $[0, 1]$.

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each f_n is continuous and diff'ble on $[0,1]$ but f is neither diff'ble nor continuous at $x=1$. ∎

7.4 Remark. *The pointwise limit of integrable functions is not necessarily integrable.*

Pf. $f_n(x) = \begin{cases} 1, & x = \frac{p}{m} \in \mathbb{Q} \text{ in reduced form, } m \le n \\ 0, & \text{otherwise,} \end{cases}$

$\forall n \in \mathbb{N}$.

راجع 5.1, 5.2

Notice that each f_n is integrable on $[0,1]$ $\left(\int_0^1 f_n(x)\,dx = 0 \right)$.

and $f_n \longrightarrow f$ pointwise, where

$f(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & \text{otherwise} \end{cases}$ is not integrable on $[0,1]$ (see 5.1).

7.5 Remark. *There exist differentiable functions f_n and f such that $f_n \to f$ pointwise on* $[0, 1]$ *but*

$$\boxed{\lim_{n\to\infty} f_n'(x) \neq \left(\lim_{n\to\infty} f_n(x)\right)'} \qquad (1)$$

for $x = 1$.

Proof. $f_n(x) = \dfrac{x^n}{n}$, $[0,1]$

$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \dfrac{x^n}{n} = 0 = f(x), \forall x \in [0,1]$.

$\therefore f_n \longrightarrow f$ pointwise on $[0,1]$.

$$f_n'(x) = \frac{n x^{n-1}}{n} = x^{n-1}, \quad f_n'(1) = (1)^{n-1} = 1$$

Now, $\lim_{n\to\infty} f_n'(x) = \lim_{n\to\infty} 1 = 1$ at $x = 1$.

but $\left(\lim_{n\to\infty} f_n(x)\right)' = f'(x) = 0$ since $f(x) = 0$.

$$\therefore \lim_{n\to\infty} f_n'(x) \neq \left(\lim_{n\to\infty} f_n(x)\right)'$$

7.6 Remark. *There exist continuous functions f_n and f such that $f_n \to f$ pointwise on $[0, 1]$ but*

$$\lim_{n\to\infty} \int_0^1 f_n(x)\,dx \neq \int_0^1 \left(\lim_{n\to\infty} f_n(x)\right) dx. \qquad (2)$$

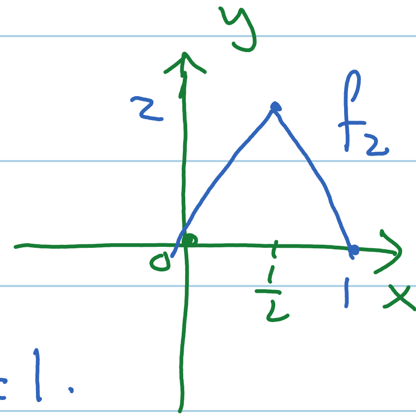
Pf. Let $f_1(x) = 1$, and for $n \geq 2$, $\{f_1, f_2, f_3, \ldots\}$

$n=1$ $\qquad = \{1,$

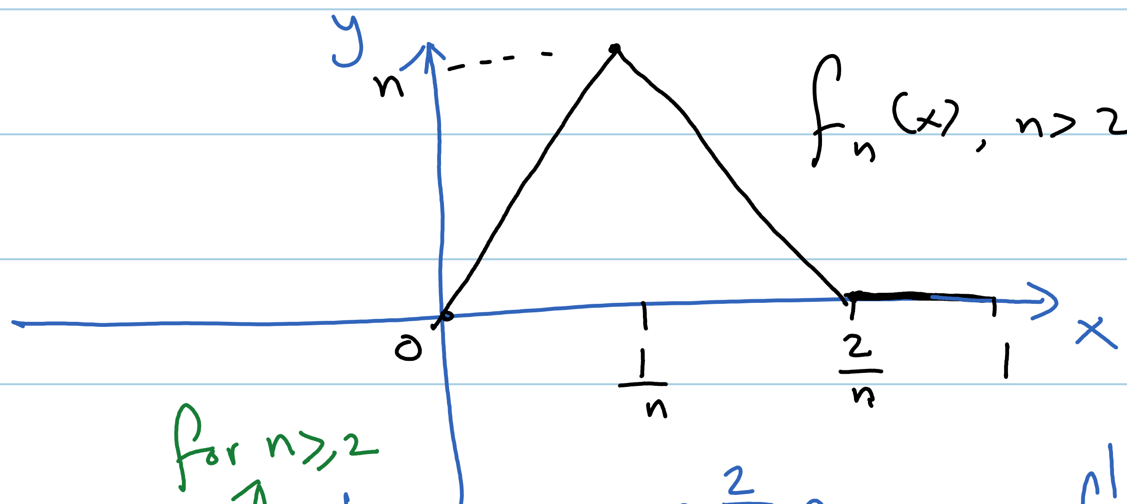
$$f_n(x) = \begin{cases} n^2 x, & 0 \leq x < \frac{1}{n} \\ 2n - n^2 x, & \frac{1}{n} \leq x < \frac{2}{n} \\ 0, & \frac{2}{n} \leq x \leq 1 \end{cases}$$

مثلاً

$$f_2(x) = \begin{cases} 4x, & 0 \leq x < \frac{1}{2} \\ 4 - 4x, & \frac{1}{2} \leq x \leq 1 \end{cases}$$



$$\int_0^1 f_2(x)\,dx = \frac{1}{2}(1)(2) = 1.$$



for $n \geq 2$

Notice that $\int_0^1 f_n(x)\,dx = \int_0^{\frac{2}{n}} f_n(x)\,dx + \cancel{\int_{\frac{2}{n}}^1 f_n(x)\,dx} = \frac{1}{2}\left(\frac{2}{n}\right)(n) = 1$

$$\therefore \int_0^1 f_n(x) = 1, \quad \forall n \in \mathbb{N}$$

$$\lim_{n\to\infty} \int_0^1 f_n(x)\,dx = \lim_{n\to\infty} 1 = 1.$$

$f_n \to f=0$ pointwise

but $$\int_0^1 \left(\lim_{n\to\infty} f_n(x)\right) dx = \int_0^1 0\,dx = 0$$

$$\Longrightarrow \lim_{n\to\infty} \int_0^1 f_n(x)\,dx \neq \int_0^1 \lim_{n\to\infty} f_n(x)\,dx \quad \square$$

7.7 Definition.

Let E be a nonempty subset of $\mathbf{R}$. A sequence of functions $f_n : E \to \mathbf{R}$ is said to *converge uniformly* on E to a function f (notation: $f_n \to f$ uniformly on E as $n \to \infty$) if and only if for every $\varepsilon > 0$ there is an $N \in \mathbf{N}$ such that

$$n \geq N \quad \text{implies} \quad |f_n(x) - f(x)| < \varepsilon$$

for all $x \in E$.

Rmk ①. the only difference between Uniform convergence and pointwise convergence is that, for uniform convergence, N must be chosen independtly of x.

② If f_n conv. uniformly on E, then f_n converges pointwise on E.

But the converse is false.

③ To prove $f_n \to f$ uniformly on a set E: Dominate $|f_n(x) - f(x)|$ by constant seq. b_n, independent of x, $b_n \to 0$ as $n \to \infty$.

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7.8 *EXAMPLE.*

Prove that $x^n \to 0$ uniformly on $[0, b]$ for any $b < 1$, and pointwise, but not uniformly, on $[0, 1)$.

Proof. • $\underset{f_n(x)}{x^n} \longrightarrow \underset{f(x)}{0}$ uniformly on $[0,b]$, $b<1$:

Let $\varepsilon > 0$. Then,

$$|f_n(x) - f(x)| = |x^n - 0| = |x^n| \le b^n$$

and $b^n \to 0$ as $n \to \infty$ since $b<1$

$\therefore$ $x^n \to 0$ uniformly for $x \in [0,b]$.

• $\boxed{x^n \not\to 0 \text{ uniformly on } [0,1)}$

Spse $x^n \to 0$ " " ". For $0 < \varepsilon < \frac{1}{2}$, $\exists\ N \in \mathbb{N}$ such that

$$|x^n - 0| < \varepsilon, \ \forall\, n \ge N.$$

In particular, $\boxed{|x^N| < \varepsilon}$, $\forall x \in [0,1)$.

Since $\lim_{x \to 1^-} x^N = 1$, $\exists\ x_0 \in (0,1)$ such that $\boxed{x_0^N > \varepsilon}$ (See the Figure next page).

Thus, $\varepsilon < x_0^N < \varepsilon$ (i.e, $\varepsilon < \varepsilon$), a contradiction.

$\therefore$ $x^n \not\to 0$ uniformly. ∎

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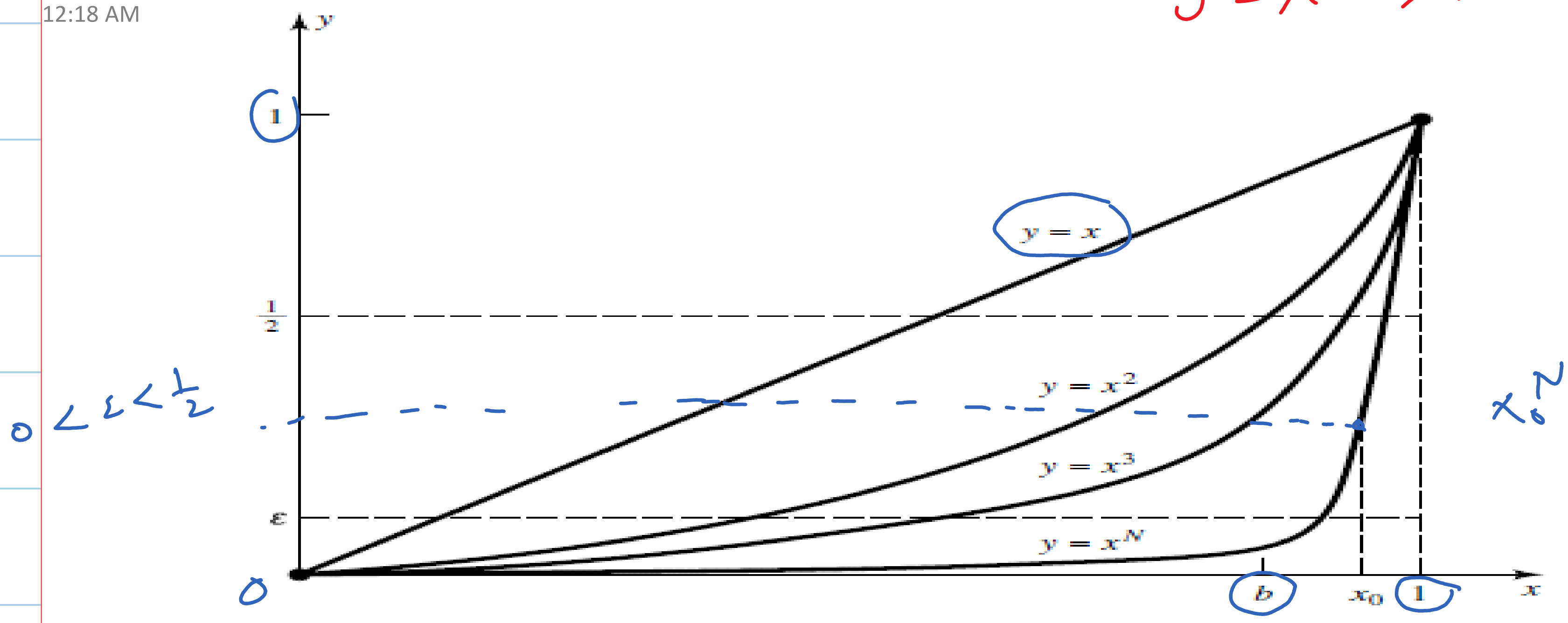
$y = x^N$, $N = 1, 2 \ldots$



$x_0 \in (b, 1) \subseteq (0, 1)$

$b < 1$

Question: If $f_n \to f$ uniformly on E.

each f_n is continuous at $x_0 \in E$

$\Longrightarrow$ f is cont. at $x_0 \in E$. ??

Ans. Yes.

7.9 Theorem. *Let E be a nonempty subset of* **R** *and suppose that $f_n \to f$ uniformly on E, as $n \to \infty$. If each f_n is continuous at some $x_0 \in E$, then f is continuous at $x_0 \in E$.*

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3.3

Proof. Let $\varepsilon > 0$, we need to find a $\delta > 0$

s.t. if $|x - x_0| < \delta$, then $|f(x) - f(x_0)| < \varepsilon$.

Since $f_n \to f$ uniformly, $\exists\, N \in \mathbb{N}$ s.t.,

$$\boxed{n \geq N,\ x \in E \Longrightarrow |f_n(x) - f(x)| < \frac{\varepsilon}{3}}$$

Since f_N is continuous at $x_0 \in E$, $\exists\, \delta > 0$ such that

$$\boxed{|x - x_0| < \delta \text{ and } x \in E \Longrightarrow |f_N(x) - f_N(x_0)| < \frac{\varepsilon}{3}}$$

Suppose $|x - x_0| < \delta$, $x \in E$. then triangle inequality.

$$|f(x)-f(x_0)| \le |f(x)-f_N(x)| + |f_N(x)-f_N(x_0)| + |f_N(x_0)-f(x_0)|$$

unif.

$$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$$

Thus, f is continuous at $x_0 \in E$. ∎

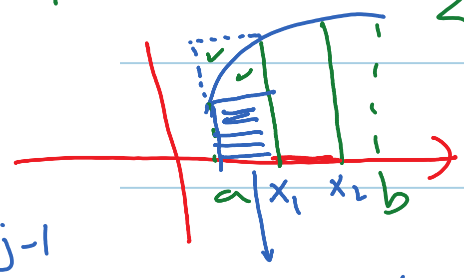
Recall, (Section 5.1)

Let $a, b \in \mathbf{R}$ with $a < b$. A function $f : [a, b] \to \mathbf{R}$ is said to be (*Riemann*) *integrable* on $[a, b]$ if and only if f is bounded on $[a, b]$, and for every $\varepsilon > 0$ there is a partition P of $[a, b]$ such that $U(f, P) - L(f, P) < \varepsilon$.

+ve

$|U(f,P) - L(f,P)| < \varepsilon$

$x_0 = a \quad x_1 \; - - - - \; b = x_n$



$$U(f, P) := \sum_{j=1}^{n} M_j(f)\, \Delta x_j,$$

$x_j - x_{j-1}$

$\Delta x_1 = x_1 - x_0$

where

$$M_j(f) := \sup f([x_{j-1}, x_j]) := \sup_{t \in [x_{j-1}, x_j]} f(t).$$

$M_1(f) = \sup_{x \in [x_0, x_1]} f(x)$

$$L(f, P) := \sum_{j=1}^{n} m_j(f)\, \Delta x_j,$$

where

$$m_j(f) := \inf f([x_{j-1}, x_j]) := \inf_{t \in [x_{j-1}, x_j]} f(t).$$

$f_1, f_2, \ldots$

7.10 Theorem. *Suppose that $f_n \to f$ uniformly on a closed interval $[a, b]$. If each f_n is integrable on $[a, b]$, then so is f and* → f is integrable and

$$\lim_{n\to\infty} \int_a^b f_n(x)\,dx = \int_a^b \left(\lim_{n\to\infty} f_n(x)\right) dx.$$

In fact, $\lim_{n\to\infty} \int_a^x f_n(t)\,dt = \int_a^x f(t)\,dt$ *uniformly for* $x \in [a, b]$.

Proof.

- f is bounded on $[a,b]$. (Exercise 7.1.3)
- f is integrable on $[a,b]$:

Let $\varepsilon > 0$, $\exists\, N \in \mathbb{N}$ s.t

$$\boxed{n \geq N \implies |f_n(x) - f(x)| < \frac{\varepsilon}{3(b-a)}} \quad (3)$$

for all $x \in [a,b]$ (since $f_n \to f$ uniformly).

In particular, $|f(x) - f_N(x)| < \dfrac{\varepsilon}{3(b-a)}$ ($n = N$ in (3)).

$$U(f - f_N, P) = \sum_{j=1}^{n} M_j(f - f_N)\,\Delta x_j,$$

$$M_j(f - f_N) = \sup_{x \in [x_{j-1}, x_j]} |f(x) - f_N(x)| \leq \frac{\varepsilon}{3(b-a)}$$

$$U(f - f_N, P) \leq \sum_{j=1}^{n} \frac{\varepsilon}{3(b-a)}\,\Delta x_j$$

$$= \frac{\varepsilon}{3(b-a)} \sum_{j=1}^{n} \Delta x_j$$

$$= \frac{\varepsilon}{3\cancel{(b-a)}} \cdot \cancel{(b-a)} = \frac{\varepsilon}{3}.$$

$\therefore\ U(f - f_N, P) \leq \dfrac{\varepsilon}{3}$. $\quad$ (i)

Similarly, $L(f - f_N, P) \geq -\dfrac{\varepsilon}{3}$. (Exercise).

for any Partition P of $[a,b]$.

Since f_N is integrable, $\exists$ a partion P such that $\boxed{U(f_N,P) - L(f_N,P) < \frac{\varepsilon}{3}}$ (ii)

It follows that

$$\left.\begin{array}{l} \sup(f+g) \le \sup f + \sup g \\ \inf(f+g) \ge \inf f + \inf g \end{array}\right\} \text{(H.w)}$$

$$U(f,P) - L(f,P) = U(\underbrace{f-f_N}+f_N,P) - L(\underbrace{f-f_N}+\underbrace{f_N},P)$$

$$\le \underbrace{U(f-f_N,P)} + U(f_N,P)$$

$$-\underbrace{L(f-f_N,P)} - L(f_N,P)$$

$$= U(f-f_N,P) - L(f-f_N,P)$$

$$+ \underbrace{U(f_N,P) - L(f_N,P)}$$

$$< \overset{(i)}{\frac{\varepsilon}{3}} + \overset{(i)}{\frac{\varepsilon}{3}} + \overset{(ii)}{\frac{\varepsilon}{3}} = \varepsilon.$$

that is f is integrable on $[a,b]$.

- $\underbrace{\int_a^x f_n(t)\,dt}_{g_n(x)} \longrightarrow \underbrace{\int_a^x f(t)\,dt}_{g(x).}$ as $n \to \infty$. :

(we need to prove that $\exists N \in \mathbb{N}$ s.t

$$n \ge N \Longrightarrow \left|\int_a^x f_n(t)\,dt - \int_a^x f(t)\,dt\right| < \varepsilon)$$

Now, $\left|\int_a^x f_n(t)\,dt - \int_a^x f(t)\,dt\right|$

$$= \left|\int_a^x (\underbrace{f_n(t) - f(t)}_{h})\,dt\right|$$

$$\left|\int_a^b h(x)dx\right| \le \int_a^b |h(x)|dx$$

$$\le \int_a^x \underbrace{|f_n(t) - f(t)|}\,dt ,$$

$$\le \int_a^x \frac{\varepsilon}{3(b-a)}\,dt \quad , \text{since } f_n \to f \text{ uniformly.}$$

$$= \frac{\varepsilon}{3(b-a)} \int_a^x dt$$

$$= \frac{\varepsilon}{3(b-a)} \cdot \underbrace{(x-a)}_{< b-a} \quad , \quad \forall x \in [a,b]$$

$$< \frac{\varepsilon}{3\cancel{(b-a)}} \cdot \cancel{(b-a)} < \varepsilon \ , \ \forall x \in [a,b] \text{ and } n \geq N$$

$$\therefore \int_a^x f_n(t)\,dt \longrightarrow \int_a^x f(t).$$

Example. Find $\lim\limits_{n\to\infty} \int_0^2 e^{\frac{x^2}{n}}\,dx$.

Sol. $f_n(x) = e^{\frac{x^2}{n}}$ integrable on $[0,2]$.

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} e^{\frac{x^2}{n}} = e^0 = 1 = f(x)$$

$\cdot$ $f_n(x) = e^{\frac{x^2}{n}} \longrightarrow 1$ uniformly on $[0,2]$.

Let $\varepsilon > 0$, $|f_n(x) - f(x)| = |\underbrace{e^{\frac{x^2}{n}} - 1}_{\text{موجب}}|$ $\quad 0 \leq x \leq 2$

$$= e^{\frac{x^2}{n}} - 1$$

$$\leq e^{\frac{4}{n}} - 1 \quad \text{since } 0 \leq x \leq 2$$

$$\longrightarrow 0 \text{ as } n \to \infty.$$

$\therefore f_n \longrightarrow f$ uniformly on $[0,2]$.

Thm 7.10 $\Rightarrow$

$$\therefore \lim_{n\to\infty} \int_0^2 f_n(x)\,dx = \int_0^2 \left[\lim_{n\to\infty} f_n(x)\right] dx$$

$$= \int_0^2 \left(\lim_{n\to\infty} e^{\frac{x^2}{n}}\right) dx = \int_0^2 1\,dx = 2.$$

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$\{x_n\}$ Cauchy $\forall \varepsilon>0, \exists N\in\mathbb{N}$ s.t
$m,n\geq N \Rightarrow |x_n-x_m|<\varepsilon.$

7.11 ***Lemma.*** [UNIFORM CAUCHY CRITERION].
Let E be a nonempty subset of $\mathbf{R}$ and let $f_n : E \to \mathbf{R}$ be a sequence of functions. Then f_n converges uniformly on E if and only if for every $\varepsilon > 0$ there is an $N \in \mathbf{N}$ such that

$$\boxed{n, m \geq N \quad \text{imply} \quad |f_n(x) - f_m(x)| < \varepsilon} \tag{4}$$

for all $x \in E$.

Pf. ($\Rightarrow$) If $f_n \longrightarrow f$ uniformly on E,
then given $\varepsilon>0$, $\exists N\in\mathbb{N}$ s.t
$n\geq N \Rightarrow |f_n(x)-f(x)|<\frac{\varepsilon}{2}$. Hence,
if both $n,m\geq N$, then

$$|f_n(x)-f_m(x)| = |\underbrace{f_n(x)-f(x)}+\underbrace{f(x)-f_m(x)}|$$
$$\leq |f_n(x)-f(x)|+|f_m(x)-f(x)|$$
$$< \frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon, \ \forall x\in E.$$

$\therefore |f_n(x)-f_m(x)|<\varepsilon, \ \forall m,n\geq N.$

($\Leftarrow$) Conversely, If (4) holds for $x\in E$, i.e.,

$$|\underbrace{f_n(x)}_{\to f(x)}-\underbrace{f_m(x)}_{\to f(x)}|<\varepsilon, \ \forall m,n\geq N, \ x\in E.$$

then $\{f_n(x)\}$ is a Cauchy seq. in $\mathbb{R}$,
therefore it is a convergent seq. that is,
$\lim_{n\to\infty} f_n(x) = f(x)$ exists, $\forall x\in E$.

If we let $m\to\infty$ in (4), we obtain
$$|f_n(x)-f(x)|\leq\frac{\varepsilon}{2}<\varepsilon, \ \forall n\geq N$$

Hence, by def'n, $f_n \to f$ uniformly on E. ∎

Here is a result about interchanging a limit sign and the derivative sign

$\{f_n(x_0)\}$ conv. $x_0 \in (a,b)$

7.12 Theorem. *Let (a, b) be a bounded interval and suppose that f_n is a sequence of functions which converges at some $x_0 \in (a, b)$. If each f_n is differentiable on (a, b), and f_n' converges uniformly on (a, b) as $n \to \infty$, then f_n converges uniformly on (a, b) and*

$$\boxed{\lim_{n\to\infty} f_n'(x) = \left(\lim_{n\to\infty} f_n(x)\right)'}$$

for each $x \in (a, b)$.

Proof. Fix $c \in (a,b)$ and defined

$$g_n(x) = \begin{cases} \dfrac{f_n(x) - f_n(c)}{x-c}, & \boxed{x \neq c} \\ f_n'(c), & x = c \end{cases}, \quad n \in \mathbb{N}.$$

Clearly,

$$\boxed{f_n(x) = f_n(c) + g_n(x)(x-c), \quad n \in \mathbb{N}, \; x \in (a,b)} \quad (5)$$

Claim. g_n conv. uniformly on (a,b).

Pf (claim). Let $\varepsilon > 0$, $m, n \in \mathbb{N}$; $x \in (a,b)$, $\boxed{x \neq c}$.

Apply the mean value thm to the diffrence $\boxed{f_n - f_m}$ on $[c,x]$. We conclude that $\exists\, \eta$ between c and x such that

$$\frac{\big(f_n(x) - f_m(x)\big) - \big(f_n(c) - f_m(c)\big)}{x-c} = (f_n - f_m)'(\eta)$$

$$\Rightarrow \quad g_n(x) - g_m(x) = f_n'(\eta) - f_m'(\eta).$$

Since f_n' conv. uniformly on (a,b), then $\exists$

$N \in \mathbb{N}$ s.t

$$(*) \quad \boxed{m,n \geq N \Longrightarrow |g_n(x) - g_m(x)| < \frac{\varepsilon}{2(b-a)}},$$

for $x \in (a,b)$ with $x \neq c$.

$(*)$ holds for $x = c$ since $g_n(c) = f_n'(c)$ for $n \in \mathbb{N}$.

$\Rightarrow g_n$ conv. uniformly on (a,b).

this proves the claim.

Since $\{f_n(x_0)\}_{n\in\mathbb{N}}$ conv., then it is a Cauchy seq. in $\mathbb{R}$. that is,

$$|f_m(x_0) - f_n(x_0)| < \frac{\varepsilon}{2}, \ \forall m,n \geq N \quad (**)$$

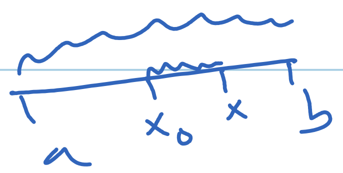
Notice that (5) holds for $c = x_0$ that is,

$$f_n(x) = f_n(x_0) + (x - x_0)\, g_n(x), \quad x \in (a,b).$$

thus, If $m,n \geq N$, we have

$$|f_n(x) - f_m(x)| = |f_n(x_0) + (x - x_0)\, g_n(x) - f_m(x_0) - (x - x_0)\, g_m(x)|$$

$$\leq \underbrace{|f_n(x_0) - f_m(x_0)|}_{(**)} + |x - x_0|\, \underbrace{|g_n(x) - g_m(x)|}_{\text{claim}}$$



$$< \frac{\varepsilon}{2} + (b-a)\frac{\varepsilon}{2(b-a)} = \varepsilon.$$

$\therefore |f_n(x) - f_m(x)| < \varepsilon, \ \forall m,n \geq N.$

$\Rightarrow f_n$ conv. uniformly by uniform Cauchy Criterion.

$$\cdot \left[\lim_{n\to\infty} f_n(x)\right]' = \lim_{n\to\infty} f_n'(x). :$$

Pf. Fix $c \in (a,b)$. Define f, g on (a,b) by $f(x) := \lim_{n\to\infty} f_n(x)$ and $g(x) := \lim_{n\to\infty} g_n(x)$

we need to prove

$$\boxed{f'(c) = \lim_{n\to\infty} f_n'(c)} \quad (6)$$

Since g_n is continuous at $x=c$ and g_n conv. uniformly on (a,b) $(c(a,m))$ as $n\to\infty$ to g, then by thm 7.9, g is continuous at $x=c$.

Since $g_n(c) = f_n'(c)$, then

$$\boxed{\lim_{n\to\infty} f_n'(c) \overset{\text{def. of } g_n}{=} \lim_{n\to\infty} g_n(c) \overset{\text{unif.}}{=} g(c) \overset{\text{continuity of } g \text{ at } c}{=} \lim_{x\to c} g(x)} \quad (i)$$

on the other hand, if $\boxed{x \neq c}$

$$\boxed{\frac{f(x)-f(c)}{x-c}} \overset{f_n\to f \text{ unif.}}{=} \lim_{n\to\infty} \frac{f_n(x)-f_n(c)}{x-c} \overset{\text{def. of } g_n}{=} \lim_{n\to\infty} g_n(x) \overset{g_n \text{ conv. unif.}}{=} \boxed{g(x)}$$

$$\underbrace{\lim_{x\to c} \frac{f(x)-f(c)}{x-c}} = \underbrace{\lim_{x\to c} g(x)}$$

$$\boxed{f'(c) = \lim_{x\to c} g(x)} \quad \dots (ii)$$

$$(i) \text{ and } (ii) \Rightarrow \lim_{n\to\infty} f_n'(c) = f'(c) = \left(\lim_{n\to\infty} f_n(c)\right)'$$

H.w All \ {4,10}.

CHAPTER 8

Euclidean Spaces

8.1 ALGEBRAIC STRUCTURE

For each $n \in \mathbf{N}$, let $\mathbf{R}^n$ denote the n-fold cartesian product of $\mathbf{R}$ with itself; that is,

$$\mathbf{R}^n := \{(x_1, x_2, \ldots, x_n) : x_j \in \mathbf{R} \text{ for } j = 1, 2, \ldots, n\}.$$ $\dim \mathbb{R}^n = n$

By a *Euclidean space* we shall mean $\mathbf{R}^n$ together with the "Euclidean inner product" defined in Definition 8.1 below. The integer n is called the *dimension* of $\mathbf{R}^n$, elements $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ of $\mathbf{R}^n$ are called *points or vectors* or *ordered n-tuples*, and the numbers x_i are called *coordinates* or *components* of $\mathbf{x}$. Two

$$x \cdot y = (x_1, \ldots, x_n) \cdot (y_1, \ldots, y_n)$$
$$= x_1 y_1 + x_2 y_2 + \cdots + x_n y_n.$$

- $x = y \Rightarrow x_i = y_i, \ \forall i = 1, \ldots, n.$
- $0 \in \mathbb{R}^n \Rightarrow 0 = (0, 0, \ldots, 0)$

$n = 2$, $\mathbb{R}^2 = \{(x, y) : x, y \in \mathbb{R}\}$. xy-plane

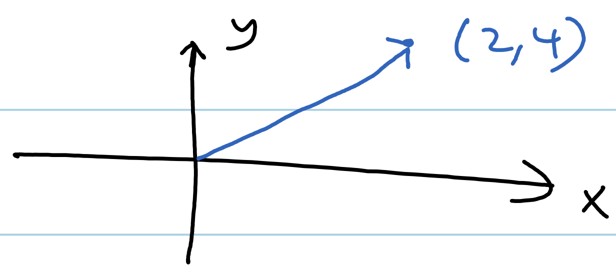
$n = 3$, $\mathbb{R}^3 = \{(x, y, z) : x, y, z \in \mathbb{R}\}$.

xyz-plane.

$n = 1$, $\mathbb{R}^1 = \mathbb{R}$ real line.

$a \in \mathbb{R}^n$ a vector starts at the origin ends at a.

$a = (2, 4) \in \mathbb{R}^2$



8.1 Definition.

Let $\mathbf{x} = (x_1, \ldots, x_n)$, $\mathbf{y} = (y_1, \ldots, y_n) \in \mathbf{R}^n$, and $\alpha \in \mathbf{R}$.

i) The *sum* of the vectors $\mathbf{x}$ and $\mathbf{y}$ is the vector

$$\mathbf{x} + \mathbf{y} := (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n).$$

ii) The *difference* of the vectors $\mathbf{x}$ and $\mathbf{y}$ is the vector

$$\mathbf{x} - \mathbf{y} := (x_1 - y_1, x_2 - y_2, \ldots, x_n - y_n).$$

iii) The *product* of the scalar α and the vector $\mathbf{x}$ is the vector

$$\alpha\mathbf{x} := (\alpha x_1, \alpha x_2, \ldots, \alpha x_n).$$

iv) The (*Euclidean*) *dot product* (or *scalar product* or *inner product*) of the vectors $\mathbf{x}$ and $\mathbf{y}$ is the scalar

$$\mathbf{x} \cdot \mathbf{y} := x_1 y_1 + x_2 y_2 + \cdots + x_n y_n.$$

8.2 Theorem. *Let* $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbf{R}^n$ *and* $\alpha, \beta \in \mathbf{R}$. *Then*

$$\alpha\mathbf{0} = \mathbf{0}, \quad 0\mathbf{x} = \mathbf{0}, \quad \mathbf{0} \cdot \mathbf{x} = 0, \quad 1\mathbf{x} = \mathbf{x}, \quad \mathbf{0} + \mathbf{x} = \mathbf{x}, \quad \mathbf{x} - \mathbf{x} = \mathbf{0},$$
$$\alpha(\beta\mathbf{x}) = \beta(\alpha\mathbf{x}) = (\alpha\beta)\mathbf{x}, \qquad \alpha(\mathbf{x} \cdot \mathbf{y}) = (\alpha\mathbf{x}) \cdot \mathbf{y} = \mathbf{x} \cdot (\alpha\mathbf{y}),$$
$$\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}, \quad \mathbf{x} + (\mathbf{y} + \mathbf{z}) = (\mathbf{x} + \mathbf{y}) + \mathbf{z}, \quad \mathbf{x} \cdot \mathbf{y} = \mathbf{y} \cdot \mathbf{x},$$
$$\alpha(\mathbf{x} + \mathbf{y}) = \alpha\mathbf{x} + \alpha\mathbf{y}, \quad \textit{and} \quad \mathbf{x} \cdot (\mathbf{y} + \mathbf{z}) = \mathbf{x} \cdot \mathbf{y} + \mathbf{x} \cdot \mathbf{z}.$$

Proof. (Exercise).

Rmk. $\underbrace{(\vec{x} - \vec{y}) \cdot (\vec{x} - \vec{y})}_{\|\vec{x} - \vec{y}\|^2} = \vec{x} \cdot \vec{x} - 2\vec{x} \cdot \vec{y} + \vec{y} \cdot \vec{y}$ ..

$$\|\vec{x} - \vec{y}\|^2 = \|\vec{x}\|^2 - 2\vec{x} \cdot \vec{y} + \|\vec{y}\|^2$$

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$\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}}$ ℓ^p-norm

8.3 Definition.

Let $\mathbf{x} \in \mathbf{R}^n$.

$\|\cdot\| = \|\cdot\|_2$

i) The (*Euclidean*) $\boxed{\textit{norm}}$ (or *magnitude*) of $\mathbf{x}$ is the scalar

$$\|x\|_2 = \left(\sum |x_i|^2\right)^{\frac{1}{2}} \quad \|\mathbf{x}\| := \sqrt{\sum_{k=1}^{n} |x_k|^2}. = \sqrt{|x_1|^2 + |x_2|^2 + \cdots + |x_n|^2}$$

ii) The ℓ^1-*norm* (read L-one-norm) of $\mathbf{x}$ is the scalar

$$\|\mathbf{x}\|_1 := \sum_{k=1}^{n} |x_k|. = |x_1| + |x_2| + \cdots + |x_k|.$$

iii) The *sup-norm* of $\mathbf{x}$ is the scalar (ℓ^∞-norm)

$$\|\mathbf{x}\|_\infty := \max\{|x_1|, \ldots, |x_n|\}. = \max_{1 \le i \le n} |x_i|$$

iv) The (*Euclidean*) *distance* between two points $\mathbf{a}, \mathbf{b} \in \mathbf{R}^n$ is the scalar

$$\operatorname{dist}(\mathbf{a}, \mathbf{b}) := \|\mathbf{a} - \mathbf{b}\|.$$

$a = (a_1, \ldots, a_n)$

$b = (b_1, \ldots, b_n)$

$$= \sqrt{|a_1 - b_1|^2 + \cdots + |a_n - b_n|^2}$$

$$= \sqrt{\sum_{i=1}^{n} |a_i - b_i|^2}$$

ex. Let $x = (1, -4, 6, -5) \in \mathbb{R}^4$. Find

$$\|x\|_\infty = \max\{|1|, |-4|, |6|, |-5|\}$$

$$= \max\{1, 4, 6, 5\} = 6.$$

$$\|x\|_1 = \sum_{i=1}^{4} |x_i| = |1| + |-4| + |6| + |-5|$$

$$= 1 + 4 + 6 + 5 = 16.$$

$$\|x\| = \sqrt{\sum_{i=1}^{4} |x_i|^2} = \sqrt{1^2 + 4^2 + 6^2 + 5^2} = \sqrt{78}$$

Rmk. (1) In $\mathbb{R}=\mathbb{R}^1$

$$\left.\begin{aligned}\|x\| &= |x|\\ \|x\|_\infty &= |x|\\ \|x\|_1 &= |x|\end{aligned}\right\} \Rightarrow \text{ in } \mathbb{R},\ \|x\| = \|x\|_\infty = \|x\|_1 = |x|.$$

(2) $\|\vec{x}\|^2 = \vec{x}\cdot\vec{x}$, $\forall\ \vec{x}\in\mathbb{R}^n$.

(3) $\forall\ (a,b)\in\mathbb{R}^2$, $(a,b) = a(1,0)+b(0,1)$

$$= ae_1 + be_2$$

$$= ai + bj.$$

$e_1=(1,0)$, $e_2=(0,1)$.

$x\in\mathbb{R}^n \Rightarrow x = (x_1, \dots, x_n)$

$$= x_1e_1 + x_2e_2 + \dots + x_ne_n$$

$$= \sum_{i=1}^{n} x_ie_i$$

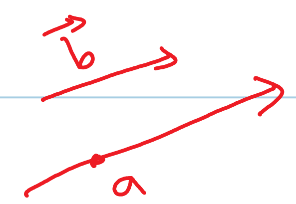
$e_1=(1,0,0,\dots,0)$, $e_2=(0,1,0,\dots,0)$, ... $e_n=(0,0,\dots,1)$

standard basis.

In $\mathbb{R}^3$, $e_1=(1,0,0)=i$, $e_2=(0,1,0)=j$

$e_3=(0,0,1)=k$.

(4) straight line in $\mathbb{R}^n$



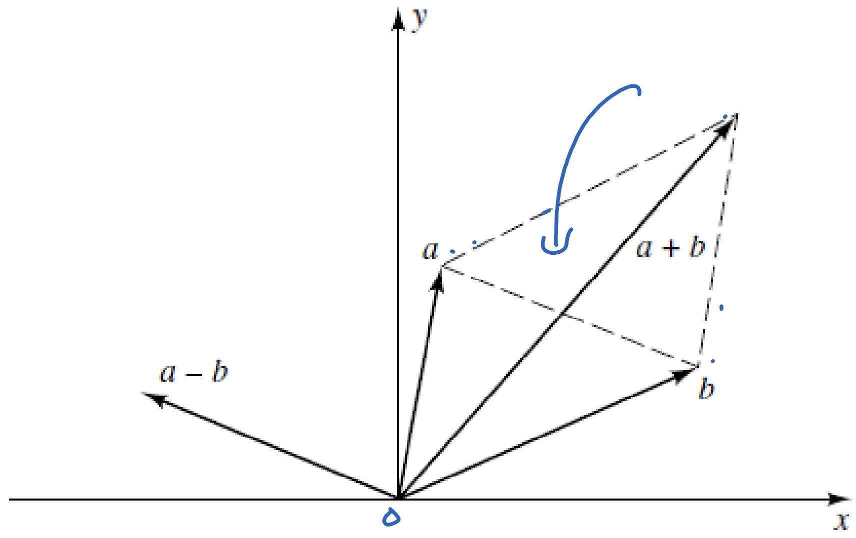
passes through $a\in\mathbb{R}^n$ in the direction $b\in\mathbb{R}^n\setminus\{0\}$ is the set

$$\boxed{l_a(b) = \left\{ a+tb : t\in\mathbb{R}\right\}}$$

(5) parallelogram $P(\vec{a},\vec{b})$ is defined

$$P(\vec{a},\vec{b}) = \{ u\vec{a} + v\vec{b} : u, v \in [0,1] \}$$

$0 \le u \le 1$
$0 \le v \le 1$

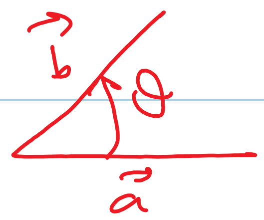


(6) line segment. From the point $a \in \mathbb{R}^n$ to $b \in \mathbb{R}^n$ is the set

$$L(\vec{a},\vec{b}) = \{ \underbrace{\overbrace{(1-t)\vec{a} + t\vec{b}}^{\psi(t)}} , 0 \le t \le 1 \}$$

$t=0 \Rightarrow \vec{a} = \psi(0)$
$t=1 \Rightarrow \vec{b} = \psi(1)$,

(7) The angle between nonzero vectors $\vec{a}, \vec{b} \in \mathbb{R}^2$ computed by :



$$\|\vec{a}-\vec{b}\|^2 = \|\vec{a}\|^2 + \|\vec{b}\|^2 - \underbrace{2\|\vec{a}\|\|\vec{b}\|\cos\theta}$$

(Law of cosines)

Also, $\|\vec{a}-\vec{b}\|^2 = (\vec{a}-\vec{b})\cdot(\vec{a}-\vec{b})$

$$= \|\vec{a}\|^2 - \underbrace{2\,\vec{a}\cdot\vec{b}} + \|\vec{b}\|^2$$

Comparing $\cos\theta = \dfrac{\vec{a}\cdot\vec{b}}{\|\vec{a}\|\|\vec{b}\|}$, $0 \le \theta \le \pi$.
$\vec{a}, \vec{b} \in \mathbb{R}^n$

8.4 Definition.

Let **a** and **b** be nonzero vectors in $\mathbf{R}^n$.

$\theta = 0, \theta = \pi$.

i) **a** and **b** are said to be *parallel* if and only if there is a scalar $t \in \mathbf{R}$ such that $\boxed{\mathbf{a} = t\mathbf{b}}$.

ii) **a** and **b** are said to be *orthogonal* if and only if $\mathbf{a} \cdot \mathbf{b} = 0$.

$\theta = \pi/2$

Rmk.
$$e_i \cdot e_j = \begin{cases} 0 & , i \neq j \\ 1 & , i = j \end{cases}$$

$\{e_1, e_2, \ldots, e_n\}$ are orthogonal basis.

8.5 Theorem. [CAUCHY–SCHWARZ INEQUALITY].
If $\mathbf{x}, \mathbf{y} \in \mathbf{R}^n$, *then*
$$|\mathbf{x} \cdot \mathbf{y}| \leq \|\mathbf{x}\| \, \|\mathbf{y}\|.$$

ex. If $\|x\| < 2$, $\|y\| < 3$, $\|z\| < 4$, $x, y, z \in \mathbb{R}^n$
Prove that $|x \cdot y - x \cdot z| < 14$.

Sol. $|x \cdot y - x \cdot z| \leq |x \cdot y| + |x \cdot z|$ (triangle ineq.)

$\leq \|x\| \|y\| + \|x\| \|z\|$

$< (2)(3) + (2)(4) = 14$.

Proof. (Cauchy-Scharz ineq.).

$\|\vec{x} - t\vec{y}\|^2 = (\vec{x} - t\vec{y}) \cdot (\vec{x} - t\vec{y})$, $t \in \mathbb{R}$, $\vec{x}, \vec{y} \in \mathbb{R}^n$.

$0 \leq \|\vec{x} - t\vec{y}\|^2 = \|\vec{x}\|^2 - 2t(\vec{x} \cdot \vec{y}) + t^2 \|\vec{y}\|^2$

$\Rightarrow \boxed{\|x\|^2 - 2t(\vec{x} \cdot \vec{y}) + t^2 \|\vec{y}\|^2 \geq 0}$ $(*)$.

Case 1 If $\vec{y} = \vec{0} \in \mathbb{R}^n$ (trivial case).

$$|\vec{x}\cdot\vec{y}| = |\vec{x}\cdot\vec{0}| = 0 \leq \|\vec{x}\|\cdot\|\vec{0}\| = 0.$$

Case 2. If $\vec{y} \neq \vec{0}$. Take

$$t = \frac{\vec{x}\cdot\vec{y}}{\|y\|^2} \quad \text{in } (*):$$

$$\|\vec{x}\|^2 - 2t(\vec{x}\cdot\vec{y}) + t^2\|\vec{y}\|^2 \geq 0$$

$$0 \leq \|\vec{x}\|^2 - 2\left(\frac{\vec{x}\cdot\vec{y}}{\|\vec{y}\|^2}\right)(\vec{x}\cdot\vec{y}) + \frac{(\vec{x}\cdot\vec{y})^2}{\|y\|^{\cancel{4}\,2}}\cdot\cancel{\|y\|^2}$$

$$0 \leq \|x\|^2 - \frac{2(\vec{x}\cdot\vec{y})^2}{\|\vec{y}\|^2} + \frac{(\vec{x}\cdot\vec{y})^2}{\|y\|^2}$$

$$\frac{(\vec{x}\cdot\vec{y})^2}{\|\vec{y}\|^2} \leq \|\vec{x}\|^2$$

$$\Rightarrow \quad (\vec{x}\cdot\vec{y})^2 \leq \|x\|^2\|y\|^2$$

$$|\vec{x}\cdot\vec{y}| \leq \|x\|\,\|y\|, \quad \vec{x}\in\mathbb{R}^n, \ \vec{y}\neq\vec{0}$$

Case 1 and Case 2 $\Rightarrow$ $|\vec{x}\cdot\vec{y}| \leq \|\vec{x}\|\,\|\vec{y}\|$, $\forall\, \vec{x},\vec{y}\in\mathbb{R}^n$. $\blacksquare$

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8.6 Theorem. *Let* $\mathbf{x}, \mathbf{y} \in \mathbf{R}^n$. *Then*

i) $\|\mathbf{x}\| \geq 0$ *with equality only when* $\mathbf{x} = \mathbf{0}$,
ii) $\|\alpha\mathbf{x}\| = |\alpha|\|\mathbf{x}\|$ *for all scalars* α,
iii) [TRIANGLE INEQUALITIES]. $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$ *and* $\|\mathbf{x} - \mathbf{y}\| \geq \|\mathbf{x}\| - \|\mathbf{y}\|$.

$\|\vec{x}-\vec{y}\| \leq \|\vec{x}\|+\|\vec{y}\|$

Proof. (i) Exercise.

(ii) $\|\alpha x\| = \left(\sum_{i=1}^{n} |\alpha x_i|^2\right)^{\frac{1}{2}}$

$= \left(\sum_{i=1}^{n} |\alpha|^2 |x_i|^2\right)^{\frac{1}{2}}$

$= \left(|\alpha|^2\right)^{\frac{1}{2}} \left(\sum_{i=1}^{n} |x_i|^2\right)^{\frac{1}{2}}$

$= |\alpha| \, \|x\|$.

(iii) $\boxed{\|\vec{x}+\vec{y}\|^2} = (\vec{x}+\vec{y})\cdot(\vec{x}+\vec{y})$

$|a| < b$

$-b < a < b$

$= \|x\|^2 + 2(\vec{x}\cdot\vec{y}) + \|\vec{y}\|^2$

Cauchy-Schwarz

$\leq \|\vec{x}\|^2 + 2\|\vec{x}\|\|\vec{y}\| + \|\vec{y}\|^2$

$= \boxed{\left(\|\vec{x}\| + \|\vec{y}\|\right)^2}$

$\Rightarrow \|\vec{x}+\vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|$.

To prove $\|\vec{x}-\vec{y}\| \geq \|\vec{x}\| - \|\vec{y}\|$

$\|\vec{x}\| = \|\vec{x}-\vec{y}+\vec{y}\|$

$\leq \|x-\vec{y}\| + \|\vec{y}\|$ by first ineq.

$\Rightarrow \|\vec{x}-\vec{y}\| \geq \|\vec{x}\| - \|\vec{y}\|$

Rmk. $\|\vec{x}-\vec{y}\| = \|\vec{x}+-\vec{y}\| \leq \|\vec{x}\| + \|-\vec{y}\| = \|\vec{x}\|+\|\vec{y}\|$. ($\alpha=-1$)

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8.7 Remark. *Let* $\mathbf{x} \in \mathbf{R}^n$. *Then*

i) $\|\mathbf{x}\|_\infty \le \|\mathbf{x}\| \le \sqrt{n}\|\mathbf{x}\|_\infty$, *and*

ii) $\|\mathbf{x}\| \le \|\mathbf{x}\|_1 \le \sqrt{n}\|\mathbf{x}\|$. ($\|\cdot\|$ and $\|\cdot\|_1$ are equivalent).

H.W.

Proof. (i) 1. $\|x\|_\infty \le \|x\| \le \sqrt{n}\,\|x\|_\infty$.

($\|\cdot\|_\infty$, $\|\cdot\|_1$ are equivelent norm).

For $1 \le j \le n$,

$\max\{|x_1|, --, |x_n|\} > |x_1|$
$> |x_2|$
$\vdots$

$$|x_j|^2 \le \underbrace{|x_1|^2 + |x_2|^2 + \cdots + |x_j|^2 + \cdots + |x_n|^2}$$

$$\le \left(\max_{1\le l\le n}|x_l|\right)^2 + \left(\max_{1\le l\le n}|x_l|\right)^2 + \cdots + \left(\max_{1\le l\le n}|x_l|\right)^2$$

$$= n\left(\|x\|_\infty\right)^2$$

$$\Rightarrow \sum_{i=1}^{n}|x_i|^2 \le n\left(\|x\|_\infty\right)^2$$

$$\|x\|^2 \le n\left(\|x\|_\infty\right)^2$$

$$\Rightarrow \boxed{\|x\| \le \sqrt{n}\,\|x\|_\infty} \quad (1)$$

. $\|x\|_\infty \le \|x\|$

Notice $|x_j|^2 \le \|x\|^2 = |x_1|^2 + \cdots + |x_n|^2$, $1 \le j \le n$

$$\Rightarrow |x_j| \le \|x\|, \quad \forall\, 1 \le j \le n$$

$$\|x\|_\infty = \max_{1\le j\le n}|x_j| \le \|x\| \Rightarrow \boxed{\|x\|_\infty \le \|x\|} \quad (2)$$

(1) + (2) prove (i). (ii) Exercise.

8.8 Definition.

The *cross product* of two vectors $\mathbf{x} = (x_1, x_2, x_3)$ and $\mathbf{y} = (y_1, y_2, y_3)$ in $\mathbf{R}^3$ is the vector defined by

$$\mathbf{x} \times \mathbf{y} := (x_2y_3 - x_3y_2, x_3y_1 - x_1y_3, x_1y_2 - x_2y_1).$$

$$\mathbf{x} \times \mathbf{y} = \det \begin{array}{ccc} + & - & + \\ \end{array} \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix} \begin{array}{l} \\ \to \vec{x} \\ \to \vec{y} \end{array}$$

$$= \begin{vmatrix} x_2 & x_3 \\ y_2 & y_3 \end{vmatrix} i - \begin{vmatrix} x_1 & x_3 \\ y_1 & y_3 \end{vmatrix} j + \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} k$$

$$= (x_2y_3 - x_3y_2)i - (x_1y_3 - x_3y_1)j + (x_1y_2 - x_2y_1)k \in \mathbb{R}^3.$$

8.9 Theorem. *Let* $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbf{R}^3$ *be vectors and* α *be a scalar. Then*

i) $\mathbf{x} \times \mathbf{x} = \mathbf{0}, \quad \mathbf{x} \times \mathbf{y} = -\mathbf{y} \times \mathbf{x},$

$$\vec{x} \times \vec{x} = \begin{vmatrix} i & j & k \\ x_1 & x_2 & x_3 \\ x_1 & x_2 & x_3 \end{vmatrix} = \vec{0} = (0,0,0).$$

ii) $(\alpha\mathbf{x}) \times \mathbf{y} = \alpha(\mathbf{x} \times \mathbf{y}) = \mathbf{x} \times (\alpha\mathbf{y}),$

iii) $\mathbf{x} \times (\mathbf{y} + \mathbf{z}) = (\mathbf{x} \times \mathbf{y}) + (\mathbf{x} \times \mathbf{z}),$

iv) $\boxed{(\mathbf{x} \times \mathbf{y}) \cdot \mathbf{z} = \mathbf{x} \cdot (\mathbf{y} \times \mathbf{z})} = \det \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix}, = \dots$

v) $\mathbf{x} \times (\mathbf{y} \times \mathbf{z}) = \underbrace{(\mathbf{x} \cdot \mathbf{z})}_{\alpha}\mathbf{y} - (\mathbf{x} \cdot \mathbf{y})\mathbf{z},$

and

vi) $\boxed{\|\mathbf{x} \times \mathbf{y}\|^2 = (\mathbf{x} \cdot \mathbf{x})(\mathbf{y} \cdot \mathbf{y}) - (\mathbf{x} \cdot \mathbf{y})^2.} = \|x\|^2\|y\|^2 - \|x\|^2\|y\|^2 \cos^2\theta$

vii) *Moreover, if* $\mathbf{x} \times \mathbf{y} \neq \mathbf{0}$, *then the vector* $\mathbf{x} \times \mathbf{y}$ *is orthogonal to* $\mathbf{x}$ *and* $\mathbf{y}$.

$(\vec{x} \times \vec{y}) \cdot \vec{x} = 0, \quad (\vec{x} \times \vec{y}) \cdot \vec{y} = 0.$

$$\|\vec{x} \times \vec{y}\|^2 = \|\vec{x}\|^2\|\vec{y}\|^2 - (\|\vec{x}\|\|\vec{y}\|\cos\theta)^2$$

$$= \|x\|^2\|y\|^2(1 - \cos^2\theta).$$

$$= \|x\|^2\|y\|^2\sin^2\theta.$$

$$\Rightarrow \boxed{\|\vec{x}\times\vec{y}\| = \|\vec{x}\|\,\|\vec{y}\|\sin\theta}$$

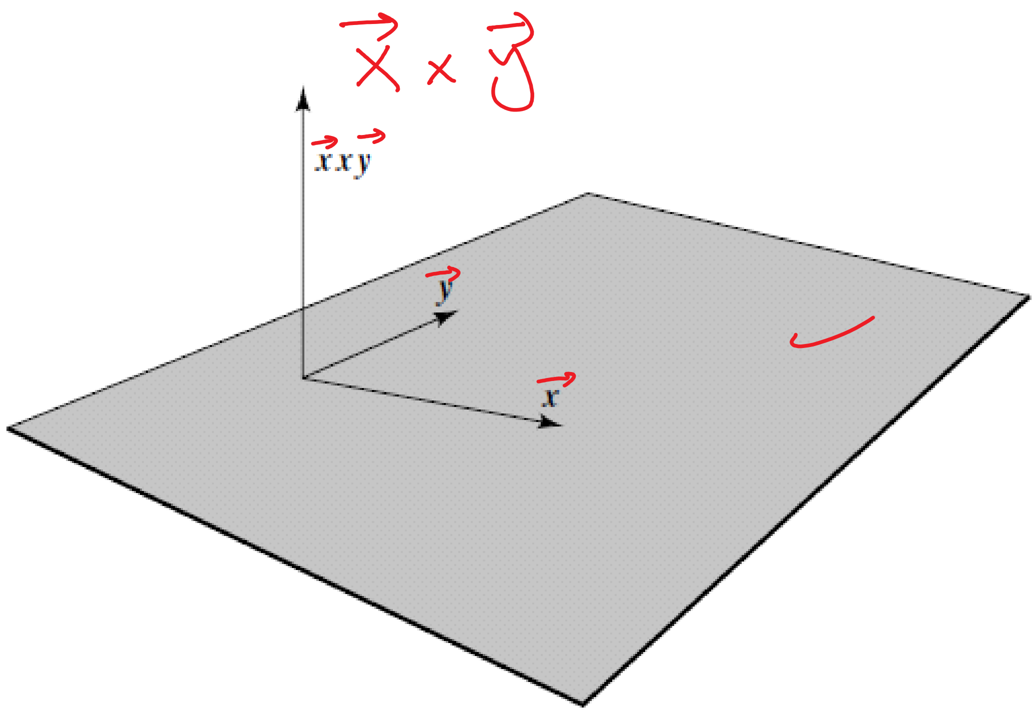
Pf. (vii) (H.W) $(\underbrace{\vec{x}\times\vec{y}}) \cdot \vec{x} \overset{(i)}{=} -(\underbrace{\vec{y}\times\vec{x}}) \cdot \vec{x}$

$\overset{(iv)}{=} -\vec{y} \cdot (\vec{x}\times\vec{x})$

$= -\vec{y}\cdot\vec{0} = \vec{0}$.

$\Rightarrow (\vec{x}\times\vec{y})\cdot\vec{x} = 0 \Rightarrow \vec{x}\times\vec{y}$ is orthogonal to $\vec{x}$.

Similarly, $(\vec{x}\times\vec{y})\cdot\vec{y} = \vec{0}$ (Exercise).



8.10 Remark. *Let* **x**, **y** *be nonzero vectors in* $\mathbf{R}^3$ *and* θ *be the angle between* **x** *and* **y**. *Then*

$$\boxed{\|\mathbf{x}\times\mathbf{y}\| = \|\mathbf{x}\|\,\|\mathbf{y}\|\,\sin\theta.}$$

$\vec{x}\cdot\vec{y} = \|x\|\,\|y\|\cos\theta$.

Proof. See P. 24.

Corollary. $\boxed{\|\vec{x}\times\vec{y}\| \leq \|\vec{x}\|\,\|\vec{y}\|}$.

Pf. From above remark

$\|\vec{x}\times\vec{y}\| = \|\vec{x}\|\,\|\vec{y}\|\,\underbrace{\sin\theta}_{\leq 1} \leq \|\vec{x}\|\,\|\vec{y}\|$

$x, y, z \in \mathbb{R}^3$

ex. prove if $n = 3$, $\|\vec{x} - \vec{y}\| < 2$, $\|\vec{z}\| < 3$,
then $\|\vec{x} \times \vec{z} - \vec{y} \times \vec{z}\| < 6$.

Pf. $\|\vec{x} \times \vec{z} - \vec{y} \times \vec{z}\| = \|(\vec{x} - \vec{y}) \times \vec{z}\|$

$\leq \underbrace{\|\vec{x} - \vec{y}\|}\ \underbrace{\|\vec{z}\|}$.

$< \quad 2 \cdot 3 = 6$ ▨

H.W's All, (1, 2, 3, 4, 5, 6, 7, 8, 9(*), 10).

8.2 PLANES AND LINEAR TRANSFORMATIONS

A plane Π in $\mathbf{R}^3$ is a set of points that is "flat" in some sense. What do we mean by flat? Any vector that lies in Π is orthogonal to a common direction, called the *normal*, which we will denote by $\mathbf{b}$. Fix a point $\mathbf{a} \in \Pi$. Since the vector $\mathbf{x} - \mathbf{a}$ lies in Π for all $\mathbf{x} \in \Pi$ and since two vectors are orthogonal when their dot product is zero, we see that $(\mathbf{x} - \mathbf{a}) \cdot \mathbf{b} = 0$ for all $\mathbf{x} \in \Pi$ (see Figure 8.4).

Using this three-dimensional case as a guide, for any $\mathbf{a}, \mathbf{b} \in \mathbf{R}^n$ with $\mathbf{b} \neq \mathbf{0}$, we call the set

$$\boxed{\Pi_{\mathbf{b}}(\mathbf{a}) := \{\mathbf{x} \in \mathbf{R}^n : (\mathbf{x} - \mathbf{a}) \cdot \mathbf{b} = 0\}}$$

the *hyperplane* in $\mathbf{R}^n$ passing through a point $\mathbf{a} \in \mathbf{R}^n$ with normal $\mathbf{b}$. (We call it a *plane* when $n = 3$.) In particular, $\Pi_{\mathbf{b}}(\mathbf{a})$ is the set of all points $\mathbf{x}$ such that $\mathbf{x} - \mathbf{a}$ is orthogonal to $\mathbf{b}$.

$\mathbb{R}^4$

$ax+by=c$ line

$ax+by+cz+dw=e$

$ax+by+cz=d$ plane

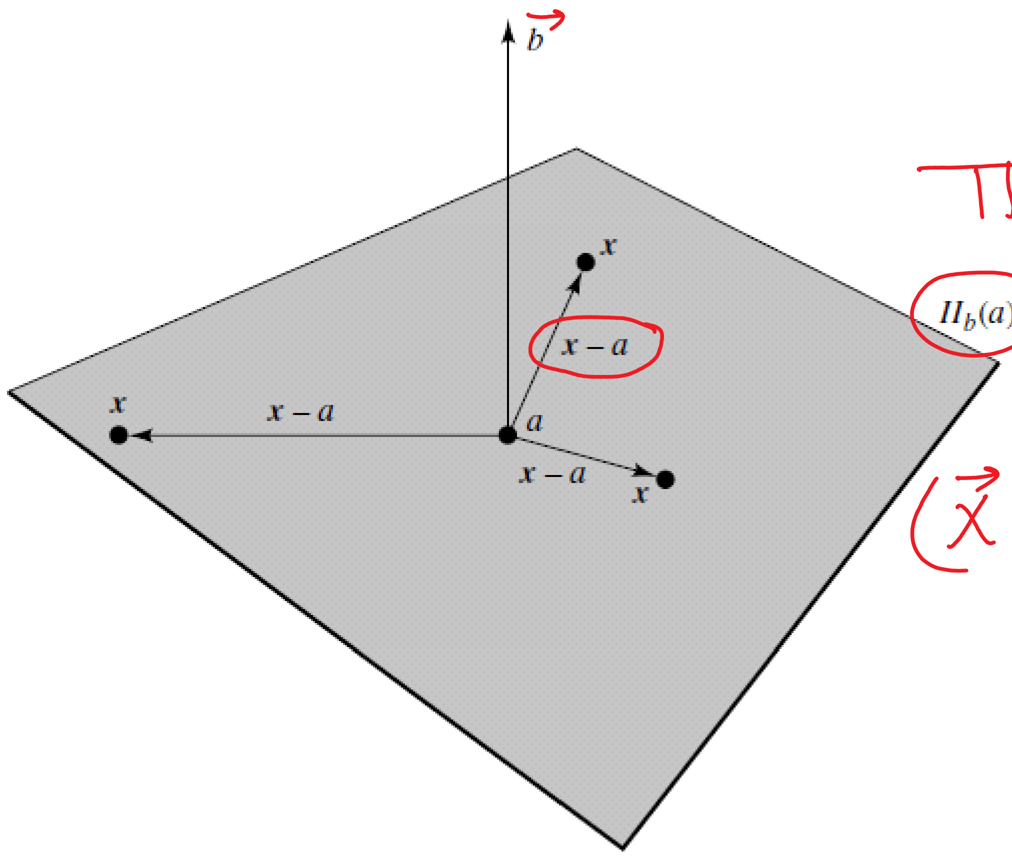
$a_1x_1 + a_2x_2 + \dots + a_nx_n = b$

hyperplane $\mathbb{R}^n$

$\mathbb{R}^2, \mathbb{R}^3 \to \mathbb{R}^n$

$n=2$ line

$n=3$ plane



$\Pi_{\vec b}(\vec a)$

$(\vec x - \vec a) \cdot \vec b = 0$

Equation of a hyperplane Π :

$$\Pi_{\vec b}(\vec a) = \{ \vec x \in \mathbb{R}^n : \underbrace{(\vec x - \vec a)\cdot \vec b = 0} \}$$

$$\vec x \cdot \vec b = \vec a \cdot \vec b$$

ثابت

$$b_1x_1 + b_2x_2 + \cdots + b_nx_n = a_1b_1 + \cdots + a_nb_n$$

$$\boxed{b_1x_1 + b_2x_2 + \cdots + b_nx_n = d}$$

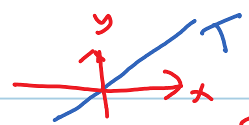
In $\mathbb{R}^3$, $\quad b_1x_1 + b_2x_2 + b_3x_3 = d$, $(b_1, b_2, b_3) = \vec n$ normal.

or $\quad$ Plane

$(ax + by + cz = d)$.

In $\mathbb{R}^2$, $\quad b_1x_1 + b_2x_2 = d$

or $\quad ax + by = d \quad$ Straight line



8.11 Remark. *Let $T : \mathbf{R} \to \mathbf{R}$. Then $T(x) = sx$ for some $s \in \mathbf{R}$ if and only if T satisfies*

$$\boxed{T(x+y) = T(x) + T(y)} \quad \text{and} \quad \boxed{T(\alpha x) = \alpha T(x)} \qquad (4)$$

for all $x, y, \alpha \in \mathbf{R}$.

Pf: $(\Longrightarrow)$ $\quad T(x) = sx$, $s \in \mathbb{R}$ Given

$$T(x+y) = s(x+y) = sx + sy = T(x) + T(y).$$

$$T(\alpha x) = s(\alpha x) = \alpha(sx) = \alpha T(x).$$

$\therefore$ (4) holds.

$(\Longleftarrow)$ conversely, If (4) holds.

$$T(x) = T(\underset{\alpha}{x} \cdot \underset{x}{1}) \overset{(4)}{=} x\,T(1),$$

$$= sx, \text{ where } s = T(1) \in \mathbb{R}$$

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$T \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$

8.12 Definition.

A function $\mathbf{T} : \mathbf{R}^n \to \mathbf{R}^m$ is said to be *linear* [notation: $\mathbf{T} \in \mathcal{L}(\mathbf{R}^n; \mathbf{R}^m)$] if and only if it satisfies

$$\mathbf{T}(\vec{\mathbf{x}} + \vec{\mathbf{y}}) = \mathbf{T}(\mathbf{x}) + \mathbf{T}(\mathbf{y}) \quad \text{and} \quad \mathbf{T}(\alpha\mathbf{x}) = \alpha\mathbf{T}(\mathbf{x})$$

for all $\mathbf{x}, \mathbf{y} \in \mathbf{R}^n$ and all scalars α.

Rmk: ① $T(\vec{0}) = \vec{0}$. where $T \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$

Pf. $T(\vec{0}) = T(\vec{0} + \vec{0})$

$T(\vec{0}) = T(\vec{0}) + T(\vec{0})$ since $T \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$

$T(\vec{0}) = 2T(\vec{0})$

$\Rightarrow 2T(\vec{0}) - T(\vec{0}) = 0$

$\Rightarrow T(\vec{0}) = \vec{0}$.

② If $F(\vec{x}) = 0$ is the eq. of a hyperplane passing through the origin, then $F \in \mathcal{L}(\mathbb{R}^n, \mathbb{R})$.

Pf. $\Pi_{\vec{b}}(\vec{0}) = \{ x \in \mathbb{R}^n : (\vec{x} - \vec{0}) \cdot \vec{b} = 0 \}$

$F(\vec{x}) = F(x_1, \dots, x_n) = b_1 x_1 + b_2 x_2 + \dots + b_n x_n = 0$

$F : \mathbb{R}^n \to \mathbb{R}$

$F \in \mathcal{L}(\mathbb{R}^n, \mathbb{R})$.

$F(\vec{x} + \vec{y}) = F(x_1 + y_1, \dots, x_n + y_n)$

$= b_1(x_1 + y_1) + \dots + b_n(x_n + y_n)$

$= (b_1 x_1 + \dots + b_n x_n) + (b_1 y_1 + \dots + b_n y_n)$

$= F(\vec{x}) + F(\vec{y})$.

$$F(\alpha\vec{x}) = F(\alpha x_1, --, \alpha x_n)$$
$$= b_1(\alpha x_1) + --+ b_n(\alpha x_n)$$
$$= \alpha(b_1 x_1 + --+ b_n x_n)$$
$$= \alpha F(x).$$

$$\therefore F \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}). \quad \blacksquare$$

Notation $\vec{x} \in \mathbb{R}^n \Rightarrow \vec{x} = (x_1, x_2, ---, x_n)$.

$\underbrace{\vec{x} := [\vec{x}]}$ $:= [x_1 \; x_2 \; ---- \; x_n]$ $1 \times n$ row matrix.

or $[\vec{x}] := [x_1 \; x_2 \; -- \; x_n]^T := \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n\times 1}$.

$$\underset{\in\mathbb{R}^{m\times n}}{B_{m\times n}} \cdot \underset{\in\mathbb{R}^{n}}{x_{n\times 1}} = \underset{\in\mathbb{R}^{m}}{(Bx)_{m\times 1}}.$$

$$T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$$

$$T(\vec{x}) = B\vec{x} \quad , \vec{x} \in \mathbb{R}^n, \; B_{m\times n}$$

T is linear transformation.

$x = [x] = [x_1 \; .-- \; x_n]$
$= \begin{bmatrix} x_1 \\ \vdots \end{bmatrix}^T$

8.13 Remark. *If* $\mathbf{x}, \mathbf{y} \in \mathbf{R}^n$ *and* α *is a scalar, then*

$$\underline{[\mathbf{x} + \mathbf{y}] = [\mathbf{x}] + [\mathbf{y}]}, \qquad [\underset{1\times 1}{\mathbf{x} \cdot \mathbf{y}}] = \underset{1\times n}{[\mathbf{x}]}\underset{n\times 1}{[\mathbf{y}]^T}, \quad \textit{and} \quad [\alpha\mathbf{x}] = \alpha[\mathbf{x}].$$

Pf. $[\vec{x}+\vec{y}] = [x_1+y_1 \quad x_2+y_2 \quad .... \quad x_n+y_n]$
$= [x_1 \; x_2 \cdots x_n] + [y_1 \; y_2 \; --- \; y_n] = [\vec{x}] + [\vec{y}]$

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$[\vec{x}\cdot\vec{y}] = [x][y]^T$:

scalar

$[\vec{x}\cdot\vec{y}] = [\,x_1y_1 + x_2y_2 + \cdots + x_ny_n\,]_{1\times 1}$

$= \underbrace{[x_1 \;\; x_2 \;\; \cdots \;\; x_n]}_{1\times n} \cdot \underbrace{\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}}_{n\times 1}$

$= [\vec{x}][\vec{y}]^T$.

$[\alpha\vec{x}] = \alpha[\vec{x}]$;

$[\alpha\vec{x}] = [\,\alpha x_1 \;\; \alpha x_2 \;\; \cdots \;\; \alpha x_n\,]$

$= \alpha[x_1 \;\; x_2 \;\; \cdots \;\; x_n] = \alpha[\vec{x}]$

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$B = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{bmatrix}_{m\times n}$

8.14 Remark. *Let $B = [b_{ij}]$ be an $m \times n$ matrix whose entries are real numbers and let $\mathbf{e}_1, \ldots, \mathbf{e}_n$ represent the usual basis of $\mathbf{R}^n$. If*

$T : \mathbb{R}^n \to \mathbb{R}^m$

$$\mathbf{T}(\mathbf{x}) = \underbrace{B}_{m\times n}\underbrace{\mathbf{x}}_{n\times 1}, \quad \mathbf{x} \in \mathbf{R}^n, \tag{6}$$

$m\times 1 \in \mathbb{R}^m$ $\quad x \in \mathbb{R}^n$

then $\mathbf{T}$ is a linear function from $\mathbf{R}^n$ to $\mathbf{R}^m$ and the jth column of B can be obtained by evaluating $\mathbf{T}$ at $\mathbf{e}_j$:

$$\boxed{(b_{1j}, b_{2j}, \ldots, b_{mj}) = T(\mathbf{e}_j), \qquad j = 1, 2, \ldots, n.} \tag{7}$$

$\{e_1, e_2, \cdots, e_n\}$ standard basis for $\mathbb{R}^n$

$T(e_1) = (b_{11}, b_{21}, \cdots, b_{m1})$

$T(e_2) = (b_{12}, b_{22}, \cdots, b_{m2})$

$\vdots$

$T(e_n) = (b_{1n}, b_{2n}, \cdots, b_{mn})$

$\begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & & b_{2n} \\ \vdots & \vdots & & \vdots \\ b_{m1} & b_{m2} & & b_{mn} \end{bmatrix}$

$\therefore B = [\,T(e_1) \quad T(e_2) \quad \cdots \quad T(e_n)\,]_{m\times n}.$

Proof: $T : \mathbb{R}^n \to \mathbb{R}^m$, $T(x) = Bx$, $x \in \mathbb{R}^n$.

• T is linear:

$T(x+y) = B[x+y] = B([x]+[y])$

$= B[x] + B[y] = T(x) + T(y).$

$T(\alpha x) = B[\alpha x] = B(\alpha[x])$

$= \alpha B[x] = \alpha T(x).$

$\therefore T$ is linear.

• $T(e_j) = (b_{1j}, b_{2j}, \cdots, b_{mj})$, $j = 1, 2, \cdots, n.$

$$T(e_j) = B e_j$$

$$= \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1j} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2j} & \cdots & b_{2n} \\ \vdots & & & & & \\ b_{m1} & b_{m2} & \cdots & b_{mj} & \cdots & b_{mn} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \begin{array}{l} \\ \\ \\ \to j\text{th} \\ \\ \\ \\ \end{array}$$

$$= \begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{mj} \end{bmatrix}, \quad j = 1, 2, \ldots, n.$$

Application.

8.2.4 c) Find the matrix representative of **T** if $\mathbf{T}(x_1, x_2, \ldots, x_n) = (x_1 - x_n, x_n - x_1)$.

$$T : \mathbb{R}^n \longrightarrow \mathbb{R}^2$$

$$T(x) = Bx \qquad B_{2\times n} \cdot X_{n\times 1}$$

$$T(e_1) = T(1, 0, \ldots, 0) = (1-0, 0-1) = (1, -1)$$

$$T(e_2) = T(\overset{x_1}{0}, \overset{x_2}{1}, 0, \ldots, \overset{x_n}{0}) = (0, 0).$$

$$T(e_3) = T(0, 0, 1, \ldots, 0) = (0, 0).$$

$$\vdots$$

$$T(e_n) = T(0, 0, \ldots, \overset{x_n}{1}) = (0-1, 1-0) = (-1, 1)$$

$$\therefore B = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & -1 \\ -1 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix}_{2\times n} = [T]_{\beta = \{e_1, \ldots, e_n\}}$$

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$T(e_j) = (b_{1j}, \ldots, b_{mj}) \cdots$ (7)

$T: \mathbb{R}^n \to \mathbb{R}^m$ linear

8.15 Theorem. *For each* $\mathbf{T} \in \mathcal{L}(\mathbf{R}^n; \mathbf{R}^m)$ *there is a matrix* $\boxed{B = [b_{ij}]_{m \times n}}$ *such that (6) holds. Moreover, the matrix* $\boxed{B \text{ is unique}}$*. Specifically, for each fixed* $\mathbf{T}$ *there is only one* B *which satisfies (6), and the columns of* B *are defined by (7).*

$Tx = Bx.$

proof. B is unique:

$Tx = Bx$

$Tx = Cx, \forall x \in \mathbb{R}^n$

$Bx = Cx, \forall x \in \mathbb{R}^n$

$Be_j = Ce_j$ in particular. $j = 1, \ldots, n$

$$\begin{pmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{mj} \end{pmatrix} = \begin{pmatrix} c_{1j} \\ \vdots \\ c_{mj} \end{pmatrix}$$

$\Rightarrow b_{1j} = c_{1j}, b_{2j} = c_{2j}, \ldots, b_{mj} = c_{mj}$

$j = 1, \ldots, n.$

That is, $b_{ij} = c_{ij}, \forall i = 1, \ldots, m$, $j = 1, \ldots, n$

$\Rightarrow B = C$

$\Rightarrow$ uniqueness.

Existence. $T: \mathbb{R}^n \to \mathbb{R}^m$

let $x \in \mathbb{R}^n$,

$$T(x) = T(x_1 e_1 + x_2 e_2 + \cdots + x_n e_n)$$

$$= x_1 T(e_1) + x_2 T(e_2) + \cdots + x_n T(e_n)$$

(T is lin.)

$$= x_1 \begin{pmatrix} b_{11} \\ b_{21} \\ \vdots \\ b_{m1} \end{pmatrix} + x_2 \begin{pmatrix} b_{12} \\ b_{22} \\ \vdots \\ b_{m2} \end{pmatrix} + \cdots + x_n \begin{pmatrix} b_{1n} \\ b_{2n} \\ \vdots \\ b_{mn} \end{pmatrix}$$

$$= \begin{pmatrix} x_1 b_{11} + x_2 b_{12} + \cdots + x_n b_{1n} \\ \vdots \\ x_1 b_{m1} + \cdots + x_n b_{mn} \end{pmatrix} = \breve{B} x.$$

Remark. $T(x) = Bx$.

The unique matrix B which satisfies (6) is called the *matrix which represents* $\mathbf{T}$.

8.16 Definition.

Let $\mathbf{T} \in \mathcal{L}(\mathbf{R}^n; \mathbf{R}^m)$. The *operator norm* of $\mathbf{T}$ is the extended real number

$T : \mathbb{R}^n \to \mathbb{R}^m$, $x \to T(x)$

$$\|\mathbf{T}\| := \sup_{\|\mathbf{x}\| \neq 0} \frac{\|\mathbf{T}(\mathbf{x})\|}{\|\mathbf{x}\|}.$$

$T(x) \in \mathbb{R}^m$, $x \in \mathbb{R}^n$

8.17 Theorem. *Let* $\mathbf{T} \in \mathcal{L}(\mathbf{R}^n; \mathbf{R}^m)$. *Then the operator norm of* $\mathbf{T}$ *is finite and satisfies*

$\|T\| < +\infty$

$$\boxed{\|\mathbf{T}(\mathbf{x})\| \leq \|\mathbf{T}\| \, \|\mathbf{x}\|} \tag{8}$$

for all $\mathbf{x} \in \mathbf{R}^n$.

Proof. Proof of (8): If $x = 0 \in \mathbb{R}^n$, $T(0) = 0$

$$\underbrace{\|T(0)\|}_{\|\vec{0}\| = 0} = \|T\| \underbrace{\|0\|}_{} \qquad \underbrace{\|T\|}_{<\infty} \cdot 0$$

$\therefore$ (8) holds for $\vec{x} = \vec{0} \in \mathbb{R}^n$.

Let $\vec{x} \neq \vec{0}$, by using the last def'n (def'n of $\|T\|$).

$$\|T\| := \sup_{x \neq 0} \frac{\|T(x)\|}{\|x\|} \geq \frac{\|T(x)\|}{\|x\|}$$

$$\Rightarrow \|T(x)\| \leq \|T\| \|x\|.$$

• pf of $\|T\|$ is finite:

$$\|T\| = \sup_{\|x\| \neq 0} \frac{\|T(x)\|}{\|x\|}$$

We need to prove $\frac{\|T(x)\|}{\|x\|} \leq \alpha$ for some α.

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$T(x) = Bx$ اقتنعنا

$T(x) = (b_1 \cdot \vec{x}, \; b_2 \cdot \vec{x}, \; \ldots, \; b_m \cdot \vec{x})$

$b_1, b_2, \ldots, b_m$ the rows of T

If $B = O$, $\|T\| = 0 < +\infty$. (Trivial).

If $B \neq O$, by using Cauchy-Scharz $\leq$,

$$\|T(x)\|^2 = (b_1 \cdot \vec{x})^2 + (b_2 \cdot \vec{x})^2 + \cdots + (b_m \cdot x)^2$$

$$\leq \|b_1\|^2 \|x\|^2 + \cdots + \|b_m\|^2 \|x\|^2$$

$a \leq \max\{a, b\}$
$b \leq \max\{a, b\}$

$$\leq \max\{\|b_j\|^2 : 1 \leq j \leq m\} \|x\|^2 + \cdots + \max\{\|b_j\|^2 : 1 \leq j \leq m\} \|x\|^2$$

$$= m \cdot \max\{\|b_j\|^2 : 1 \leq j \leq m\} \|x\|^2$$

$$\Rightarrow \|T(x)\| \le \underbrace{m \cdot \max\{\|b_j\|^2 : 1 \le j \le m\}}_{C} \|x\|^2$$

$$\|T(x)\|^2 \le C \cdot \|x\|^2, \text{ where}$$

$$C := \max\{\|b_j\|^2 : 1 \le j \le m\}$$

$$\frac{\|T(x)\|}{\|x\|} \le \sqrt{C}, \quad \forall x \in \mathbb{R}^n, \|x\| \ne 0.$$

$$\Rightarrow \frac{\|T(x)\|}{\|x\|} \text{ is bdd above by } \sqrt{C}.$$

1.3 or 1.4.

It follows from the Completeness Axiom that $\|T\| := \sup_{\|x\| \ne 0} \frac{\|T(x)\|}{\|x\|}$ exists and is finite.

Notation. $\|B\|, \ \|T\|$

Rmk. we will refer to the number $\|T\|$ as $\|B\|$.

***8.18 Remark.** *If* $\mathbf{T} : \mathbf{R}^n \to \mathbf{R}^m$ *and* $\mathbf{U} : \mathbf{R}^m \to \mathbf{R}^p$ *are linear, then so is* $\mathbf{U} \circ \mathbf{T}$. *In fact, if* B *is the* $m \times n$ *matrix which represents* $\mathbf{T}$, *and* C *is the* $p \times m$ *matrix which represents* $\mathbf{U}$, *then* CB *is the matrix which represents* $\mathbf{U} \circ \mathbf{T}$.

Te_j

Proof. (H.W)

Exercise.

$$\begin{tikzcd} \mathbb{R}^n \arrow[r, "B"', "T"] \arrow[rr, bend right, ""] & \mathbb{R}^m \arrow[r, "C"', "U"] & \mathbb{R}^p \end{tikzcd}$$

$$\underbrace{U \circ T}_{CB.} : \mathbb{R}^n \longrightarrow \mathbb{R}^p.$$

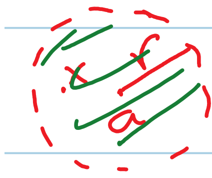
H.W's 1, 2, 3, 4, 5, 6, 7, 9, 11(a, b)

8.3 TOPOLOGY OF $\mathbf{R}^n$

$a-\delta \quad a \quad x \quad a+\delta$

$|x-a|<\delta$

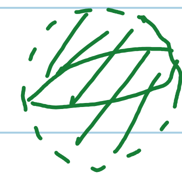
$\mathbb{R}^2$



$\|x-a\|<r$

ball

$\mathbb{R}^3$



8.19 Definition.

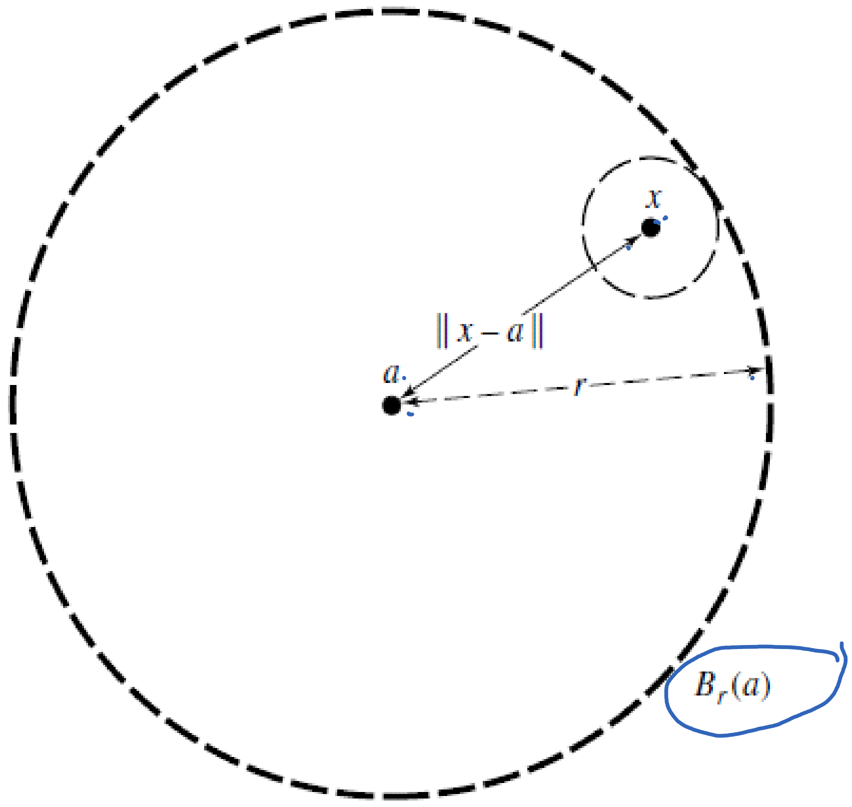
Let $\mathbf{a} \in \mathbf{R}^n$.

i) For each $r > 0$, the *open ball* centered at $\mathbf{a}$ of radius r is the set of points

$$B_r(\mathbf{a}) := \{\mathbf{x} \in \mathbf{R}^n : \|\mathbf{x}-\mathbf{a}\| < r\}.$$

ii) For each $r \geq 0$, the *closed ball* centered at $\mathbf{a}$ of radius r is the set of points

$$\{\mathbf{x} \in \mathbf{R}^n : \|\mathbf{x}-\mathbf{a}\| \leq r\}.$$



$n=1$, $\mathbb{R}$, open ball centered at a of radius r is the open interval

$|x-a|<r \Rightarrow -r < x-a < r$

$B_r(a) = \{x : a-r < x < a+r\}$

$=(a-r, a+r)$.

closed ball in $\mathbb{R}$ is closed interval

$[a-r, a+r] := \{x \in \mathbb{R} : a-r \leq x \leq a+r\}$

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$\mathbb{R}^n$ $V \subseteq \mathbb{R}^n$

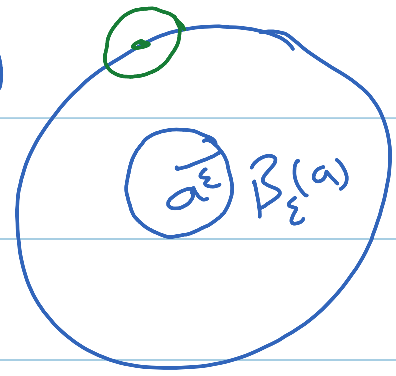
8.20 Definition.

Let $n \in \mathbf{N}$.

i) A subset V of $\mathbf{R}^n$ is said to be $\boxed{\textit{open}}$ (in $\mathbf{R}^n$) if and only if for every $\mathbf{a} \in V$ there is an $\varepsilon > 0$ such that $B_\varepsilon(\mathbf{a}) \subseteq V$.

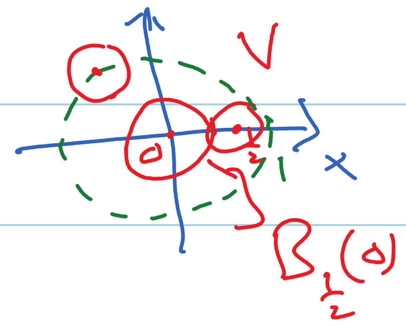
ii) A subset E of $\mathbf{R}^n$ is said to be *closed* (in $\mathbf{R}^n$) if and only if $E^c := \mathbf{R}^n \backslash E$ is open. $= \mathbb{R}^n - E$.

Let $x \in V$, $\exists \varepsilon > 0$ $B_\varepsilon(x) \subseteq V$



$B_\varepsilon(a) \subseteq V$.

$\mathbb{R}^2$ $V = \{ x^2 + y^2 < 1 \}$



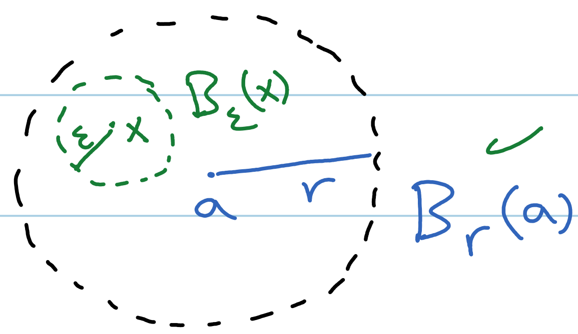
$[2,3]$ closed interval since $[2,3]^c = \boxed{(-\infty,2) \cup (3,\infty)}$ open.

open.

8.21 Remark. *For every* $\mathbf{x} \in B_r(\mathbf{a})$ *there is an* $\varepsilon > 0$ *such that* $B_\varepsilon(\mathbf{x}) \subseteq B_r(\mathbf{a})$.

(That is, $B_r(a)$ is an open).

proof. Let $x \in B_r(a)$. Then $\|x-a\| < r$ (by def'n).



Let $\boxed{\varepsilon = r - \|x-a\|} > 0$

Let $y \in B_\varepsilon(x) \Rightarrow \|y-x\| < \varepsilon$.

$y \in B_r(a)$

$\|y-a\| < r$

$\|y-a\| = \|y-x+x-a\| \leq \|y-x\| + \|x-a\|$
$< \varepsilon + \|x-a\| = r$

$\Rightarrow \|y-a\| < r \Rightarrow y \in B_r(a)$.

$\Rightarrow B_\varepsilon(x) \subseteq B_r(a)$ ∎

8.22 Remark. *If* $\mathbf{a} \in \mathbf{R}^n$, *then* $\mathbf{R}^n \backslash \{\mathbf{a}\}$ *is open and* $\{\mathbf{a}\}$ *is closed.*

(That is every singleton $E := \{a\}$ is closed)

ex. $\{(1,2)\}$ in $\mathbb{R}^2$ is closed. ($\vec{a}$)

ex. $\{4\}$ in $\mathbb{R}$ is closed

$\{(1,0,0,0,0)\}$ in $\mathbb{R}^5$ is closed.

Proof. We need to prove $E^c := \mathbb{R}^n \backslash \{\vec{a}\}$ is open.

Let $x \in E^c$ ($\exists\, \varepsilon > 0$ s.t $B_\varepsilon(x) \subseteq E^c$) ما نريد اثباته

$E = \{a\}$, $x \in \{a\}^c$, $x \neq a$

Let $\varepsilon := \|x-a\|$. Then by defn.

$a \notin B_\varepsilon(x) \Rightarrow B_\varepsilon(x) \subseteq E^c$

$x = a$ That is E^c is open.

$\Rightarrow E = \{a\}$ is closed ∎

Rmk (T) or (F)

Every set is either open or closed.

Ans. (False)

ex. $[1,4)$ is neither open nor closed in $\mathbb{R}^2$.

8.23 Remark. *For each $n \in \mathbf{N}$, the empty set $\emptyset$ and the whole space $\mathbf{R}^n$ are both open and closed.* (clopen).

proof. Exercise. p. 290.

8.24 Theorem. *Let $n \in \mathbf{N}$.*

i) *If $\{V_\alpha\}_{\alpha \in A}$ is any collection of open subsets of $\mathbf{R}^n$, then*

$$\bigcup_{\alpha \in A} V_\alpha$$

is open.

ii) *If $\{V_k : k = 1, 2, \ldots, p\}$ is a finite collection of open subsets of $\mathbf{R}^n$, then*

$$V_1 \cap V_2 \cap \cdots \cap V_p = \bigcap_{k=1}^{p} V_k := \bigcap_{k \in \{1,2,\ldots,p\}} V_k$$

is open.

iii) *If $\{E_\alpha\}_{\alpha \in A}$ is any collection of closed subsets of $\mathbf{R}^n$, then*

$$\bigcap_{\alpha \in A} E_\alpha$$

is closed.

iv) *If $\{E_k : k = 1, 2, \ldots, p\}$ is a finite collection of closed subsets of $\mathbf{R}^n$, then*

$$\bigcup_{k=1}^{p} E_k := \bigcup_{k \in \{1,2,\ldots,p\}} E_k$$

is closed.

v) *If V is open and E is closed, then $V \setminus E$ is open and $E \setminus V$ is closed.*

Proof:

i) *If* $\{V_\alpha\}_{\alpha\in A}$ *is any collection of open subsets of* $\mathbf{R}^n$, *then*

$$\bigcup_{\alpha\in A} V_\alpha$$

is open.

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Let $x \in \bigcup_{\alpha\in A} V_\alpha$ $\left(\exists\, r>0 \text{ s.t. } B_r(x) \subseteq \bigcup_{\alpha\in A} V_\alpha \right)$

$\Rightarrow x \in V_\alpha$ for some $\alpha \in A$.

Since V_α is open, it follows $\exists$ an $r>0$ s.t. $B_r(x) \subseteq V_\alpha \subseteq \bigcup_{\alpha\in A} V_\alpha$

Thus, $B_r(x) \subseteq \bigcup_{\alpha\in A} V_\alpha$, $\forall x \in \bigcup_{\alpha\in A} V_\alpha$

that is $\bigcup_{\alpha\in A} V_\alpha$ is open.

ii) *If* $\{V_k : k = 1, 2, \dots, p\}$ *is a finite collection of open subsets of* $\mathbf{R}^n$, *then*

$$\bigcap_{k=1}^{p} V_k := \bigcap_{k\in\{1,2,\dots,p\}} V_k$$

is open.

Pf. Let $x \in \bigcap_{k=1}^{p} V_k \Rightarrow x \in V_k$, $\forall k = 1, 2, \dots, p$

Since V_k is open, it follows that

$\exists\, r_k > 0$ s.t. $B_{r_k}(x) \subseteq V_k$

توضيح,
$\exists\, r_1 > 0 \quad B_{r_1}(x) \subseteq V_1$

$\exists\, r_2 > 0 \quad B_{r_2}(x) \subseteq V_2$,

$\vdots \qquad \vdots$

$\exists\, r_k > 0 \quad B_{r_k}(x) \subseteq V_k$, $k = 1, \dots, p$

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Let $r := \min\{r_1, r_2, \ldots, r_p\}$

then $r > 0$ and

$B_r(x) \subseteq V_k$, for all $k = 1, \ldots, p$.

$\Rightarrow B_r(x) \subseteq \bigcap_{k=1}^{p} V_k$

$\therefore \bigcap_{k=1}^{p} V_k$ is open.

iii) *If $\{E_\alpha\}_{\alpha \in A}$ is any collection of closed subsets of $\mathbf{R}^n$, then*

$$\bigcap_{\alpha \in A} E_\alpha$$

is closed.

Pf. $\left(\bigcap_{\alpha \in A} E_\alpha\right)^c = \bigcup_{\alpha \in A} E_\alpha^c$ is open by (ii)

open

$\Rightarrow \left(\bigcap_{\alpha \in A} E_\alpha\right)^c$ is open

$\Rightarrow \bigcap_{\alpha \in A} E_\alpha$ is closed by def.

iv) *If $\{E_k : k = 1, 2, \ldots, p\}$ is a finite collection of closed subsets of $\mathbf{R}^n$, then*

$$\bigcup_{k=1}^{p} E_k := \bigcup_{k \in \{1,2,\ldots,p\}} E_k$$

is closed.

open

Pf. $\boxed{\left(\bigcup_{k=1}^{p} E_k\right)^c} = \bigcap_{k=1}^{p} E_k^c$ is open by (iii)

$\Rightarrow \bigcup_{k=1}^{p} E_k$ is closed by def'n.

v)

v) *If V is open and E is closed, then $V \setminus E$ is open and $E \setminus V$ is closed.*

$V \setminus E = \underbrace{V}_{\text{open}} \cap \underbrace{E^c}_{\text{open}}$ is open by (ii)

$E \setminus V = \underbrace{E}_{\text{closed}} \cap \underbrace{V^c}_{\text{closed}}$ is closed by (iii)

8.25 Remark. *Statements ii) and iv) of Theorem 8.24 are false if arbitrary collections are used in place of finite collections.*

any

(ii) $\bigcap_{k=1}^{p} V_k$ is open where V_k is open $\forall k = 1, \dots, p$.

False if we say $\bigcap_{k \in A} V_k$ is open where V_k is open $\forall k \in A$.

Ex: $\bigcap_{k=1}^{\infty} \underbrace{\left(-\frac{1}{k}, \frac{1}{k}\right)}_{\text{open.}}$

$= (-1, 1) \cap \left(-\frac{1}{2}, \frac{1}{2}\right) \cap \left(-\frac{1}{3}, \frac{1}{3}\right) \cap \dots\dots$

$= \{0\}$. closed.

Each $\left(-\frac{1}{k}, \frac{1}{k}\right)$, $k \in \mathbb{N}$ is open

but $\bigcap_{k \in \mathbb{N}.} \left(-\frac{1}{k}, \frac{1}{k}\right)$ is closed

(iv) $\bigcup_{i=1}^{p} A_i$ is closed if A_i closed $i = 1, \dots, p$

ex. $\bigcup_{k \in \mathbb{N}} \underbrace{\left[\frac{1}{k+1}, \frac{k}{k+1}\right]}_{\text{closed}} = (0, 1)$ open

$\frac{1}{k+1} \to 0$, $\frac{k}{k+1} \to 1$

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(ε-δ) def'n

Recall, $f: E \to \mathbb{R}$ is continuous at $a \in E$ iff $\forall \varepsilon > 0$, $\exists$ a $\delta > 0$ such that

$$|x-a| < \delta,\ x \in E \implies \underbrace{|f(x) - f(a)| < \varepsilon}_{f(x) \in B_\varepsilon(f(a))}.$$

"ball Language"

$-\delta < x - a < \delta$

$a - \delta < x < a + \delta$

$x \in B_\delta(a)$, $x \in E$

$x \in E \cap B_\delta(a)$

f is continuous at $a \in E$

if $x \in \underbrace{E \cap B_\delta(a)}_{A}$, then $f(x) \in \underbrace{B_\varepsilon(f(a))}_{B}$

that is, $f(E \cap B_\delta(a)) \subseteq B_\varepsilon(f(a))$.

$x \in A \Rightarrow f(x) \in B$

$f(A) \subseteq B$

$f: E \to \mathbb{R}$ is continuous at $a \in E$

"ball Language" if and only if $\boxed{E \cap B_\delta(a) \subseteq f^{-1}(B_\varepsilon(f(a)))}$.

Thus, f is continuous at $a \in E$ if and only if for all $a \in \mathbf{R}$, the inverse image under f of every open ball centered at $f(a)$ contains the intersection of E and an open ball centered at a.

U } $E \subseteq \mathbb{R}^n$

$U = E \cap A$, A is open

8.26 Definition.

Let $E \subseteq \mathbf{R}^n$.

i) A set $U \subseteq E$ is said to be *relatively open* in E if and only if there is an open set A such that $U = E \cap A$.

ii) A set $C \subseteq E$ is said to be *relatively closed* in E if and only if there is a closed set B such that $C = E \cap B$.

C } E $\quad C = E \cap B$, B is closed

Ex. Exercise 8.3.4

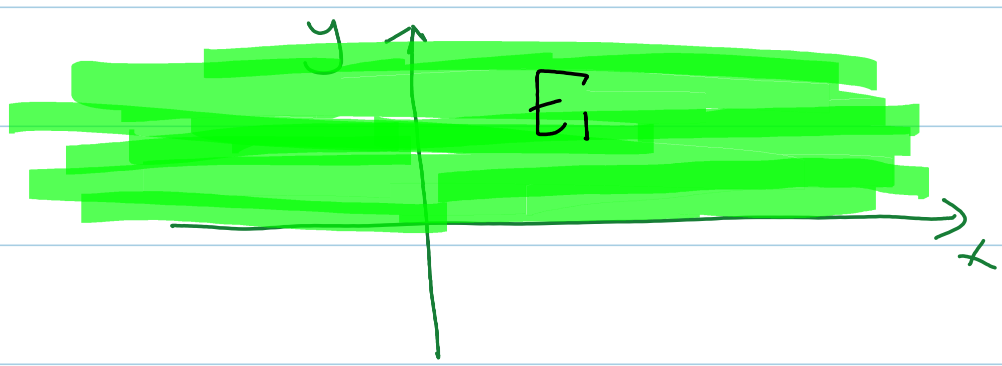
$\subseteq \mathbb{R}^2$ closed $\quad \subseteq \mathbb{R}^2$ open

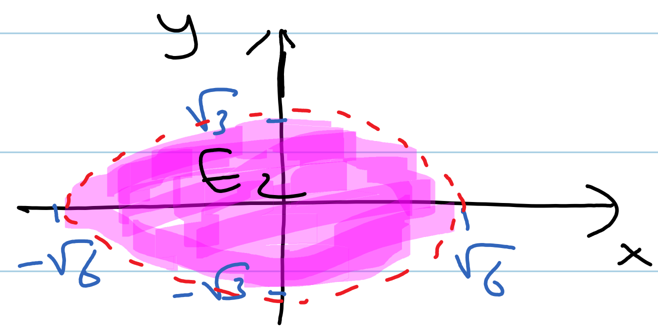
8.3.4. a) Set $E_1 := \{(x, y) : y \geq 0\}$ and $E_2 := \{(x, y) : x^2 + 2y^2 < 6\}$, and sketch a graph of the set

$$U := \{(x, y) : x^2 + 2y^2 < 6 \quad \text{and} \quad y \geq 0\}. = E_1 \cap E_2$$

b) Decide whether U is relatively open or relatively closed in E_1. Explain your answer.

c) Decide whether U is relatively open or relatively closed in E_2. Explain your answer.





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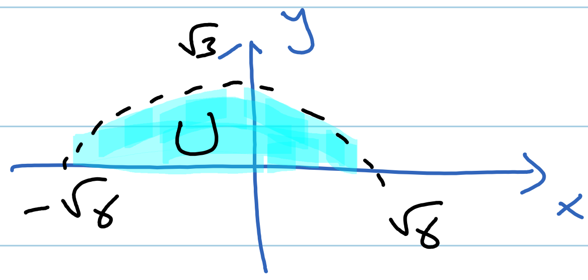

closed

$U = E_1 \cap E_2 \Rightarrow U$ is relatively closed in E_2.

open

$U = E_1 \cap E_2 \Rightarrow U$ is relatively open in E_1

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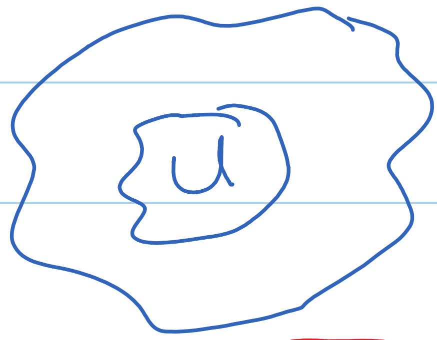

$E \subseteq \mathbb{R}^n$.

8.27 Remark. *Let $U \subseteq E \subseteq \mathbf{R}^n$.*

i) *Then U is relatively open in E if and only if for each $\mathbf{a} \in U$ there is an $r > 0$ such that $B_r(\mathbf{a}) \cap E \subset U$.*

ii) *If E is open, then U is relatively open in E if and only if U is (plain old vanilla) open (in the usual sense).*

Proof. (i) $\Rightarrow$ Spse that U is relatively open in E, then by def'n,

$U = E \cap A$ for some open set A.

Since A is open, $a \in A$, then there is an $r > 0$ such that $B_r(a) \subset A$

$$E \cap B_r(a) \subset E \cap A = U$$

$$\Rightarrow E \cap B_r(a) \subset U.$$

Conversely, spse that $\forall a \in U$, $\exists r > 0$ s.t $B_r(a) \cap E \subset U$. we need to prove that U is relatively open

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in E (i.e, $U = E \cap A$ for some open set A)

Now, for each $a \in U$, $\exists$ $r(a) > 0$ (Notation) s.t $B_{r(a)}(a) \cap E \subseteq U$

$$\left(\bigcup_{a \in U} B_{r(a)}(a) \right) \cap E \subseteq U.$$

Since this union is taken $\forall a \in U$, then

$$\left(\bigcup_{a\in A} B_{r(a)}(a)\right)\cap E = U.$$

Let $A = \bigcup_{a\in A} B_{r(a)}(a)$ is open by thm 8.24 (each $B_{r(a)}(a)$ is open)

$\therefore\ U = A\cap E$, $A = \bigcup_{a\in A} B_{r(a)}(a)$ open

by def'n, U is relatively open in E.

Rmk. A is open since it is the union of open balls which is open.

ii) *If E is open, then U is relatively open in E if and only if U is (plain old vanilla) open (in the usual sense).*

Pf. Assume $U \subseteq E \subseteq \mathbb{R}^n$ and E is open.

$\Rightarrow$ Let U be relatively open in E, then

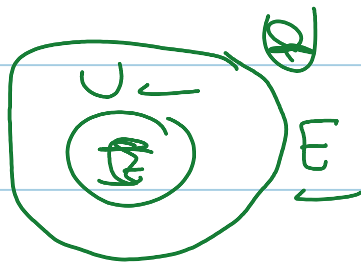
$U = \underbrace{E}_{\substack{\text{open}\\ \text{(Given)}}} \cap \underbrace{A}_{\text{open by def'n}}$ for some open set A.

Since E and A are open, then

$U = E\cap A$ is open.

Conversely, If U is open,

$U = E\cap A$, where $A = U$ open



$U \subseteq E$

$\Rightarrow$ U is relatively open in E. $\blacksquare$

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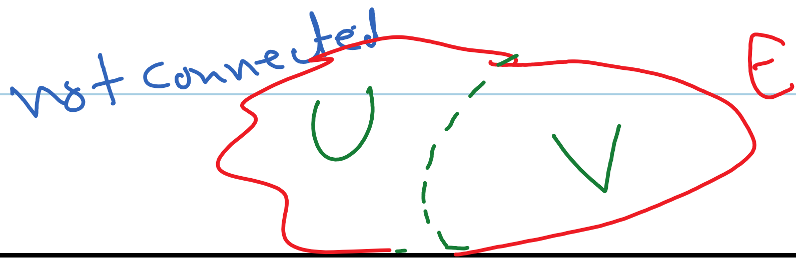

$E = U \cup V$

8.28 Definition.

Let E be a subset of $\mathbf{R}^n$.

$U \neq \phi$, $V \neq \phi$

i) A pair of sets U, V is said to *separate* E if and only if U and V are nonempty, relatively open in E, $E = U \cup V$, and $U \cap V = \emptyset$.

ii) E is said to be *connected* if and only if E cannot be separated by any pair of relatively open sets U, V.

Ex 1 Let $E = \phi \subseteq \mathbb{R}^n$. prove that $E = \phi$ is connected.

Pf. Spse not, $\boxed{\phi = U \cup V}$, where $U \neq \phi$, $V \neq \phi$

U, V relatively open, $U \cap V = \phi$.

a contradiction.

$\therefore$ ϕ is connected.

Ex 2 Every singleton set $E = \{a\} \subseteq \mathbb{R}^n$ is connected.

Pf. Spse not, $E = \{a\} = U \cup V$, where $U \neq \phi$, $V \neq \phi$, $U \cap V = \phi$

U & V are relatively open in E.

then E has at least two elements, a contradiction.

Rmk. A set E is not connected if there are open sets A, B such that $E \cap A \neq \phi$, $E \cap B \neq \phi$ and

$$E = (E \cap A) \cup (E \cap B), \quad A \cap B = \phi.$$

Ex. Show that $E = \mathbb{Q}$ is not connected.

Pf. Let $A = (-\infty, \sqrt{2})$, $B = (\sqrt{2}, \infty)$.
A, B open sets in $\mathbb{R}$.

$E \cap A = \mathbb{Q} \cap (-\infty, \sqrt{2}) \neq \phi$ (1 is there).

$E \cap B = \mathbb{Q} \cap (\sqrt{2}, \infty) \neq \phi$ (2 is there).

$A \cap B = (-\infty, \sqrt{2}) \cap (\sqrt{2}, \infty) = \phi$.

$$\begin{aligned}(E \cap A) \cup (E \cap B) &= E \cap (A \cup B) \\ &= \mathbb{Q} \cap \left((-\infty, \sqrt{2}) \cup (\sqrt{2}, \infty) \right) = \mathbb{Q} = E.\end{aligned}$$

$\therefore$ $E = \mathbb{Q}$ is not connected.

"rational"

Ex. Let $E = \{ (x,y) : -1 \leq x \leq 1, \ -|x| < y < |x| \}$ in $\mathbb{R}^2$. Prove that E is not connected.

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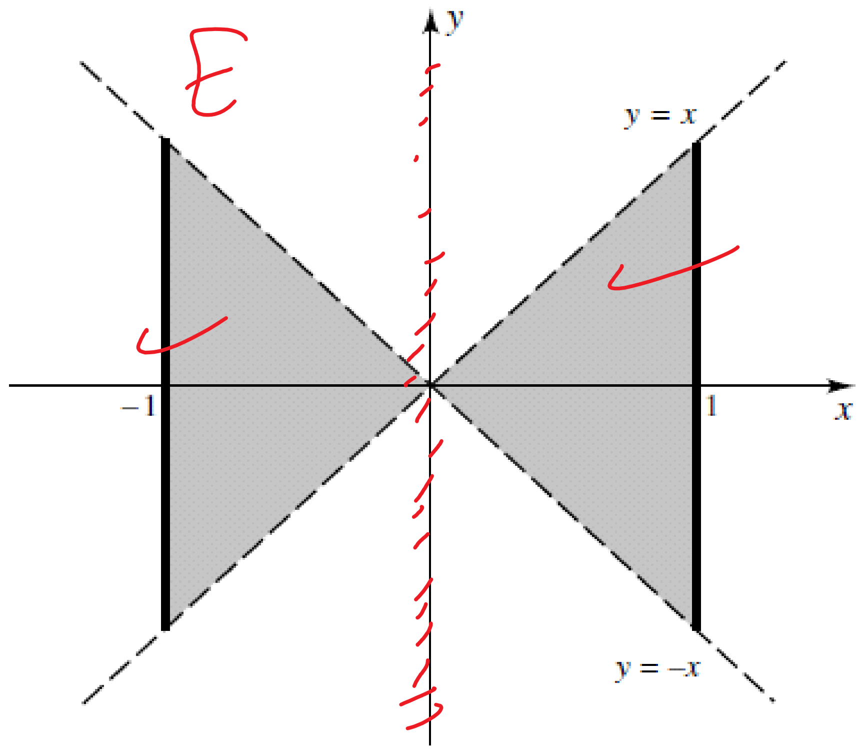
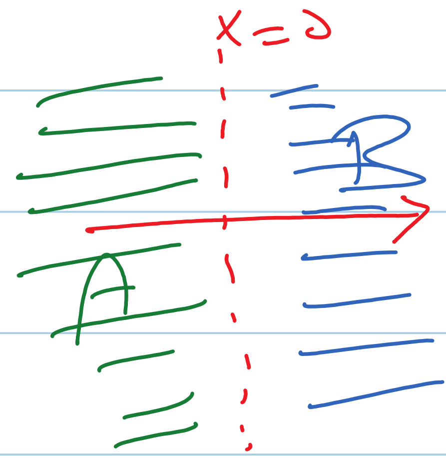


Let $A = \{ (x,y) : x < 0 \}$ and

$B = \{ (x,y) : x > 0 \}$



$E \cap A \neq \phi$, $E \cap B \neq \phi$

$A \cap B = \phi$, A and B are open sets in $\mathbb{R}^2$.

$$(E \cap A) \cup (E \cap B) = E \cap (A \cup B)$$
$$= E \cap (\mathbb{R}^2 \setminus \{x=0\})$$
$$= E.$$

$\therefore$ E is not connected.

<u>thm 8.30</u> A subset E of $\mathbb{R}$ is connected iff E is an interval.

Pf (H.W).

ex. $[1,2]$ is connected in $\mathbb{R}$.

H.W's (8.3)
1, 2, 3, 4, 5, 6, 8, 9, 10

8.3.10. Graph generic open balls in $\mathbf{R}^2$ with respect to each of the "non-Euclidean" norms $\|\cdot\|_1$ and $\|\cdot\|_\infty$. What shape are they?

Sol. $B_r(x) = \{ y \in \mathbb{R}^n : \|x-y\| < r \}$. شكل عام.

$$B_r(0) = \{ y \in \mathbb{R}^n : \|y\| < r \}.$$

In $\mathbb{R}^2$, $B_1(0) = \{ (x,y) \in \mathbb{R}^2 : \|(x,y)\| < 1 \}$

ℓ^1-norm, $B_1(0) = \{ (x,y) \in \mathbb{R}^2 : \|(x,y)\|_1 < 1 \}$

ℓ^1-ball at $(0,0)$

$$\boxed{= \{ (x,y) \in \mathbb{R}^2 : |x| + |y| < 1 \}}$$

ℓ^∞-norm, $B_1(0) = \{ (x,y) \in \mathbb{R}^2 : \|(x,y)\|_\infty < 1 \}$

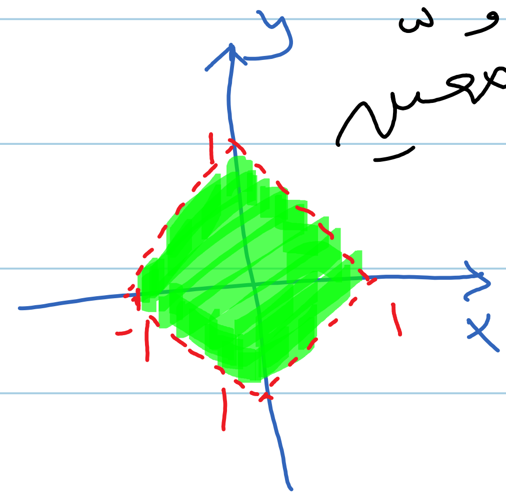
ℓ^∞-ball at $(0,0)$.

$$\boxed{= \{ (x,y) \in \mathbb{R}^2 : \max\{|x|, |y|\} < 1 \}}$$

Graph ℓ^1-ball at $(0,0)$

$$|x| + |y| < 1$$

$x, y \geq 0 \implies x + y < 1$



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~~$x \geq 0, y$~~

$x < 0, y \geq 0$ $\quad -x + y < 1$

$y < 1 + x$

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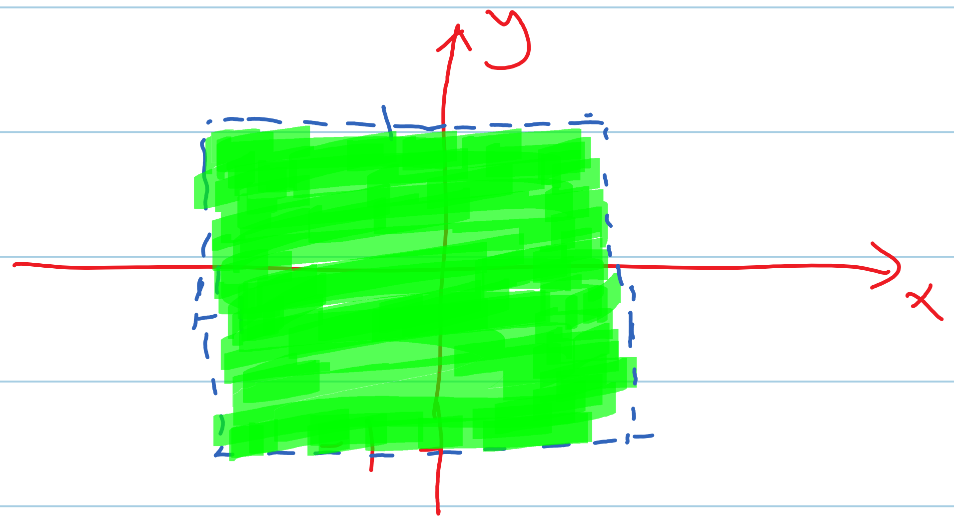
ℓ^∞- ball at (0,0)

$\max\{|x|, |y|\} < 1$

$\Rightarrow |x| < 1$ and $|y| < 1$

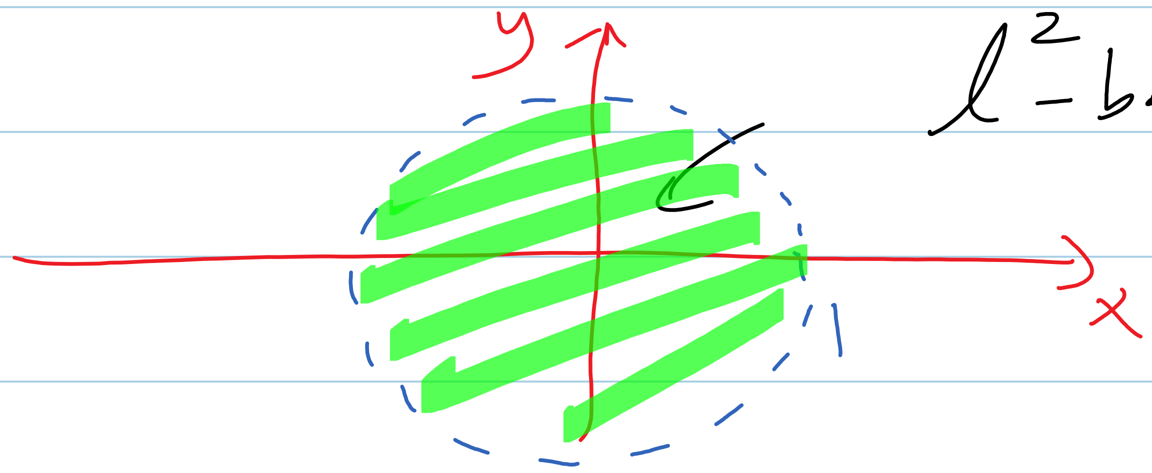
$-1 < x < 1$ and $-1 < y < 1$



ℓ^2- ball at (0,0)

$B_1(0) = \{ (x,y) \in \mathbb{R}^2 : \|(x,y)\| < 1\}$

$= \{ (x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$



ℓ^2-ball at (0,0).

(Disk).

8.4 INTERIOR, CLOSURE, AND BOUNDARY

8.31 Definition.

Let E be a subset of a Euclidean space $\mathbf{R}^n$.

i) The *interior* of E is the set

$$E^o := \bigcup\{V : V \subseteq E \text{ and } V \text{ is open in } \mathbf{R}^n\}.$$

ii) The *closure* of E is the set

$$\overline{E} := \bigcap\{B : B \supseteq E \text{ and } B \text{ is closed in } \mathbf{R}^n\}.$$

Rmk: ① E° is open set and $\overline{E}$ is closed set.

② E° is the largest open set contained in E

$\overline{E}$ is the smallest closed set containing E.

ex. Let $E = (1, 2]$ in $\mathbb{R}$.

$E^\circ = (1,2)$, $\overline{E} = [1,2]$.

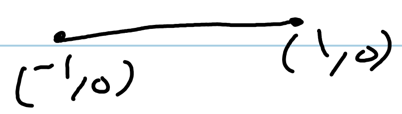
ex: Find E°, $\overline{E}$ of the set

(a) $E = \{(x,y) : -1 \leq x \leq 1, -|x| < y < |x|\}$

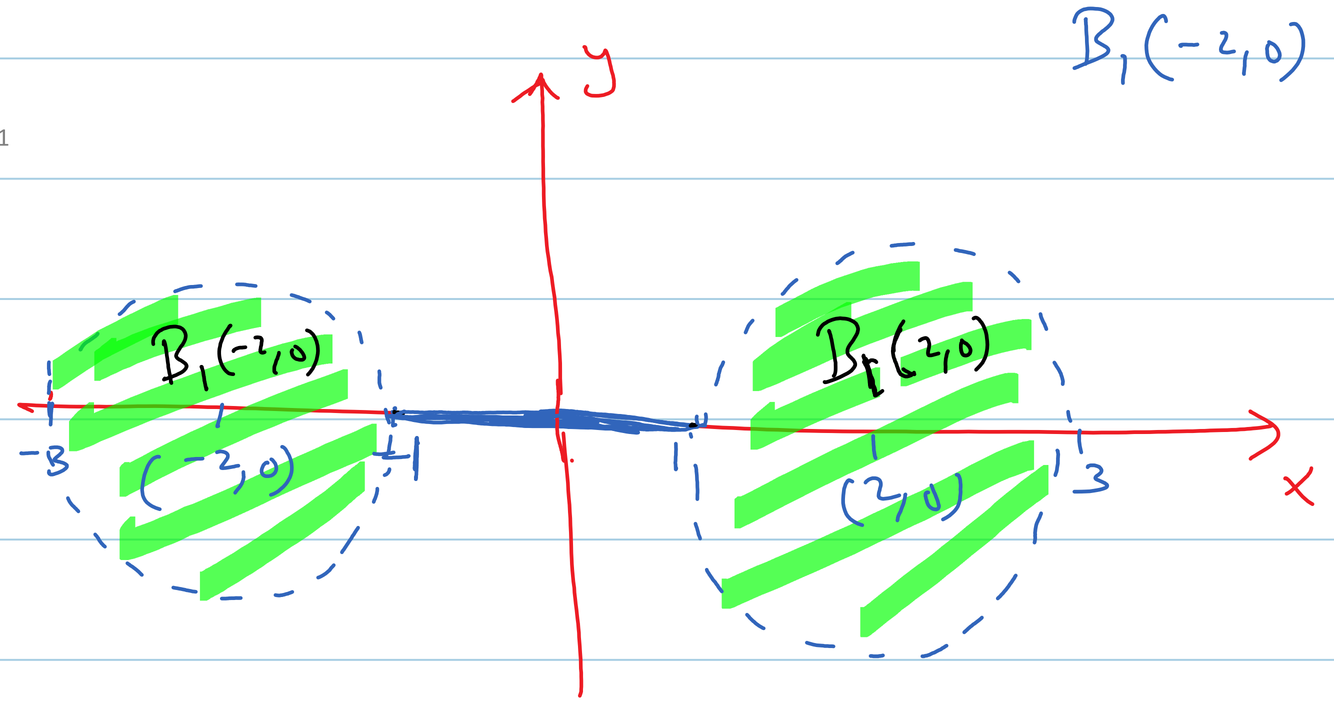
Sol: $E^\circ = \{(x,y) : -1 < x < 1, -|x| < y < |x|\}$.

$\overline{E} = \{(x,y) : -1 \leq x \leq 1, -|x| \leq y \leq |x|\}$.

(b) $E = B_1(-2,0) \cup B_1(2,0) \cup \{(x,0) : -1 \leq x \leq 1\}$.



$B_1(-2,0)$



bow-tie-shaped

$$E^o = B_1(-2,0) \cup B_1(2,0)$$

$$\overline{E} = \overline{B_1(-2,0)} \cup \overline{B_1(2,0)} \cup \{(x,0) : -1 \le x \le 1\}$$

$$= \{(x,y) : (x+2)^2 + (y-0)^2 \le 1\}$$

$$\cup \{(x,y) : (x-2)^2 + y^2 \le 1\}$$

$$\cup \{(x,0) : -1 \le x \le 1\}.$$

8.32 Theorem. *Let $E \subseteq \mathbf{R}^n$. Then*

i) $E^o \subseteq E \subseteq \overline{E}$,

ii) *if V is open and $V \subseteq E$, then $V \subseteq E^o$, and*

iii) *if C is closed and $C \supseteq E$, then $C \supseteq \overline{E}$.*

Proof. (Exercise) use the def'n.

Rmk. ① the interior of abounded interval with end points a and b ((a,b), $(a,b]$, $[a,b)$, $[a,b]$). is (a,b).

the closure $= [a,b]$.

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(2) $E^\circ = E$ iff E is open.

$\bar{E} = E$ iff E is closed

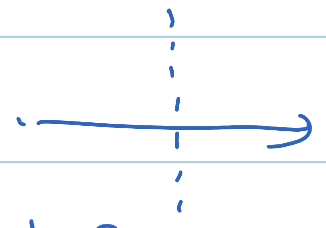
(Pf. Exercise).

ex. $E = (1,2)$ $E^\circ = (1,2) = E$

$E = [1,2]$ $\bar{E} = E$

Rmk.

the interior of a nice enough set E in $\mathbf{R}^2$ can be obtained by removing all its "edges," and the closure of E by adding all its "edges."



ex. $E = \{(x,y) : x \neq 0\} = \mathbb{R}^2 \setminus \{y\text{-axis}\}$

$E^\circ = E$ since E is open.

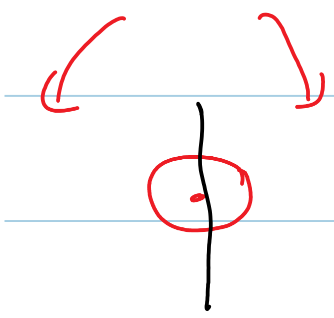
$\bar{E} = \mathbb{R}^2$.

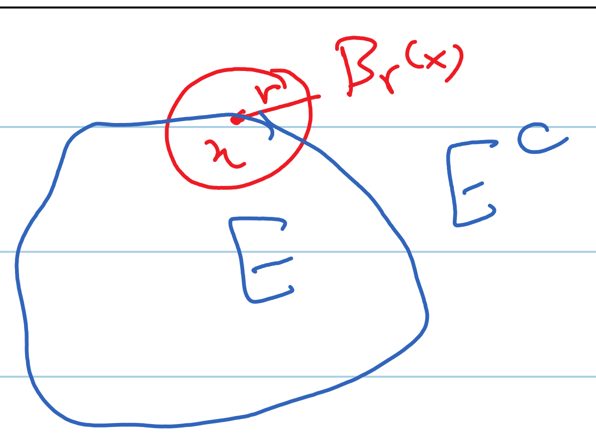
8.34 Definition. ∂E

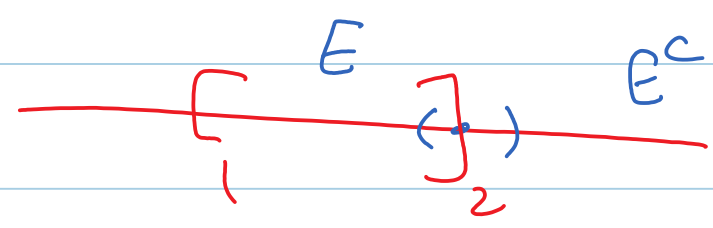
Let $E \subseteq \mathbf{R}^n$. The *boundary* of E is the set

$$\partial E := \{\mathbf{x} \in \mathbf{R}^n : \text{ for all } r > 0, \quad B_r(\mathbf{x}) \cap E \neq \emptyset \text{ and } B_r(\mathbf{x}) \cap E^c \neq \emptyset\}.$$

[We will refer to the last two conditions in the definition of ∂E by saying that $B_r(\mathbf{x})$ *intersects* E and E^c.]







$B_\varepsilon(2) \cap E \neq \phi$

and $B_\varepsilon(2) \cap E^c \neq \phi$

ex. $E = (1,2)$

$\partial E = \{1,2\}$.

ex. $E = \{ (x,y) : x^2 + 4y^2 \le 1 \}$.

Find E°, $\bar{E}$, ∂E.

Sol. Since E is closed, $\bar{E} = E$.

$E^\circ = \{ (x,y) : x^2 + 4y^2 < 1 \}$

$\partial E = \{ (x,y) : x^2 + 4y^2 = 1 \}$

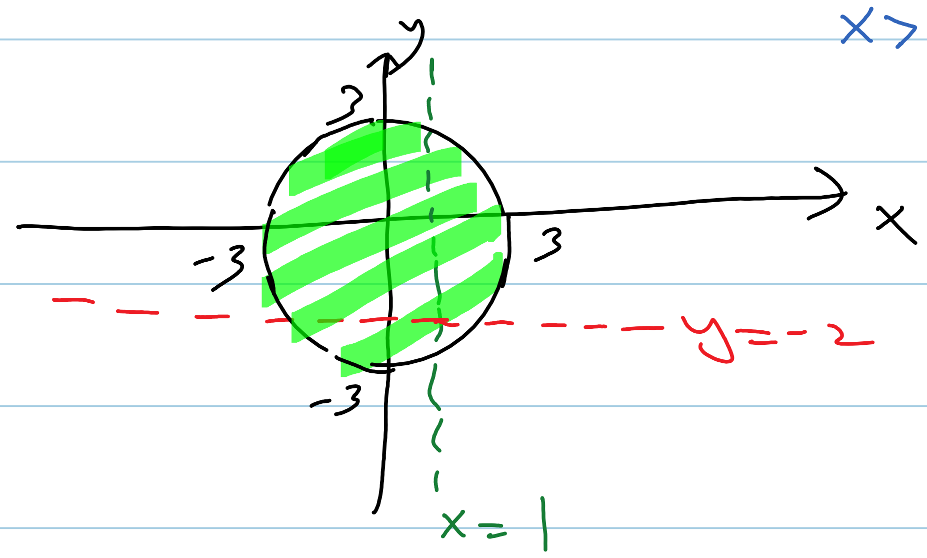
8.35 ***EXAMPLE.***

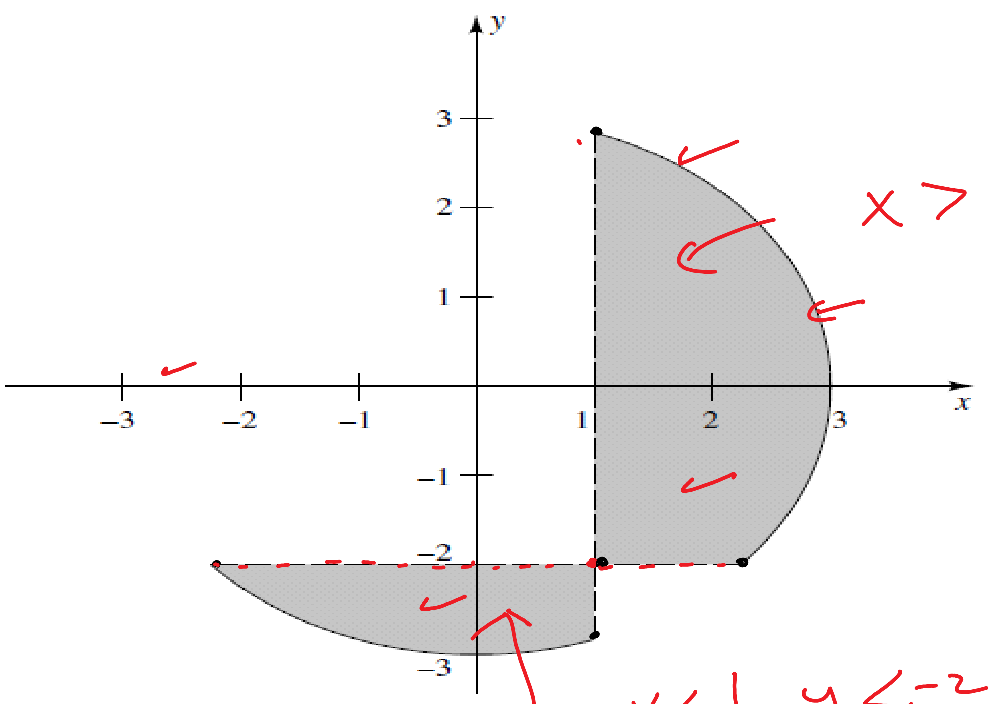
Describe the boundary of the set (∂E)

$$E = \{(x, y) : x^2 + y^2 \le 9 \text{ and } (x - 1)(y + 2) > 0\}.$$

$x > 1$ and $y > -2$

or $x < 1$, $y < -2$





$x > 1$ and $y > -2$

$x < 1$, $y < -2$

$\partial E = \{ (x,y) : x^2 + y^2 \le 9 \text{ and } (x-1)(y+2) \ge 0 \}$
$\setminus \{ (x,y) : x^2 + y^2 < 9 \text{ and } (x-1)(y+2) > 0 \}$

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8.36 Theorem. *Let $E \subseteq \mathbf{R}^n$. Then $\partial E = \overline{E} \setminus E^o$.*

ex. $E = (1,2]$ in $\mathbb{R}$, $E^o = (1,2)$, $\overline{E} = [1,2]$

$\overline{E} \setminus E^o = \{1,2\} = \partial E$

proof. $\overline{E} \setminus E^o = \overline{E} \cap (E^o)^c$

we need to show $\partial E = \underbrace{\overline{E} \cap (E^o)^c}_{x}$

It sufficies to show

$x \in \overline{E} \iff B_r(x) \cap E \neq \phi, \ \forall r > 0$ — (1)

and $x \notin E^o \iff B_r(x) \cap E^c \neq \phi, \ \forall r > 0$ — (2)

we will prove (1) and leave the prof of (2) to the student.

($\Rightarrow$) let $\boxed{x \in \overline{E}}$ we need to show

$B_r(x) \cap E \neq \phi, \ \forall r > 0$

Spse not, $\exists r_0 > 0$ s.t $B_{r_0}(x) \cap E = \phi$

Then $\underbrace{(B_{r_0}(x))^c}_{\text{closed}} \supseteq E$

$B_{r_0}(x) \subseteq E^c$

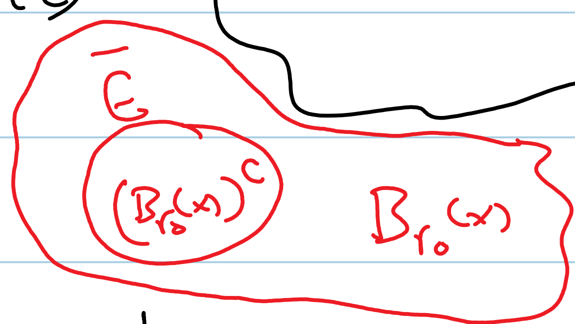
$(B_{r_0}(x))^c \supseteq E$

$A \subseteq B \Rightarrow A^c \supseteq B^c$

that is $(B_{r_0}(x))^c$ is closed set contains E.

hence by thm 8.32 (iii)

$(B_{r_0}(x))^c \supseteq \overline{E}$



$\Rightarrow \overline{E} \cap B_{r_0}(x) = \phi$

$\Rightarrow x \notin \overline{E}$, a contradiction.

($\Leftarrow$)
Conversely of (1) :

spse that $B_r(x) \cap E \neq \phi$, $\forall r > 0$
we need to show $x \in \bar{E}$
Spse not, i.e, $x \notin \bar{E} \Rightarrow x \in (\bar{E})^c$ open

$\exists$ an $r_0 > 0$ such that

$B_{r_0}(x) \subseteq (\bar{E})^c$

$\Rightarrow$

$\phi = B_{r_0}(x) \cap \bar{E} \supseteq B_{r_0}(x) \cap E$

U is open
$\forall x \in U, \exists r > 0$
s.t $B_r(x) \subseteq U$

$\Rightarrow B_{r_0}(x) \cap E = \phi$, for some $r_0 > 0$, a contradiction.

$\therefore x \in \bar{E}$ this proves (1)

proof of (2) exercise

Rmk: (1) let $E \subseteq \mathbb{R}^n$. then $(\bar{A})^c = (A^c)^\circ$ and $(A^\circ)^c = \overline{(A^c)}$

(2) If $A \subseteq B \subseteq \mathbb{R}^n$. then
(i) $A^\circ \subseteq B^\circ$
(ii) $\bar{A} \subseteq \bar{B}$ (Exercise).

Pf. (2) (i) $A^\circ \subseteq B^\circ$.

Pf. We have $A^\circ \subseteq A \subseteq B$ given.

$\Rightarrow A^\circ \subseteq B$

A° is open set contained in B
by thm 8.32 (ii), $A^\circ \subseteq B^\circ$

$(A\cup B)^\circ \neq A^\circ \cup B^\circ$

8.37 Theorem. *Let A, $B \subseteq \mathbf{R}^n$. Then*

i) $(A\cup B)^o \supseteq A^o \cup B^o, \quad (A\cap B)^o = A^o \cap B^o,$

ii) $\overline{A\cup B} = \overline{A}\cup\overline{B}, \quad \overline{A\cap B} \subseteq \overline{A}\cap\overline{B},$ $\quad \overline{A\cap B} \neq \overline{A}\cap\overline{B}$

iii) $\partial(A\cup B) \subseteq \partial A\cup\partial B,$ *and* $\partial(A\cap B) \subseteq \partial A\cup\partial B.$

$\partial(A\cup B) \neq \partial A \cup \partial B$

$\partial(A\cap B) \neq \partial A \cap \partial B.$

Proof. (i) $(A\cup B)^\circ \supseteq A^\circ \cup B^\circ$

we have $A \subseteq A\cup B$, $B \subseteq A\cup B$

$\overset{\text{Last remark}}{\Longrightarrow} A^\circ \subseteq (A\cup B)^\circ, B^\circ \subseteq (A\cup B)^\circ$

$\Rightarrow A^\circ \cup B^\circ \subseteq (A\cup B)^\circ$

Similarly, $A\cap B \subseteq A$, $A\cap B \subseteq B$

$\overset{\text{Last remark}}{\Longrightarrow} (A\cap B)^\circ \subseteq A^\circ, (A\cap B)^\circ \subseteq B^\circ$

$\Rightarrow \boxed{(A\cap B)^\circ \subseteq A^\circ \cap B^\circ.}$ — (*)

we need to show $A^\circ\cap B^\circ \subseteq (A\cap B)^\circ$:

We have $A^\circ \cap B^\circ \subseteq A^\circ \subseteq A$

$A^\circ \cap B^\circ \subseteq B^\circ \subseteq B$

$\Rightarrow A^\circ \cap B^\circ \subseteq A\cap B$

open set

We have $A^\circ \cap B^\circ$ is an open set contained in $A\cap B$

by thm 8.32 (ii), $\boxed{A^\circ \cap B^\circ \subseteq (A\cap B)^\circ}$ (**)

$(*)$ & $(**)$ give $(A\cap B)^\circ = A^\circ \cap B^\circ$

(ii) $\overline{A\cap B} \subseteq \bar{A}\cap\bar{B}$

$A\cap B \subseteq A \subseteq \bar{A}$
$A\cap B \subseteq B \subseteq \bar{B}$ $\Rightarrow A\cap B \subseteq \bar{A}\cap\bar{B}$

$\Rightarrow \bar{A}\cap\bar{B}$ is closed set contains $A\cap B$

$\xRightarrow{\text{thm 32. (iii)}}$ $\bar{A}\cap\bar{B} \supseteq \overline{A\cap B}$

Similarly, $\bar{A}\cup\bar{B}$ is closed set contains $A\cup B$

Thm 32 (ii) $\Rightarrow$ $\bar{A}\cup\bar{B} \supseteq \overline{A\cup B}$

We need to show $\bar{A}\cup\bar{B} \subseteq \overline{A\cup B}$

let $x \in \bar{A}\cup\bar{B} \Rightarrow x\in\bar{A}$ or $x\in\bar{B}$

we need to show $x\in\overline{A\cup B}$

Spse not, $x\notin\overline{A\cup B}$, by def'n,

$\exists$ closed set $E \supseteq A\cup B$ such that $x\notin E$.

i.e, $E\supseteq A$ and $E\supseteq B$, $x\notin E$

$\Rightarrow x\notin\bar{A}$ and $x\notin\bar{B}$

$\Rightarrow x\notin\bar{A}\cap\bar{B}$, a contradiction

$\therefore x\in\overline{A\cup B}$

$\therefore \bar{A}\cup\bar{B} \subseteq \overline{A\cup B}$.

(iii) $\partial(A\cup B) \subseteq \partial A \cup \partial B$

Let $x \in \partial(A\cup B)$, by def'n

$B_r(x) \cap (A\cup B) \neq \phi$ and

$B_r(x) \cap \underbrace{(A\cup B)^c}_{A^c \cap B^c} \neq \phi$, $\forall r>0$.

$\Rightarrow B_r(x) \cap A^c \neq \phi$ and $B_r(x) \cap B^c \neq \phi$

$\Rightarrow$ $B_r(x) \cap A \neq \phi$ and $B_r(x) \cap A^c \neq \phi$

OR $\underbrace{B_r(x) \cap B \neq \phi \text{ and } B_r(x) \cap B^c \neq \phi}$

$x \in \partial A \cup \partial B$.

$\Rightarrow \partial(A\cup B) \subseteq \partial A \cup \partial B$.

$\partial(A\cap B) \subseteq \partial A \cup \partial B$ (exercise)

is not connected

Section 8.4

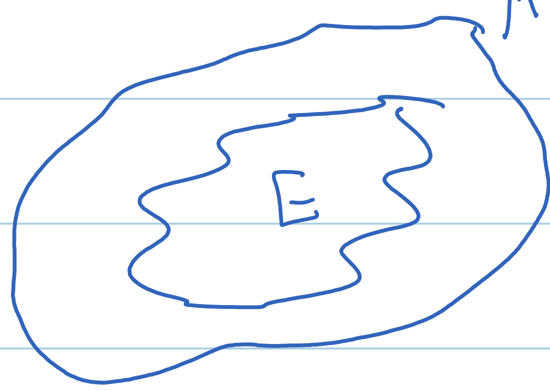
***8.38 Theorem.** *Let $E \subseteq \mathbf{R}^n$. If there exist nonempty, relatively open sets U, V which separate E, then there is a pair of open sets A, B such that $A \cap E \neq \emptyset$, $B \cap E \neq \emptyset$, $A \cap B = \emptyset$, and $E \subseteq A \cup B$.*

توضيح

$\underbrace{(A\cap E) \cup (B\cap E)} = E$

$(A\cup B) \cap E$

AUB



If $E \subseteq A\cup B$,

$(A\cup B) \cap E = E$

Section 8.3

8.29 Remark. *Let $E \subseteq \mathbf{R}^n$. If there exists a pair of open sets A, B such that $E \cap A \neq \emptyset$, $E \cap B \neq \emptyset$, $E \subseteq A \cup B$, and $A \cap B = \emptyset$, then E is not connected.*

Proof Remark 8.29 (H.W's). p. 293.

H.W's (8.4) 1, 2, 3, 5, 6, 7, 9, 10.

The End of ch8 (8.1 — 8.4).

CHAPTER 9

Convergence in $\mathbf{R}^n$

Revision of ch2 and ch3

CH3

In this chapter we generalize the concepts of limits and continuity from $\mathbf{R}$ to $\mathbf{R}^n$. We begin, as we did in Chapter 2, with sequences. CH2

9.1 LIMITS OF SEQUENCES

$x_k = (x_k^{(1)}, x_k^{(2)}, \dots, x_k^{(n)})$, $k = 1, 2, 3 \dots$

$x_1 = (x_1^{(1)}, x_1^{(2)}, \dots, x_1^{(n)})$

$x_2 = (x_2^{(1)}, x_2^{(2)}, \dots, x_2^{(n)})$

9.1 Definition. Let $\{\mathbf{x}_k\}$ be a sequence of points in $\mathbf{R}^n$ $= \{x_1, x_2, x_3, \dots\}$

i) $\{\mathbf{x}_k\}$ is said to *converge* to some point $\mathbf{a} \in \mathbf{R}^n$ (called the *limit* of $\mathbf{x}_k$) if and only if for every $\varepsilon > 0$ there is an $N \in \mathbf{N}$ such that

$x_k \to a \in \mathbb{R}^n$

$$k \ge N \quad \text{implies} \quad \|\mathbf{x}_k - \mathbf{a}\| < \varepsilon.$$

Notation: $\mathbf{x}_k \to \mathbf{a}$ as $k \to \infty$ or $\mathbf{a} = \lim_{k\to\infty} \mathbf{x}_k$.

$|x_k| \le M$

ii) $\{\mathbf{x}_k\}$ is said to be *bounded* if and only if there is an $M > 0$ such that $\|\mathbf{x}_k\| \le M$ for all $k \in \mathbf{N}$.

iii) $\{\mathbf{x}_k\}$ is said to be *Cauchy* if and only if for every $\varepsilon > 0$ there is an $N \in \mathbf{N}$ such that

$$k, m \ge N \quad \text{imply} \quad \|\mathbf{x}_k - \mathbf{x}_m\| < \varepsilon.$$

ex. use the last Def'n (i) to prove that

$$x_k = \left(\frac{1}{k}, \underbrace{1 - \frac{1}{k^2}}_{x_k^{(2)}}\right) \longrightarrow (0,1) = a \text{ as } k \to \infty$$

$\frac{1}{k} = x_k^{(1)}$, $(0,1) \in \mathbb{R}^2$

Proof. Let $\varepsilon > 0$ we need to find $N \in \mathbb{N}$ s.t if $k \ge N$, then

$$\|x_k - a\| = \left\|\left(\frac{1}{k}, 1 - \frac{1}{k^2}\right) - (0,1)\right\| < \varepsilon.$$

Now, $\|x_k - a\|^2 = \left\|\left(\frac{1}{k}, 1 - \frac{1}{k^2} - 1\right)\right\|^2$

$$= \left\|\left(\frac{1}{k}, -\frac{1}{k^2}\right)\right\|^2$$

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$$= \left(\frac{1}{k}\right)^2 + \left(\frac{-1}{k^2}\right)^2$$

$$= \frac{1}{k^2} + \frac{1}{k^4} \leq \frac{1}{k^2} + \frac{1}{k^2} = \frac{2}{k^2} < \varepsilon^2$$

For $\varepsilon > 0$, $\exists\, N \in \mathbb{N}$ s.t $k > N \Rightarrow k > \frac{\sqrt{2}}{\varepsilon}$

If $k > N$, then

$$\|x_k - a\|^2 = \frac{2}{k^2} < 2\left(\frac{\varepsilon}{\sqrt{2}}\right)^2 = \varepsilon^2$$

i.e, $\|x_k - a\| < \varepsilon, \ \forall k > N$.

Thus, $k > N \Rightarrow \left\|\left(\frac{1}{k}, 1 - \frac{1}{k^2}\right) - (0, 1)\right\| < \varepsilon$.

9.2 Theorem. *Let* $\mathbf{a} = (a_1, a_2, \ldots, a_n) \in \mathbf{R}^n$ *and* $\{\mathbf{x}_k = (x_k^{(1)}, x_k^{(2)}, \ldots, x_k^{(n)})\}_{k \in \mathbf{N}}$ *be a sequence in* $\mathbf{R}^n$. *Then* $\mathbf{x}_k \to \mathbf{a}$, *as* $k \to \infty$, *if and only if for each* $j \in \{1, 2, \ldots, n\}$, *the component sequence* $x_k^{(j)} \to a_j$ *as* $k \to \infty$.

Application. Let $x_k = \left(\underbrace{1}_{x_k^{(1)}}, \underbrace{\sin k\pi}_{x_k^{(2)}}, \underbrace{\cos\left(\tfrac{1}{k}\right)}_{x_k^{(3)}}\right)$

find $\lim_{k \to \infty} x_k$.

Sol. $x_k^{(1)} = 1 \longrightarrow 1$ as $k \to \infty$.

$x_k^{(2)} = \sin(k\pi) = 0, \ \forall k$

$\therefore \ x_k^{(2)} \longrightarrow 0$ as $k \to \infty$.

$x_k^{(3)} = \cos\left(\frac{1}{k}\right) \longrightarrow \cos 0 = 1$ as $k \to \infty$

$$\therefore \lim_{k\to\infty} x_k = (1,0,1) \in \mathbb{R}^3.$$

ex. $x_k = (\underbrace{\log(k+1) - \log k}_{x_k^{(1)}}, \underbrace{2^{-k}}_{x_k^{(2)}}) \in \mathbb{R}^2$

find $\lim_{k\to\infty} x_k$

Sol. $x_k^{(1)} = \log\left(\frac{k+1}{k}\right) \longrightarrow \log 1 = 0$ as $k \to \infty$

$x_k^{(2)} = 2^{-k} \longrightarrow 0$ as $k \to \infty$

$$\therefore \lim_{k\to\infty} x_k = (0,0) \in \mathbb{R}^2.$$

Ex. (prove the last two examples by def'n).

Proof of thm 9.2 Let $a = (a_1, a_2, --, a_n) \in \mathbb{R}^n$.

$$\{x_k = (x_k^{(1)}, x_k^{(2)}, --, x_k^{(n)})\}_{k\in\mathbb{N}} \in \mathbb{R}^n.$$

($\Rightarrow$) Suppose that $x_k \longrightarrow a$ as $k \to \infty$.

we need to prove $x_k^{(j)} \longrightarrow a_j$ as $k \to \infty$ and $j \in \{1, 2, --, n\}$.

(Recall remark 8.7 (i)

$$\|x\|_\infty \le \underbrace{\|x\| \le \sqrt{n}\,\|x\|_\infty}$$
)

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$$\underbrace{\|x_k - a\|}_{(} \le \sqrt{n}\, \underbrace{\|x_k - a\|_\infty}_{\text{def'n}}$$

$\forall j = 1, \cdots, n$

$$|x_k^{(j)} - a_j| \le \sqrt{\sum_{j=1}^{n} |x_k^{(j)} - a_j|} \le \sqrt{n} \max_{1 \le j \le n} |x_k^{(j)} - a_j|$$

$\Rightarrow \forall j = 1, \cdots, n,$ $\boxed{|x_k^{(j)} - a| \le \|x_k - a\| \le \sqrt{n} \max |x_k^{(j)} - a_j|}$

$$-\|x_k - a\| \le x_k^{j} - a \le \|x_k - a\|$$

$-\|x_k-a\| \to 0$ as $k \to \infty$; $\|x_k - a\| \to 0$ (given) as $k \to \infty$

$\Rightarrow$ by squeeze thm, $x_k^{(j)} - a \to 0$

i.e, $x_k^{(j)} \to a$ as $k \to \infty$

Conversely, $\underbrace{\|x_k - a\|} \le \sqrt{n} \max_{1 \le j \le n} |x_k^{(j)} - a| \to 0$

If $x_k^{(j)} \to a$ as $k \to \infty$, by squeeze thm, $x_k - a \to 0$ as $k \to \infty$

i.e, $x_k \to a$ as $k \to \infty$

$(\frac{1}{2}, \frac{3}{2}, 1) \in \mathbb{Q}^3$

9.3 Theorem. *Let* $\mathbf{Q}^n := \{\mathbf{x} \in \mathbf{R}^n : x_j \in \mathbf{Q}$ *for* $j = 1, 2, \dots, n\}$. *For each* $\mathbf{a} \in \mathbf{R}^n$ *there is a sequence* $\mathbf{x}_k \in \mathbf{Q}^n$ *such that* $\mathbf{x}_k \to \mathbf{a}$ *as* $k \to \infty$.

Proof. Let $a = (a_1, a_2, \cdots, a_n) \in \mathbb{R}^n$

For $j = 1$, $a_1 \in \mathbb{R}$, $\exists\, r_k^{(1)} \in \mathbb{Q}$ s.t $r_k^{(1)} \to a_1$ (thm in ch 2).

Similarly $\exists\, r_k^{(2)} \in \mathbb{Q}$ s.t $r_k^{(2)} \to a_2$

$\forall\ 1 \le j \le n$, $\exists\ r_k^{(j)} \in \mathbb{Q}$ s.t

$r_k^{(j)} \longrightarrow a_j$ (in $\mathbb{R}$) as $k \to \infty$

by Thm 9.2, $x_k := (r_k^{(1)}, r_k^{(2)}, \ldots, r_k^{(n)}) \in \mathbb{Q}^n$

converges to $(a_1, a_2, \ldots, a_n) = a$ (in $\mathbb{R}^n$).

as $k \to \infty$. $\blacksquare$

9.4 Theorem. *Let $n \in \mathbf{N}$.*

i) *A sequence in $\mathbf{R}^n$ can have at most one limit.*

ii) *If $\{\mathbf{x}_k\}_{k\in\mathbf{N}}$ is a sequence in $\mathbf{R}^n$ which converges to $\mathbf{a}$ and $\{\mathbf{x}_{k_j}\}_{j\in\mathbf{N}}$ is any subsequence of $\{\mathbf{x}_k\}_{k\in\mathbf{N}}$, then $\mathbf{x}_{k_j}$ converges to $\mathbf{a}$ as $j \to \infty$.*

iii) *Every convergent sequence in $\mathbf{R}^n$ is bounded, but not conversely.*

iv) *Every convergent sequence in $\mathbf{R}^n$ is Cauchy.* and conversely.

v) *If $\{\mathbf{x}_k\}$ and $\{\mathbf{y}_k\}$ are convergent sequences in $\mathbf{R}^n$ and $\alpha \in \mathbf{R}$, then*

$$\lim_{k\to\infty} (\mathbf{x}_k + \mathbf{y}_k) = \lim_{k\to\infty} \mathbf{x}_k + \lim_{k\to\infty} \mathbf{y}_k,$$

CH2
$$\lim_{k\to\infty} (\alpha \mathbf{x}_k) = \alpha \lim_{k\to\infty} \mathbf{x}_k,$$

and $\mathbb{R}^n$

$$\lim_{k\to\infty} (\mathbf{x}_k \cdot \mathbf{y}_k) = (\lim_{k\to\infty} \mathbf{x}_k) \cdot (\lim_{k\to\infty} \mathbf{y}_k).$$

Moreover, when $n = 3$,

$$\lim_{k\to\infty} (\mathbf{x}_k \times \mathbf{y}_k) = (\lim_{k\to\infty} \mathbf{x}_k) \times (\lim_{k\to\infty} \mathbf{y}_k).$$

- $\lim_{k\to\infty} \|x_k\| = \|\lim_{k\to\infty} x_k\|$, if x_k converges.

(i.e, if $x_k \longrightarrow a$, then $\|x_k\| \longrightarrow \|a\|$ as $k \to \infty$ in $\mathbb{R}^n$).

Pf. $\lim_{k\to\infty} \|x_k\|^2 = \lim_{k\to\infty} (x_k \cdot x_k) = (\lim_{k\to\infty} x_k) \cdot (\lim_{k\to\infty} x_k) = a \cdot a$

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$$\Rightarrow \underbrace{\lim_{k\to\infty} \|x_k\|^2} = \|a\|^2$$

$$\Rightarrow \lim_{k\to\infty}\left(\|x_k\| \cdot \|x_k\|\right) = \|a\|^2$$

$$\lim_{k\to\infty} \|x_k\| \lim_{k\to\infty} \|x_k\| = \|a\|^2$$

$$\left(\lim_{k\to\infty} \|x_k\|\right)^2 = \|a\|^2$$

$$\Rightarrow \boxed{\lim_{k\to\infty} \|x_k\| = \|a\|}$$

9.5 Theorem. [**Bolzano–Weierstrass Theorem for $\mathbf{R}^n$**]. *Every bounded sequence in $\mathbf{R}^n$ has a convergent subsequence.*

Proof. Exercise.

9.6 Theorem. *A sequence $\{\mathbf{x}_k\}$ in $\mathbf{R}^n$ is Cauchy if and only if it converges.*

Proof. Exercise.

$x_k \to a$ $\forall \varepsilon>0, \exists N\in\mathbb{N}$ s.t

$k\geq N \Rightarrow \underbrace{\|x_k-a\|<\varepsilon}_{x_k\in B_\varepsilon(a)}$

9.7 Theorem. *Let $\mathbf{x}_k \in \mathbf{R}^n$. Then $\mathbf{x}_k \to \mathbf{a}$ as $k \to \infty$ if and only if for every open set V which contains $\mathbf{a}$ there is an $N \in \mathbf{N}$ such that $k \geq N$ implies $\mathbf{x}_k \in V$.*

Proof. ($\Rightarrow$) Spse that $x_k \to a$ as $k\to\infty$ and

Let V be an open set contains a.

We need to prove that $\exists$ an $N\in\mathbb{N}$ s.t

$k\geq N \Rightarrow x_k \in V$.

Since $a\in V$ "open", then by def'n, $\exists$ an $\varepsilon>0$

s.t $\boxed{B_\varepsilon(a) \subseteq V}$ Given this ε,

Since $x_k \to a$, by def'n, $\exists$ $N\in\mathbb{N}$ s.t

$k\geq N \Rightarrow \underbrace{\|x_k-a\|<\varepsilon}$

i.e; $x_k \in B_\varepsilon(a) \subseteq V$, $\forall k\geq N$.

$\Rightarrow x_k \in V$, $\forall k\geq N$.

Conversely, ($\Leftarrow$) Spse that for every open set V which contains a, $x_k\in V$, $\forall k\geq N$.

We need to show $x_k \to a$.

Let $\varepsilon>0$, Notice that

$B_\varepsilon(a)$ is open set which contains a.

Hence, by hypothesis, $\exists$ an $N\in\mathbb{N}$ s.t

$k\geq N \Rightarrow x_k\in B_\varepsilon(a)$

In particular, $\|x_k-a\|<\varepsilon$, $\forall k\geq N$

this means $x_k \to a$ as $k\to\infty$

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$x_k \in E$ conv.
$\lim x_k \in E$
$x_1, x_2 \cdots \to x$
E

9.8 Theorem. *Let $E \subseteq \mathbf{R}^n$. Then E is closed if and only if E contains all its limit points; that is, if and only if $\mathbf{x}_k \in E$ and $\mathbf{x}_k \to \mathbf{x}$ imply that $\mathbf{x} \in E$.*

Proof. ($\Rightarrow$) Let $E \subseteq \mathbb{R}^n$. If $E = \Phi$, then the thm is satisfied.

Let $E \neq \Phi$ be closed. We need to show that if $x_k \in E$ and $x_k \to x$, then $x \in E$.

Spse not, i.e., $E \neq \Phi$ is closed but some sequence $x_k \in E$, $x_k \to x$ but $x \in E^c$.

Since E is closed, then E^c is open.

thus by the last thm, $\exists$ an $N \in \mathbb{N}$ s.t $k \geq N \Rightarrow x_k \in E^c$.

i.e., $x_k \notin E$, $\forall k \geq N$, a contradiction.

hence $x \in E$.

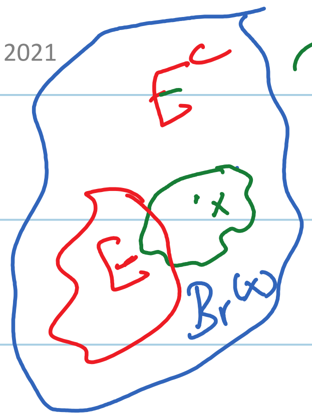
($\Leftarrow$) Conversely, spse that $E \neq \Phi$ which contains all limit points (i.e, if $x_k \in E$, $x_k \to x$, then $x \in E$).

We need to show that E is closed.

Spse that E is not closed, then $E \neq \mathbb{R}^n$. and by def'n, $\boxed{E^c \neq \Phi}$ and not open.

تذكر :

V is open
$\forall x \in V, \exists r > 0$
s.t $B_r(x) \subseteq V$



Thus, $\exists$ at least one point $\boxed{x \in E^c}$ such that $\underline{\underline{\text{no}}}$ ball $B_r(x)$ is contained in E^c

let $x_1 \in B_1(x) \cap E$

$x_2 \in B_{\frac{1}{2}}(x) \cap E$

$x_3 \in B_{\frac{1}{3}}(x) \cap E$

$\vdots$

Let $x_k \in B_{\frac{1}{k}}(x) \cap E$, $k = 1, 2, 3, ---$

then $x_k \in E$ and $\underbrace{x_k \in B_{\frac{1}{k}}(x)}$, $\forall k \in \mathbb{N}$

$x_k \in E$ and $0 \leq \underbrace{\|x_k - x\|} < \frac{1}{k}$, $\forall k \in \mathbb{N}$

Now, by squeeze thm, $\|x_k - x\| \to 0$ as $k \to \infty$

(i.e., $x_k \to x$ as $k \to \infty$).

Thus, $x_k \in E$, $x_k \to x$ but $x \notin E$.,

a contradiction.

Hence E is closed $\blacksquare$

$\underline{\text{HW's}}$ 1, 2, 3, 4, 5, 6.

9.2 HEINE–BOREL THEOREM

$y_1 \in E \quad r(y_1) > 0 \quad r_1$

$E = [a,b] \quad \mathbb{R}$

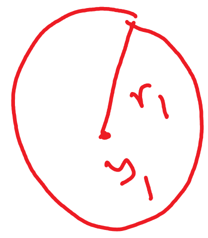
$y_1 \longrightarrow r(y_1) = r_1$

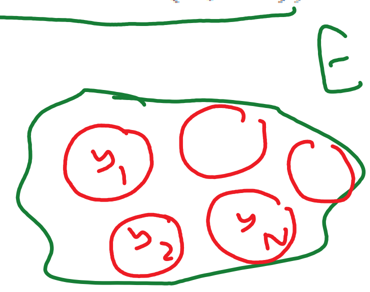
$r : E \longrightarrow (0, \infty) \qquad r_2 = r(y_2)$

9.9 ***Lemma.*** **[Borel Covering Lemma].**
Let E be a closed, bounded subset of $\mathbf{R}^n$. If r is any function from E into $(0, \infty)$, then there exist finitely many points $\mathbf{y}_1, \ldots, \mathbf{y}_N \in E$ such that

$$E \subseteq \bigcup_{j=1}^{N} B_{r(\mathbf{y}_j)}(\mathbf{y}_j).$$

$B_{r_j}(y_j)$





9.10 Definition.

Let E be a subset of $\mathbf{R}^n$. $\qquad E \subseteq \mathbb{R}^n$

i) An *open covering* of E is a collection of sets $\{V_\alpha\}_{\alpha \in A}$ such that each V_α is open and

$$\boxed{E \subseteq \bigcup_{\alpha \in A} V_\alpha.}$$

ii) The set E is said to be *compact* if and only if ~~every open covering of E has a *finite subcovering*~~; that is, if and only if given any open covering $\{V_\alpha\}_{\alpha \in A}$ of E, there is a finite subset $A_0 = \{\alpha_1, \ldots, \alpha_N\}$ of A such that

$$\boxed{E \subseteq \bigcup_{j=1}^{N} V_{\alpha_j}.} \qquad E \subseteq V_{\alpha_1} \cup V_{\alpha_2} \cup \cdots \cup V_{\alpha_N}$$

Ex. Let $E = (0,1) \subseteq \mathbb{R}$. Let $\{V_n\}_{n \in \mathbb{N}}$ s.t $V_n = (\frac{1}{n}, 1 - \frac{1}{n})$ be a collection of open sets. Prove that $\{V_n\}$ is an open covering of E.

Pf. $\bigcup_{n=1}^{\infty} V_n = \bigcup_{n=1}^{\infty} (\frac{1}{n}, 1 - \frac{1}{n}) = (0,1) = E$

$\therefore \{V_n\}$ is an infinite open covering of E.

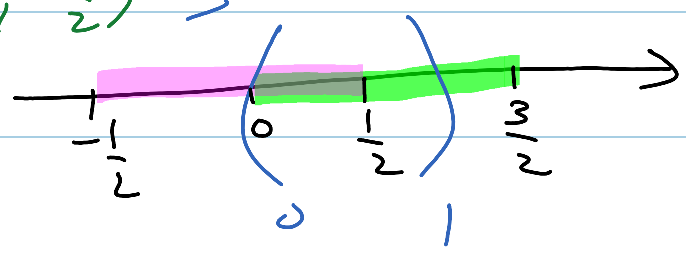
Notice that $(0,1)$ is not compact.

ex. Let $E = [1, \infty) \subseteq \mathbb{R}$. Then the Collection $\{V_n\}_{n \in \mathbb{N}}$ s.t $V_n = (1 - \frac{1}{n}, n)$ is an an open covering of E. since

$$\bigcup_{n\in\mathbb{N}} V_n = \bigcup_{n\in\mathbb{N}} \left(1-\tfrac{1}{n}, n\right) \supseteq [1,\infty) = E.$$

ex. $E = (0,1) \subseteq \mathbb{R}$. Then the collection $\{V_1, V_2\}$, where $V_1 = \left(-\tfrac{1}{2}, \tfrac{1}{2}\right)$, $V_2 = \left(0, \tfrac{3}{2}\right)$.
Then $\{V_1, V_2\}$ is a finite open cover of E

Pf: $V_1 \cup V_2 = \left(-\tfrac{1}{2}, \tfrac{1}{2}\right) \cup \left(0, \tfrac{3}{2}\right) \supseteq (0,1) = E$



ex. $V_n = (-n, n) : n \in \mathbb{N}$ is an open covering of $\mathbb{R}$. which is infinite.

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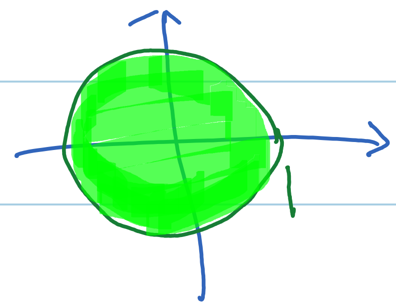
9.11 Theorem. [Heine–Borel Theorem]. $\Leftrightarrow$
Let E be a subset of $\mathbf{R}^n$. Then E is $\boxed{\text{compact}}$ if and only if E is closed and bounded.

ex. Let $E=(0,1)$ is not compact since it is not closed.

ex. $E=[1,\infty)$ is not compact since it is unbounded.

ex. $E=[1,2]$ is compact since it is bounded and closed.

ex. Let $E=\{(x,y)\in\mathbb{R}^2 : x^2+y^2\le 1\} = B_1(0,0)$ is compact since it is closed and bounded



Proof. $(\Rightarrow)$ Let $E\subseteq\mathbb{R}^n$ be compact. We need to show that E is closed and bounded.

$B_n(0)$
$(-n,n)$

Since $\{B_k(0)\}_{k\in\mathbb{N}}$ is an open covering of $\mathbb{R}^n$. $\left(\bigcup_{k\in\mathbb{N}} B_k(0)\supseteq\mathbb{R}^n\supseteq E\right)$, hence of E. Since E is compact, $\exists\, N\in\mathbb{N}$ s.t $E\subseteq\bigcup_{k=1}^{N}B_k(0)\Rightarrow \|E\|\le N$

$|x|<N$
$-N, N$
$|B_k(0)|<N$

E is closed $\iff \forall\, x_k \in E,\ x_k \to x$, then $x \in E$.

$\therefore E$ is bounded.

To verify that E is closed, spse not (E is not closed) and $E \neq \emptyset$

by thm 9.8, $\exists$ a convergent seq. $x_k \in E$, $x_k \longrightarrow x$ but $x \notin E$. For each $y \in E$,

set $\boxed{r(y) := \frac{\|x-y\|}{2}}$. Since $x \notin E$, $r(y) > 0$.

Thus, $B_{r(y)}(y)$ is open and contains y.

That is $\left\{ B_{r(y)}(y) : y \in E \right\}$ is an open covering of E. Since E is compact,

$\exists\ y_j$ and $r_j := r(y_j)$, $j = 1, 2, ---, M$

s.t $E \subseteq \bigcup_{j=1}^{M} B_{r_j}(y_j)$.

$x_k \in E \subseteq \cup B_r$

$x_k \in B_{r_1}$ or B_{r_2} ----

Set $r := \min\{ r_1, r_2, ---, r_M \}$

Since $x_k \longrightarrow x$ as $k \to \infty$, $x_k \in B_r(x)$ for large k.

But $x_k \in B_r(x) \cap E \Rightarrow x_k \in B_{r_j}(y_j)$ for some $j = 1, ---, M$.

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$x_k \in B_{r_j}(y_j)$

Thus, $r_j > \|x_k - y_j\| = \|x_k - x + x - y_j\|$

$= \|(x-y_j) - (x - x_k)\|$

توضيح

$r(y) = \frac{\|x-y\|}{2}$

$r_j = r(y_j) = \frac{\|x-y_j\|}{2}$

$\geq \|x-y_j\| - \|x_k - x\|$

$= 2r_j - \|x_k - x\|$

$x_k \in B_r(x)$

$\|x_k - x\| < r$

$> 2r_j - r$

$r = \min\{r_1, \dots, r_m\}$

$r \leq r_j, \forall j$

$-r \geq -r_j$

$\geq 2r_j - r_j$

$= r_j$

$\Rightarrow r_j > r_j$, a contradiction.

$\therefore E$ is closed.

($\Leftarrow$) Conversely, Spse that E is closed and bounded. We need to show E is compact.

Let $\{V_\alpha\}_{\alpha \in A}$ be an open covering of E.

(i.e, $\bigcup_{\alpha \in A} V_\alpha \supseteq E$). Let $x \in E$

$\Rightarrow x \in V_\alpha$ (open) for some $\alpha \in A$.

open set. تعريف

$\Rightarrow \exists\, r(x) > 0$ s.t $B_{r(x)}(x) \subseteq V_\alpha$.

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thus, by the Borel covering Lemma,

$\exists$ finite many points $x_1, x_2, ---, x_N$
and $r_j = r(x_j)$ s.t

$$E \subseteq \bigcup_{j=1}^{N} B_{r_j}(x)$$

and we have, $\forall j$, $\exists$ $\alpha_j \in A$
s.t $\boxed{B_{r_j}(x_j) \subset V_{\alpha_j}}$

$$\Rightarrow E \subseteq \bigcup_{j=1}^{N} V_{\alpha_j}$$

this means $\{V_{\alpha_j}\}_{j=1}^{N}$ is a finite
subcovering of E.
$\therefore$ E is compact. $\blacksquare$

Rmk. The Heine-Borel thm no longer holds if either "closed" or bounded is dropped

ex. (1) $E=(0,1)$ is bounded but not closed

and $E=(0,1) = \bigcup_{n\in\mathbb{N}} \left(\frac{1}{n}, 1-\frac{1}{n}\right)$

$\therefore \left\{\left(\frac{1}{n}, 1-\frac{1}{n}\right)\right\}_{n\in\mathbb{N}}$ is an open covering of E but has no finite subcovering.

(2) $V=[1,\infty)$ is closed but unbounded.

and $[1,\infty) \subseteq \bigcup_{n\in\mathbb{N}} \left(1-\frac{1}{n}, n\right)$

has no finite subcovering,

Applications. (Heine-Borel thm).

9.12 *EXAMPLE.*

Suppose that E is a closed, bounded subset of $\mathbf{R}$. If for every $x \in E$ there exist a nonnegative function $f = f_x$ and a number $r = r(x) > 0$ such that f is differentiable on $\mathbf{R}$, $f(t) > 0$ for $t \in (x-r, x+r)$, and $f(t) = 0$ for $t \notin (x-2r, x+2r)$, prove that there exist a differentiable function f and an open set V which contains E such that f is nonzero and bounded on E and $f(x) = 0$ for $x \notin V$.

proof. Given E is closed & bounded $\subseteq \mathbb{R}$.

$\forall x\in E$, $\exists$ $f=f_x \geq 0$ and $r=r(x)>0$

such that f_x is diff'ble on $\mathbb{R}$

$x-r$ $x+r$

and $f(x)>0$, $\forall x\in I_r(x) := (x-r, x+r)$

0 0 >0 0 0

$x-2r$ $x-r$ $x+r$ $x+2r$

$f(x)=0$, $\forall x \notin J_r(x) := (x-2r, x+2r)$

We need to prove that $\exists$ a diff'ble function f and an open set $V \supseteq E$ such that

f is bounded and $f \neq 0$ on E

$f(x) = 0, \ \forall x \notin V$.

Since $\{I_r(x)\}_{x \in E}$ covers E, which is compact, by the Heine-Borel thm,

$\exists$ finitely many x_j's s.t

$$E \subseteq \bigcup_{j=1}^{N} I_{r_j}(x_j), \quad r_j := r(x_j).$$

Set $\boxed{f = \sum_{j=1}^{N} f_{x_j}}$ and $\boxed{V = \bigcup_{j=1}^{N} J_{r_j}(x_j).}$

Then f is diff'ble since it is a finite sum of diff'ble functions. Clearly, $V \supseteq E$, V is open since it is a union of open intervals.

If $x \in E$, then $x \in I_{r_j}(x_j)$, for some j

So, $\boxed{f_{x_j}(x) > 0}$. thus,

$$f(x) = f_{x_1}(x) + f_{x_2}(x) + \cdots + f_{x_j}(x) + \cdots + f_{x_N}(x)$$

$$\geq 0 + 0 + \cdots + \overbrace{f_{x_j}(x)}^{>0} + \cdots + 0 > 0, \ \forall x \in E$$

Since f_{x_k} is continuous on

$$H := \bigcup_{k=1}^{N} [x_k - r_k, x_k + r_k],$$

the extreme value thm $\Longrightarrow$

f_{x_k} is bounded. That is, $\exists M_k$

s.t $|f_{x_k}| \leq M_k$ on H, $\forall k$.

Thus, $|f(x)| = |f_{x_1}(x) + f_{x_2}(x) + \cdots + f_{x_N}(x)|$

$$\leq |f_{x_1}(x)| + \cdots + |f_{x_N}(x)|$$

$$\leq M_1 + M_2 + \cdots + M_N := M$$

$\Rightarrow |f(x)| \leq M$, $\forall x \in H \supset E$.

Finally, if $x \notin V$, then $x \notin J_{r_j}(x_j)$ for all j (i.e., $f_{x_j}(x) = 0$ given).

Thus, $f(x) = f_{x_1}(x) + f_{x_2}(x) + \cdots + f_{x_N}(x)$

$$= 0 + 0 + \cdots + 0 = 0$$

$\blacksquare$

H.W's 1, 2, 3, 4, 7.

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(Review of ch3)

9.3 LIMITS OF FUNCTIONS

vector function $f : A \longrightarrow \mathbb{R}^m$, where $A \subseteq \mathbb{R}^n$, m, n are positive integers

Since $f(x) \in \mathbb{R}^m$, $\forall x \in A \subseteq \mathbb{R}^n$, $\exists$ functions $f_j : A \to \mathbb{R}$ s.t

$$f(x) = (f_1(x), f_2(x), \dots, f_m(x)), \forall x \in A.$$

f_j : coordinate or component of f.

when $m = 1$: $f : A \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$.

f has only one component

we shall call f real valued function.

. If $f = (f_1, \dots, f_m)$ is a vector function, then the maximal domain of f is defined by the intersection of the domains of the f_j's.

9.13 *EXAMPLES.*

i) Find the maximal domain of

$$\mathbf{f}(\underbrace{x, y}_{\in \mathbb{R}^2}) = (\underbrace{\log(xy - y + 2x - 2)}_{f_1(x,y)}, \underbrace{\sqrt{9 - x^2 - y^2}}_{f_2(x,y)}).$$

$f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

Domain of $f_1(x,y) = \{(x,y) : \underbrace{xy - y}+2x-2 > 0\}$

$= \{(x,y) : (x-1)(y+2) > 0\}$

Domain of $f_2(x,y) = \{(x,y) : 9 - x^2 - y^2 \geq 0\}$

$$= \{ (x,y) : x^2+y^2 \le 9 \}.$$

$$\text{Domain of } f = D_{f_1} \cap D_{f_2}$$

$$= \{ (x,y) : (x-1)(y+2) > 0 \text{ and } x^2+y^2 \le 9 \}.$$

ii) Find the maximal domain of

$$\mathbf{g}(x, y) = (\sqrt{1 - x^2}, \log(x^2 - y^2), \sin x \cos y).$$

$$g : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$$

$$g_1(x,y) = \sqrt{1-x^2}, \quad g_2(x,y) = \log(x^2-y^2)$$

$$g_3(x,y) = \sin x \cos y.$$

$$D_{g_1} = \{ (x,y) : 1-x^2 \geq 0 \} = \{ (x,y) : |x| \le 1 \}$$

$$= \{ (x,y) : -1 \le x \le 1 \}.$$

$$D_{g_2} = \{ (x,y) : x^2-y^2 > 0 \} = \{ (x,y) : |y| < |x| \}$$

$$= \{ (x,y) : -|x| < y < |x| \}$$

$$D_{g_3} = \mathbb{R}^2.$$

$$\Rightarrow D_g = \bigcap_{i=1}^{3} D_{g_i}$$

$$= \{ (x,y) : -1 \le x \le 1 \text{ and } -|x| < y < |x| \}.$$

$f: E \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^m$

To set up notation for the algebra of vector functions, let $E \subseteq \mathbf{R}^n$ and suppose that $\mathbf{f}, \mathbf{g} : E \to \mathbf{R}^m$. For each $\mathbf{x} \in E$, the *scalar product* of an $\alpha \in \mathbf{R}$ with $\mathbf{f}$ is defined by

$$(\alpha \mathbf{f})(\mathbf{x}) := \alpha \mathbf{f}(\mathbf{x}),$$

the *sum* of $\mathbf{f}$ and $\mathbf{g}$ is defined by

$$(\mathbf{f} + \mathbf{g})(\mathbf{x}) := \mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x}),$$

the (*Euclidean*) *dot product* of $\mathbf{f}$ and $\mathbf{g}$ is defined by

$$\boxed{(\mathbf{f} \cdot \mathbf{g})(\mathbf{x}) := \mathbf{f}(\mathbf{x}) \cdot \mathbf{g}(\mathbf{x}),}$$

and (when $m = 3$) the *cross product* of $\mathbf{f}$ and $\mathbf{g}$ is defined by

$$(\mathbf{f} \times \mathbf{g})(\mathbf{x}) := \mathbf{f}(\mathbf{x}) \times \mathbf{g}(\mathbf{x}).$$

$\lim_{x \to a} f(x) = L$

$f(x) \to L$ as $x \to a$

9.14 Definition. $f: I \to \mathbb{R}$

Let $n, m \in \mathbf{N}$ and $\mathbf{a} \in \mathbf{R}^n$, let V be an open set which contains $\mathbf{a}$, and suppose that $\mathbf{f} : V \setminus \{\mathbf{a}\} \to \mathbf{R}^m$. Then $\mathbf{f}(\mathbf{x})$ is said to *converge to* $\mathbf{L}$, *as* $\mathbf{x}$ *approaches* $\mathbf{a}$, if and only if for every $\varepsilon > 0$ there is a $\delta > 0$ (which in general depends on ε, $\mathbf{f}$, V, and $\mathbf{a}$) such that

$$0 < \|\mathbf{x} - \mathbf{a}\| < \delta \quad \text{implies} \quad \|\mathbf{f}(\mathbf{x}) - \mathbf{L}\| < \varepsilon.$$

9.14 Definition. (Continued)

In this case we write $\mathbf{f}(\mathbf{x}) \to \mathbf{L}$ as $\mathbf{x} \to \mathbf{a}$ or

$$\mathbf{L} = \lim_{\mathbf{x} \to \mathbf{a}} \mathbf{f}(\mathbf{x})$$

and call $\mathbf{L}$ the *limit* of $\mathbf{f}(\mathbf{x})$ as $\mathbf{x}$ approaches $\mathbf{a}$.

ex. Use the (ε-δ) def'n to prove that

$$\lim_{(x,y) \to (0,0)} f(x,y) = 0, \text{ where } f(x,y) = \frac{3x^2 y}{x^2 + y^2}$$

$\vec{x} = (x,y)$, $\vec{a} = (0,0)$, $L = 0 \in \mathbb{R}$

Pf. Notice that $f: \mathbb{R}^2 \longrightarrow \mathbb{R}$

Let $\varepsilon > 0$. We need to find a $\delta > 0$

s.t

$$0 < \|(x,y)-(0,0)\| < \delta \implies$$

$$|f(x,y) - 0| < \varepsilon$$

i.e. $\left(0 < \|(x,y)\| < \delta \implies \left| \frac{3x^2y}{x^2+y^2} \right| < \varepsilon \right)$

Set $\delta = \frac{\varepsilon}{2}$. If $0 < \|(x,y)\| < \delta$,

then

$(x,y) \neq (0,0)$

$$|f(x,y) - L| = \left| \frac{3x^2y}{x^2+y^2} \right| = \frac{3x^2|y|}{x^2+y^2}$$

$$< \frac{3x^2 \cancel{|y|}}{2|x|\cancel{|y|}} \quad \left(\text{since } x^2+y^2 > 2|x||y| \right)$$

$$< 2|x|$$

$$< 2\sqrt{x^2+y^2}$$

$$= 2\|(x,y)\|$$

$$< 2\delta = 2\frac{\varepsilon}{2} = \varepsilon.$$

Thus, by def'n,

$$\lim_{(x,y)\to(0,0)} f(x,y) = 0.$$

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$f: \mathbb{R}^n \to \mathbb{R}^m$

9.15 Theorem. *Let* $\mathbf{a} \in \mathbf{R}^n$, *let* V *be an open set which contains* $\mathbf{a}$, *and suppose that* $\mathbf{f}, \mathbf{g} : V \setminus \{\mathbf{a}\} \to \mathbf{R}^m$.

i) *If* $\mathbf{f}(\mathbf{x}) = \mathbf{g}(\mathbf{x})$ *for all* $\mathbf{x} \in V \setminus \{\mathbf{a}\}$ *and if* $\mathbf{f}(\mathbf{x})$ *has a limit as* $\mathbf{x} \to \mathbf{a}$, *then* $\mathbf{g}(\mathbf{x})$ *has a limit as* $\mathbf{x} \to \mathbf{a}$, *and*

$$\lim_{\mathbf{x}\to\mathbf{a}} \mathbf{g}(\mathbf{x}) = \lim_{\mathbf{x}\to\mathbf{a}} \mathbf{f}(\mathbf{x}).$$

ii) [Sequential Characterization of Limits]. $\mathbf{L} = \lim_{\mathbf{x}\to\mathbf{a}} \mathbf{f}(\mathbf{x})$ *exists if and only if* $\mathbf{f}(\mathbf{x}_k) \to \mathbf{L}$ *as* $k \to \infty$ *for every sequence* $\mathbf{x}_k \in V \setminus \{\mathbf{a}\}$ *which converges to* $\mathbf{a}$ *as* $k \to \infty$.

$x_k \to a$
$f(x_k) \to L$

iii) *Suppose that* $\alpha \in \mathbf{R}$. *If* $\mathbf{f}(\mathbf{x})$ *and* $\mathbf{g}(\mathbf{x})$ *have limits, as* $\mathbf{x}$ *approaches* $\mathbf{a}$, *then so do* $(\mathbf{f}+\mathbf{g})(\mathbf{x})$, $(\alpha\mathbf{f})(\mathbf{x})$, $(\mathbf{f}\cdot\mathbf{g})(\mathbf{x})$, *and* $\|f(\mathbf{x})\|$. *In fact,*

$$\lim_{\mathbf{x}\to\mathbf{a}} (\mathbf{f}+\mathbf{g})(\mathbf{x}) = \lim_{\mathbf{x}\to\mathbf{a}} \mathbf{f}(\mathbf{x}) + \lim_{\mathbf{x}\to\mathbf{a}} \mathbf{g}(\mathbf{x}),$$

$$\lim_{\mathbf{x}\to\mathbf{a}} (\alpha\mathbf{f})(\mathbf{x}) = \alpha \lim_{\mathbf{x}\to\mathbf{a}} \mathbf{f}(\mathbf{x}),$$

$$\lim_{\mathbf{x}\to\mathbf{a}} (\mathbf{f}\cdot\mathbf{g})(\mathbf{x}) = \left(\lim_{\mathbf{x}\to\mathbf{a}} \mathbf{f}(x)\right)\cdot\left(\lim_{\mathbf{x}\to\mathbf{a}} \mathbf{g}(\mathbf{x})\right),$$

and

$$\left\|\lim_{\mathbf{x}\to\mathbf{a}} \mathbf{f}(\mathbf{x})\right\| = \lim_{\mathbf{x}\to\mathbf{a}} \|\mathbf{f}(\mathbf{x})\|.$$

Moreover, when $m = 3$, $f: \mathbb{R}^n \to \mathbb{R}^3$

$$\lim_{\mathbf{x}\to\mathbf{a}} (\mathbf{f}\times\mathbf{g})(\mathbf{x}) = \left(\lim_{\mathbf{x}\to\mathbf{a}} \mathbf{f}(\mathbf{x})\right)\times\left(\lim_{\mathbf{x}\to\mathbf{a}} \mathbf{g}(\mathbf{x})\right),$$

and when $m = 1$ *and the limit of* $\mathbf{g}$ *is nonzero,* $\lim g \neq 0$

$\frac{f(x)}{g(x)} \in \mathbb{R}$

$$\lim_{\mathbf{x}\to\mathbf{a}} \mathbf{f}(\mathbf{x})/\mathbf{g}(\mathbf{x}) = \left(\lim_{\mathbf{x}\to\mathbf{a}} \mathbf{f}(\mathbf{x})\right)/\left(\lim_{\mathbf{x}\to\mathbf{a}} \mathbf{g}(\mathbf{x})\right).$$

iv) [Squeeze Theorem for Functions]. *Suppose that* $f, g, h : V \setminus \{\mathbf{a}\} \to \mathbf{R}$ *and that* $g(\mathbf{x}) \le h(\mathbf{x}) \le f(\mathbf{x})$ *for all* $\mathbf{x} \in V \setminus \{\mathbf{a}\}$. *If*

$$\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = \lim_{\mathbf{x}\to\mathbf{a}} g(\mathbf{x}) = L,$$

then the limit of h *also exists, as* $\mathbf{x} \to \mathbf{a}$, *and*

$$\lim_{\mathbf{x}\to\mathbf{a}} h(\mathbf{x}) = L.$$

v) *Suppose that* U *is open in* $\mathbf{R}^m$, *that* $\mathbf{L} \in U$, *and that* $\mathbf{h} : U \to \mathbf{R}^p$ *for some* $p \in \mathbf{N}$. *If* $\mathbf{L} = \lim_{\mathbf{x}\to\mathbf{a}} \mathbf{g}(\mathbf{x})$ *and* $\mathbf{h}$ *is continuous at* $\mathbf{L}$. *Then*

$$\lim_{\mathbf{x}\to\mathbf{a}} (\mathbf{h}\circ\mathbf{g})(\mathbf{x}) = \mathbf{h}(\mathbf{L}).$$

ex. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^3-y^3}{x^2+y^2} = 0$, $(x,y) \neq (0,0)$

$$\left|\frac{x^3-y^3}{x^2+y^2}\right| \le |x|\underbrace{\frac{x^2}{x^2+y^2}}_{<1} + |y|\underbrace{\frac{y^2}{x^2+y^2}}_{<1} \le |x|+|y|$$

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$$|f(x,y)| \leq |x|+|y|$$

$$-(|x|+|y|) \leq f(x,y) \leq |x|+|y|$$

Since $\lim_{(x,y)\to(0,0)} -(|x|+|y|) = \lim_{(x,y)\to(0,0)} (|x|+|y|) = 0,$

then $\lim_{(x,y)\to(0,0)} f(x,y) = 0$ by squeeze thm.

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$f: A \subseteq \mathbb{R}^n \to \mathbb{R}^m$

$f(x) = (f_1, f_2, \ldots, f_m)$

$\lim_{x \to a} f(x) = L = (L_1, L_2, \ldots, L_m) \Leftrightarrow \forall \varepsilon > 0, \exists \delta > 0$ s.t $\|x - a\| < \delta \Rightarrow \|f(x) - L\| < \varepsilon$

$(f_1(x), \ldots, f_m(x))$

9.16 Theorem. *Let* $\mathbf{a} \in \mathbf{R}^n$*, let* V *be an open set which contains* $\mathbf{a}$*, and suppose that* $\mathbf{f} = (f_1, \ldots, f_m) : V \setminus \{\mathbf{a}\} \to \mathbf{R}^m$*. Then*

$$\lim_{\mathbf{x} \to \mathbf{a}} \mathbf{f}(\mathbf{x}) = \mathbf{L} := (L_1, L_2, \ldots, L_m) \tag{1}$$

exists in $\mathbf{R}^m$ *if and only if*

$$\lim_{\mathbf{x} \to \mathbf{a}} f_j(\mathbf{x}) = L_j \tag{2}$$

exists in $\mathbf{R}$ *for each* $j = 1, 2, \ldots, m$.

Pf. Exercise. P. 315.

ex. Find $\lim_{(x,y) \to (0,0)} (\underbrace{3xy + 1}_{f_1(x,y)}, \underbrace{e^y + 2}_{f_2(x,y)})$

$$= \left(\lim_{(x,y) \to (0,0)} (3xy + 1), \lim_{(x,y) \to (0,0)} (e^y + 2) \right)$$

$$= (1, 3)$$

Rmk. If f_j are real functions cont. at a_j for $j = 1, 2, \ldots, n$. Then

$$F(x_1, x_2, \ldots, x_n) = f_1(x_1) + f_2(x_2) + \cdots + f_n(x_n)$$

$$G(x_1, x_2, \ldots, x_n) = f_1(x_1) f_2(x_2) \cdots f_n(x_n)$$

both have limits at $a = (a_1, \ldots, a_n)$

and $\lim_{x \to a} F(x) = F(a)$ and

$\lim_{x \to a} G(x) = G(a)$.

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ex: $\lim\limits_{(x,y)\to(0,0)} \dfrac{2+x-y}{1+2x^2+3y^2} = \dfrac{2}{1} = 2.$

$\left(\frac{0}{0}\right)$ → $(\varepsilon-\delta)$ def'n.
→ Sandwich thm.
→ $\|f(x)-L\| \le g(x)$, $g(x) > 0$
and $g(x) \to 0$ as $x \to a$

$\Rightarrow \lim\limits_{x\to a} f(x) = L$

$\lim\limits_{x\to a} f(x) = L$

ex: $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^3-y^3}{x^2+y^2} = 0$ $\left(\frac{0}{0}\right)$

$|f(x,y)-L| = \left|\dfrac{x^3-y^3}{x^2+y^2} - 0\right|$ $(x,y)\neq(0,0)$

$= \left|\dfrac{x^3}{x^2+y^2} - \dfrac{y^3}{x^2+y^2}\right|$

$\le \dfrac{|x|\cdot x^2}{x^2+y^2} + |y|\dfrac{y^2}{x^2+y^2}$ (each fraction ≤ 1)

$\le |x|+|y| = g(x,y)$

$|f(x,y)-L| \le g(x,y) = |x|+|y| > 0$
and $g(x,y)\to 0$ as $(x,y)\to(0,0)$

$\therefore f(x,y) \to L$ as $(x,y)\to(0,0).$

$\therefore \lim\limits_{(x,y)\to(0,0)} \dfrac{x^3-y^3}{x^2+y^2} = 0$

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ex. Prove that $f(x,y) = \frac{2xy}{x^2+y^2}$

has no limit as $(x,y) \to (0,0)$.

Proof. Spse that f has a limit, i.e.,

$\lim_{(x,y)\to(0,0)} f(x,y) = L$ exists.

Along $x=0$,

$$\lim_{(x,y)\to(0,0)} \frac{2xy}{x^2+y^2} = \lim_{y\to 0} \frac{0}{y^2}$$

$$= \lim_{y\to 0} 0 = \boxed{0}$$

Along $y=x$,

$$\lim_{(x,y)\to(0,0)} \frac{2xy}{x^2+y^2} = \lim_{x\to 0} \frac{2x^2}{x^2+x^2} = \boxed{1}$$

Since $\lim_{(x,y)\to(0,0)} f(x,y)$ exists, then $0 = 1$,

a contradiction.

$\therefore \lim_{(x,y)\to(0,0)} f(x,y)$ DNE.

ex. $\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4}$ DNE. Since

Along $x=0$, $\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4} = \lim_{y\to 0} \frac{0}{y^4}$

$= \lim_{y\to 0} 0 = 0$

Along $x = y^2$,

$$\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4} = \lim_{y\to 0} \frac{y^4}{2y^4} = \frac{1}{2} \neq 0$$

Therefore, $\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4}$ DNE.

Caution:

$$\lim_{(x,y)\to(a,b)} \frac{g(x,y)}{h(x,y)} \overset{?}{=} \lim_{(x,y)\to(a,b)} \frac{g_x(x,y)+g_y(x,y)}{h_x(x,y)+h_y(x,y)}.$$

No. there is No L'Höpital Rule in $\mathbb{R}^2$.

Counterexample: $\lim_{(x,y)\to(0,0)} \frac{2xy}{x^2+y^2}$ has no limit. ($g = 2xy$, $h = x^2+y^2$)

$g_x + g_y = 2x+2y$, $h_x + h_y = 2x+2y$

$$\lim_{(x,y)\to(0,0)} \frac{g_x+g_y}{h_x+h_y} = \lim_{(x,y)\to(0,0)} \frac{2x+2y}{2x+2y} = 1$$

but $\lim_{(x,y)\to(0,0)} \frac{2xy}{x^2+y^2}$ DNE.

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Iterated limits of $f(x,y)$ at (a,b).

$$\lim_{y\to b}\lim_{x\to a} f(x,y) := \lim_{y\to b}\Big(\underbrace{\lim_{x\to a} f(x,y)}_{h(y)}\Big)$$

and

$$\lim_{x\to a}\lim_{y\to b} f(x,y) := \lim_{x\to a}\Big(\overbrace{\lim_{y\to b} f(x,y)}^{g(x)}\Big)$$

ex. Evaluate the iterated limits of

$$f(x,y) = \frac{x^2}{x^2+y^2} \quad \text{at } (0,0).$$

Sol. $\cdot$

$$\lim_{x\to 0}\lim_{y\to 0}\frac{x^2}{x^2+y^2} = \lim_{x\to 0}\left(\lim_{y\to 0}\frac{x^2}{x^2+y^2}\right)$$

$$= \lim_{x\to 0}\left(\frac{x^2}{x^2}\right)$$

$$= \lim_{x\to 0} 1 = \boxed{1}$$

$\cdot$

$$\lim_{y\to 0}\lim_{x\to 0}\frac{x^2}{x^2+y^2} = \lim_{y\to 0}\left(\lim_{x\to 0}\frac{x^2}{x^2+y^2}\right)$$

$$= \lim_{y\to 0}\left(\frac{0}{y^2}\right) \qquad y\to 0$$

$$= \lim_{y\to 0} 0 = \boxed{0}$$

This leads us to ask, When are the iterated limits equal? The following result shows that if f has a limit as $(x, y) \to (a, b)$ and both iterated limits exist, then these limits must be equal.

$\lim_{(x,y)\to(a,b)} f(x,y)$ exists

$p \Rightarrow q$

$\sim q \Rightarrow \sim p$

9.22 Remark. *Suppose that I and J are open intervals, that $a \in I$ and $b \in J$, and that $f : (I \times J) \setminus \{(a, b)\} \to \mathbf{R}$. If*

$$g(x) := \lim_{y \to b} f(x, y)$$

exists for each $x \in I \setminus \{a\}$, if $\lim_{x \to a} f(x, y)$ exists for each $y \in J \setminus \{b\}$, and if $f(x, y) \to L$ as $(x, y) \to (a, b)$ (in $\mathbf{R}^2$), then

$$L = \lim_{x \to a} \lim_{y \to b} f(x, y) = \lim_{y \to b} \lim_{x \to a} f(x, y).$$

α: $\lim_{(x,y) \to (a,b)} f(x,y) = L$ exists

Proof. Let $\varepsilon > 0$. $\exists$ a $\delta > 0$ such that

$$\boxed{0 < \|(x,y) - (a,b)\| < \delta \Rightarrow |f(x,y) - L| < \varepsilon} \quad *$$

Let $x \in I$ s.t $0 < |x - a| < \frac{\delta}{\sqrt{2}}$. then

for any y satisfies $0 < |y - b| < \frac{\delta}{\sqrt{2}}$,

we have $\|(x,y) - (a,b)\| = \sqrt{(x-a)^2 + (y-b)^2}$

$$< \sqrt{\left(\frac{\delta}{\sqrt{2}}\right)^2 + \left(\frac{\delta}{\sqrt{2}}\right)^2} = \delta$$

Hence,

$$|g(x) - L| = |g(x) - f(x,y) + f(x,y) - L|$$

$$\leq |g(x) - f(x,y)| + |f(x,y) - L|$$

$$\boxed{|g(x) - L| < |g(x) - f(x,y)| + \varepsilon}$$

$$\lim_{y \to b} |g(x) - L| < \lim_{y \to b} \cancel{|g(x) - f(x,y)|} + \varepsilon$$

$\to 0$ (Given)

$$|g(x) - L| \leq \varepsilon, \quad \forall x \in I \text{ which satisfy } |x - a| < \delta/\sqrt{2}.$$

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$$\therefore \lim_{x\to a} g(x) = L$$

$$\Rightarrow \lim_{x\to a}\left(\lim_{y\to b} f(x,y)\right) = L$$

$$\therefore \lim_{x\to a}\lim_{y\to b} f(x,y) = L.$$

Similarly, you can prove the other iterated limit als equals L, i.e.,

$$\lim_{y\to b}\lim_{x\to a} f(x,y) = L \quad \text{(exercise)}.$$

H.W's 1, 2, 3, 4, 5, 7, 8.

9.4 CONTINUOUS FUNCTIONS

Review (sections 3.3 and 3.4) $f: E \to \mathbb{R}$
$E \subseteq \mathbb{R}$.

9.23 Definition.

Let E be a nonempty subset of $\mathbf{R}^n$ and $\mathbf{f}: E \to \mathbf{R}^m$.

$\lim_{x \to a} f(x) = f(a)$

i) $\mathbf{f}$ is said to be *continuous at* $\mathbf{a} \in E$ if and only if for every $\varepsilon > 0$ there is a $\delta > 0$ (which in general depends on ε, $\mathbf{f}$, E, and $\mathbf{a}$) such that

$$\|x - \mathbf{a}\| < \delta \quad \text{and} \quad \mathbf{x} \in E \quad \text{imply} \quad \|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{a})\| < \varepsilon. \tag{3}$$

ii) $\mathbf{f}$ is said to be *continuous on* E (notation: $\mathbf{f}: E \to \mathbf{R}^m$ is continuous) if and only if $\mathbf{f}$ is continuous at every $\mathbf{x} \in E$.

Rmk. (1) $\emptyset \neq E \subseteq \mathbb{R}^n$.
f is continuous at $a \in E \iff f(x_k) \to f(a)$ for all $x_k \in E$ and $x_k \to a$. (Exercise).

(2) If f and g are cont. at $a \in E$ (resp. on E), then
$f+g$, αf (α scalar), $f \cdot g$, $\|f\|$, and (when $m=3$) $f \times g$ are continuous. (exercise).

(3) If $f: E \to \mathbb{R}^m$ is cont. at $a \in E$ and $g: f(E) \to \mathbb{R}^p$ is continuous at $f(a) \in f(E)$, then $g \circ f$ is continuous at $a \in E$ (exercise).

9.24 Definition.

Let E be a nonempty subset of $\mathbf{R}^n$ and $\mathbf{f} : E \to \mathbf{R}^m$. Then $\mathbf{f}$ is said to be *uniformly continuous* on E (notation: $\mathbf{f} : E \to \mathbf{R}^m$ is uniformly continuous) if and only if for every $\varepsilon > 0$ there is a $\delta > 0$ such that

$$\|\mathbf{x} - \mathbf{a}\| < \delta \quad \text{and} \quad \mathbf{x}, \mathbf{a} \in E \quad \text{imply} \quad \|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{a})\| < \varepsilon.$$

Rmk. Continuity and uniform continuity of a vector function are equivalent on closed, bounded sets.

9.25 Theorem. *Let E be a nonempty compact subset of $\mathbf{R}^n$. If $\mathbf{f}$ is continuous on E, then $\mathbf{f}$ is uniformly continuous on E.*

Proof. Let $\phi \neq E \subseteq \mathbb{R}^n$ be compact and f be continuous on E. Let $\varepsilon > 0$, and $a \in E$. Since f is cont. at $a \in E$, $\exists\ \delta(a) > 0$ s.t

$$\underbrace{\|x - a\| < \delta(a)}, \ x \in E \Rightarrow \|f(x) - f(a)\| < \frac{\varepsilon}{2}$$

$\left(\text{i.e., } \ x \in B_{\delta(a)}(a), \ x \in E \Rightarrow \|f(x) - f(a)\| < \frac{\varepsilon}{2} \right)$

Since $\frac{\delta(a)}{2} > 0$, $\forall a \in E$, then

9.2 the collection $\left\{ B_{\frac{\delta(a)}{2}}(a) \right\}_{a \in E}$ is an open covering of E. $\left(\text{i.e., } E \subseteq \bigcup_{a \in E} B_{\frac{\delta(a)}{2}}(a) \right)$

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Since E is compact, $\exists$ finitely many $a_j \in E$ and $\delta_j := \frac{\delta(a_j)}{2}$ such that $E \subset \bigcup_{j=1}^{N} B_{\delta_j}(a_j)$. $(*)$

Set $\delta := \min\{\delta_1, \delta_2, --, \delta_N\}$

clearly $\delta > 0$.

Suppose that $x, a \in E$ with $\|x-a\| < \delta$.

By $(*)$, $x \in B_{\delta_j}(a_j)$ for some $1 \le j \le N$.

(i.e, $\|x-a_j\| < \delta_j$)

Hence,
$$\|a-a_j\| \le \|a-x\| + \|x-a_j\|$$
$$< \delta_j + \delta_j = 2\delta_j = \delta(a_j)$$

i.e, $\|a-a_j\| < \delta(a_j)$ this means $a \in B_{\delta(a_j)}(a_j)$. It follows,

$$\|f(x)-f(a)\| \le \|f(x)-f(a_j)\| + \|f(a_j)-f(a)\|$$
$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

This proves that f is uniformly continuous on E.

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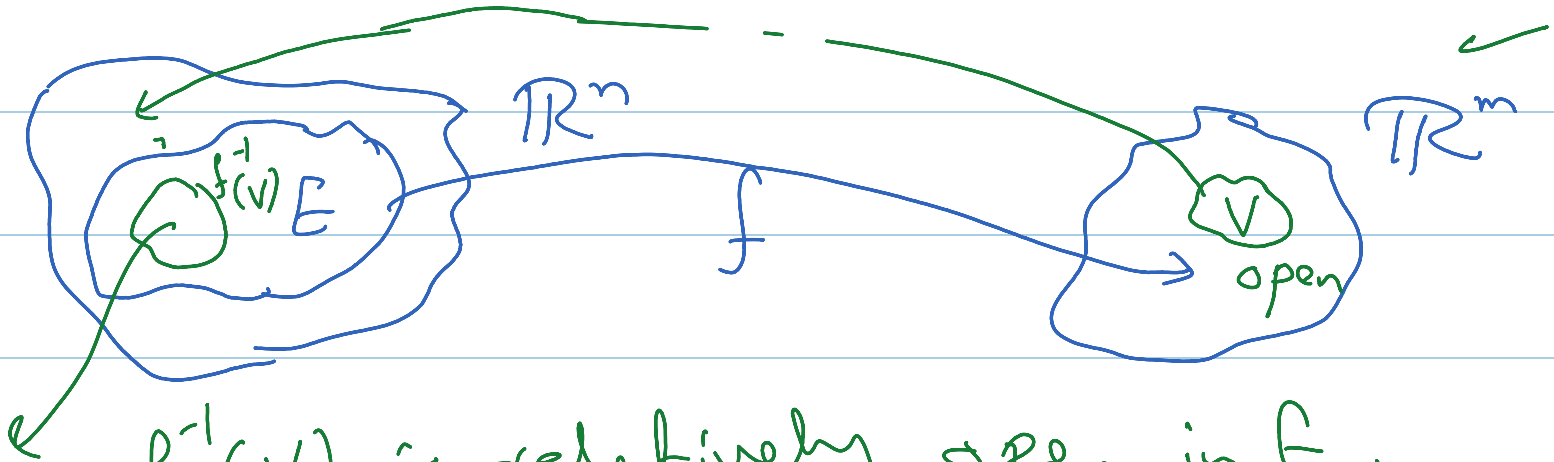
Recall, U is relatively open in E if $U = E \cap V$, V is open.

or iff $\forall a \in U$, $\exists r > 0$ s.t. $B_r(a) \cap E \subseteq U$

Rmk 8.27

9.26 Theorem. *Suppose that $E \subseteq \mathbf{R}^n$ and that $\mathbf{f} : E \to \mathbf{R}^m$. Then $\mathbf{f}$ is continuous on E if and only if $\mathbf{f}^{-1}(V)$ is relatively open in E for every V open in $\mathbf{R}^m$.*



$f^{-1}(V)$ is relatively open in E.

Proof. ($\Rightarrow$) spse that $f : E \subseteq \mathbb{R}^n \to \mathbb{R}^m$ is continuous on E and that V is open in $\mathbb{R}^m$, we need to show $f^{-1}(V)$ is relatively open in E.

(i.e., $\forall a \in f^{-1}(V)$, $\exists \delta > 0$ s.t $B_\delta(a) \cap E \subseteq f^{-1}(V)$).

If $f^{-1}(V) = \Phi$, then $f^{-1}(V)$ is open.

Spse $a \in f^{-1}(V) \Rightarrow f(a) \in V$

Since $f(a) \in V$, V is open, then by defn, $\exists$ an $\varepsilon > 0$ s.t

$$\boxed{B_\varepsilon(f(a)) \subseteq V.}$$

تذكر

E is open

$\forall a \in E, \exists r > 0$ s.t $B_r(a) \subseteq E$

Since f is continuous at $a \in E$, then $\exists\, \delta > 0$

s.t $\boxed{\|x-a\| < \delta \text{ and } x \in E \Rightarrow \|f(x)-f(a)\| < \varepsilon}$

i.e, $\boxed{x \in B_\delta(a) \cap E \Rightarrow f(x) \in B_\varepsilon(f(a))}$

تذكر i.e,

$x \in A \Rightarrow f(x) \in B$

$f(A) \subseteq B$

$$f(B_\delta(a) \cap E) \subseteq B_\varepsilon(f(a)) \subseteq V$$

$$\Rightarrow B_\delta(a) \cap E \subseteq f^{-1}(V).$$

$\xrightarrow{\text{by Rmk 8.27}}$ $f^{-1}(V)$ is relatively open in E.

($\Leftarrow$) Conversely, spse that $f^{-1}(V)$ is relatively open in E for every V open in $\mathbb{R}^n$.

We need to prove that f is continuous on E.

Let $\varepsilon > 0$ and $a \in E$.

Take $V = B_\varepsilon(f(a))$. Notice that V is open in $\mathbb{R}^m$.

By hypothesis $f^{-1}(V)$ is relatively open in E.

By Rmk 8.27, $\exists$ a $\delta > 0$ s.t.,

$$\boxed{B_\delta(a) \cap E \subset f^{-1}(B_\varepsilon(f(a)))} \quad (**)$$

We conclude that if $\|x-a\| < \delta$ and $x \in E$ (i.e $x \in B_\delta(a) \cap E$), then

by $(**)$ $\quad x \in f^{-1}(B_\varepsilon(f(a)))$

i.e, $f(x) \in B_\varepsilon(f(a))$

i.e, $\|f(x) - f(a)\| < \varepsilon$

it follows f is continuous at $a \in E$. ∎

$f: E \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^m$

Rmk. (1) when f is cont. on E, f^{-1} takes open sets to relatively open sets in E.

If E is open, then thm 9.26 will be:
" spse that A is open in $\mathbb{R}^n$ and $f: A \longrightarrow \mathbb{R}^m$. Then f is cont. on A iff $f^{-1}(V)$ is open in $\mathbb{R}^n$ for every open subset V of $\mathbb{R}^m$.

(when f is cont. and A is open, f^{-1} takes open sets to open sets in A).

(Exercise 9.4.3 page 326).

open sets are invariant under inverse images by cont. functions

(2) **9.4.5.** Suppose that $E \subseteq \mathbf{R}^n$ and that $\mathbf{f}: E \to \mathbf{R}^m$.

a) Prove that $\mathbf{f}$ is continuous on E if and only if $\mathbf{f}^{-1}(B)$ is ~~relatively~~ closed in E for every closed subset B of $\mathbf{R}^m$.

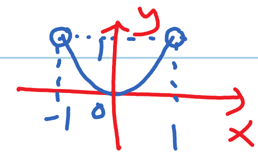
(3) **9.4.4.** Suppose that A is closed in $\mathbf{R}^n$ and $\mathbf{f}: A \to \mathbf{R}^m$. Prove that $\mathbf{f}$ is continuous on A if and only if $\mathbf{f}^{-1}(E)$ is closed in $\mathbf{R}^n$ for every closed subset E of $\mathbf{R}^m$.

We now turn our attention from inverse images (f^{-1}) of sets to images of sets (f). Are open sets and closed sets invariant under images (f) by continuous functions? The following examples show that the answers to these questions are also no.

$f: H \longrightarrow \mathbb{R}^m$ cont. on H and H open
$\Longrightarrow$ $f(H)$ is open in $\mathbb{R}^m$??

H closed $\Longrightarrow$ $f(H)$ closed in $\mathbb{R}^m$??

Ans. No



Ex. $f(x) = x^2$, $V = (-1,1)$

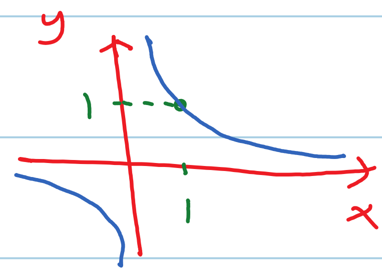
$[0,\infty)$

f is cont. on V and V is open

but $f(V) = f(-1,1) = [0,1)$ is neither open nor closed.

Ex. $f(x) = \frac{1}{x}$, $E = [1,\infty)$ closed

f is cont. on E



$f(E) = f([1,\infty)) = (0,1]$

is neither open nor closed

Question Are bounded sets and connected sets invarient under inverse images by continuous functions?

Ans. No.

ex. $f(x) = \frac{1}{x^2+1}$, $E = (0,1]$

f is continuous on E and E is bounded

but $f^{-1}(E) = f^{-1}((0,1]) = (-\infty,\infty)$ unbounded.

توضيح

$x \in f^{-1}((0,1])$

$f(x) \in (0,1]$

$0 < f(x) \le 1 \Rightarrow 0 < \frac{1}{x^2+1} \le 1$ ✓

ex. $f(x) = x^2$, $E = (1,4)$ connected

f is cont. on on $\mathbb{R}$ and hence on E and

E is connected but

$f^{-1}(E) = f^{-1}((1,4)) = (-2,-1) \cup (1,2)$

disconnected.

$f(x) \in (1,4)$

$1 < f(x) < 4 \Rightarrow 1 < x^2 < 4$

$1 < |x| < 2$

we need this thm later. (chapter 1 page 34).

1.37 Theorem. *Let X and Y be sets and $f : X \to Y$.*

i) If $\{E_\alpha\}_{\alpha\in A}$ is a collection of subsets of X, then

$$f\left(\bigcup_{\alpha\in A} E_\alpha\right) = \bigcup_{\alpha\in A} f(E_\alpha) \quad \text{and} \quad f\left(\bigcap_{\alpha\in A} E_\alpha\right) \subseteq \bigcap_{\alpha\in A} f(E_\alpha).$$

ii) If B and C are subsets of X, then $f(C\setminus B) \supseteq f(C)\setminus f(B)$.

iii) If $\{E_\alpha\}_{\alpha\in A}$ is a collection of subsets of Y, then

$$f^{-1}\left(\bigcup_{\alpha\in A} E_\alpha\right) = \bigcup_{\alpha\in A} f^{-1}(E_\alpha) \quad \text{and} \quad f^{-1}\left(\bigcap_{\alpha\in A} E_\alpha\right) = \bigcap_{\alpha\in A} f^{-1}(E_\alpha).$$

iv) If B and C are subsets of Y, then $f^{-1}(C\setminus B) = f^{-1}(C)\setminus f^{-1}(B)$.

v) If $\boxed{E \subseteq f(X)}$, then $f(f^{-1}(E)) = E$, but if $\boxed{E \subseteq X}$, then $f^{-1}(f(E)) \supseteq E$.

$f : X \to Y$ $\qquad$ $f : X \to Y$

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9.29 Theorem. *If H is compact in $\mathbf{R}^n$ and $\mathbf{f} : H \to \mathbf{R}^m$ is continuous on H, then $\mathbf{f}(H)$ is compact in $\mathbf{R}^m$.*

Proof. Spse that $\{V_\alpha\}_{\alpha\in A}$ is an open covering of $f(H)$ (i.e., $f(H) \subseteq \bigcup_{\alpha\in V} V_\alpha$)

Then, by thm 1.37 (parts iii and v)

$$H \overset{(v)}{\subseteq} f^{-1}(f(H)) \subseteq f^{-1}\Big(\bigcup_{\alpha\in V} V_\alpha\Big) \overset{(iii)}{=} \bigcup_{\alpha\in A} f^{-1}(V_\alpha)$$

$$\Rightarrow \quad H \subseteq \bigcup_{\alpha\in V} f^{-1}(V_\alpha)$$

i.e., $\{f^{-1}(V_\alpha)\}_{\alpha\in A}$ is an open covering of H. By thm 9.26, $f^{-1}(V_\alpha)$ are relatively open in H, i.e., $\exists$ open sets O_α s.t.,

$$\boxed{f^{-1}(V_\alpha) = O_\alpha \cap H}$$

. Since $\{O_\alpha\}_{\alpha\in A}$ is an open covering of H and H is compact, then $\exists\ \alpha_j \in A$ such that

$$H \subset \bigcup_{j=1}^{N} O_{\alpha_j}$$

. By thm 1.37 (part i and v)

$$f(H) \subseteq f\Big(\bigcup_{j=1}^{N} O_{\alpha_j} \cap H\Big) = f\Big(\bigcup_{j=1}^{N} f^{-1}(V_{\alpha_j})\Big)$$

$$\overset{(i)}{=} \bigcup_{j=1}^{N} f(f^{-1}(V_{\alpha_j})) \overset{(v)}{=} \bigcup_{j=1}^{N} V_{\alpha_j}$$

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$f: E \to \mathbb{R}^m$ $\quad$ f cont. $\quad$ V is connected $\quad$ $f^{-1}(V)$

Connected sets are also invariant under images by continuous functions.

9.30 Theorem. *If E is connected in $\mathbf{R}^n$ and $\mathbf{f} : E \to \mathbf{R}^m$ is continuous on E, then $\mathbf{f}(E)$ is connected in $\mathbf{R}^m$.*

Proof. Spse that $f(E)$ is not connected. By defn, $\exists$ U and V relatively open sets in $f(E)$ s.t $U \cap f(E) \neq \phi$, $V \cap f(E) \neq \phi$,

$$f(E) = U \cup V, \text{ and } U \cap V = \phi.$$

By Exercise 9.4.5(b), $f^{-1}(U)$ and $f^{-1}(V)$ are relatively open in E.

Since $\boxed{f(E) = U \cup V}$ and both $f^{-1}(U)$ and $f^{-1}(V)$ are subsets of E, then by thm 1.37 (iii),

$$\boxed{E = f^{-1}(U) \cup f^{-1}(V)}$$

Finally, $U \cap V = \phi \Rightarrow f^{-1}(U) \cap f^{-1}(V) = \phi$

$x \in f^{-1}(U)$ & $x \in f^{-1}(V)$
$f(x) \in U$ $\quad$ $f(x) \in V$
$f(x) \in U \cap V = \phi$
a contradiction.

Thus, $f^{-1}(U)$ and $f^{-1}(V)$ is a pair of relatively open which seperates E i.e., E is not connected, a contradiction $\blacksquare$

Ex. $y = f(x)$ continuous on $[a,b]$

$F(x) = (x, f(x))$

$F: [a,b] \longrightarrow \mathbb{R}^2$

$F([a,b]) = \{\text{graph of } y = f(x)\}$

$[a,b]$ is connected and F is continuous, by the last thm,

$F([a,b]) = \text{graph of } y = f(x)$ is connected.

Thus, the graph $y = f(x)$ of a continuous function on $[a,b]$ is connected.

$f: [a,b] \to \mathbb{R}$

9.32 Theorem. [EXTREME VALUE THEOREM]. *Suppose that H is a nonempty subset of $\mathbf{R}^n$ and that $f : H \to \mathbf{R}$. If H is compact, and f is continuous on H, then*

$$M := \sup\{f(\mathbf{x}) : \mathbf{x} \in H\} \quad \text{and} \quad m := \inf\{f(\mathbf{x}) : \mathbf{x} \in H\}$$

are finite real numbers. Moreover, there exist points $\mathbf{x}_M, \mathbf{x}_m \in H$ such that $M = f(\mathbf{x}_M)$ and $m = f(\mathbf{x}_m)$.

Proof: $\emptyset \neq H \subseteq \mathbb{R}^n$, $f: H \longrightarrow \mathbb{R}$, H compact

f is continuous on H.

Since H is compact and f is compact,

then $f(H)$ is compact by Thm 9.29.

Thus, $f(H)$ is closed and bounded

by the Heine-Borel thm.

Since $f(H)$ is bounded, M is finite

By the Approximation Property, $\exists\ x_k \in H$ such that $f(x_k) \longrightarrow M$

Since $f(H)$ is closed, $\lim_{k\to\infty} f(x_k) = \boxed{M \in f(H)}$

. Therefore, $\exists$ an $x_M \in H$ s.t. $f(x_M) = M$.

Similarly, you can prove m is finite and $m = f(x_m)$, for some $x_m \in H$. ∎

تذكر/

Rule. . $\sup A = \beta$, $\exists$ a seq. $x_k \in A$ s.t $x_k \longrightarrow \sup A = \beta$.

. $\inf A = \alpha$, $\exists$ a seq. $y_k \in A$ s.t $y_k \longrightarrow \inf A = \alpha$.

. A is closed $\Leftrightarrow$ $\forall\ x_k$ in A s.t $x_k \to x$, then $\lim_{k\to\infty} x_k = x \in A$.

9.33 Theorem. *If H is a compact subset of $\mathbf{R}^n$ and $\mathbf{f} : H \to \mathbf{R}^m$ is 1–1 and continuous, then $\mathbf{f}^{-1}$ is continuous on $\mathbf{f}(H)$.*

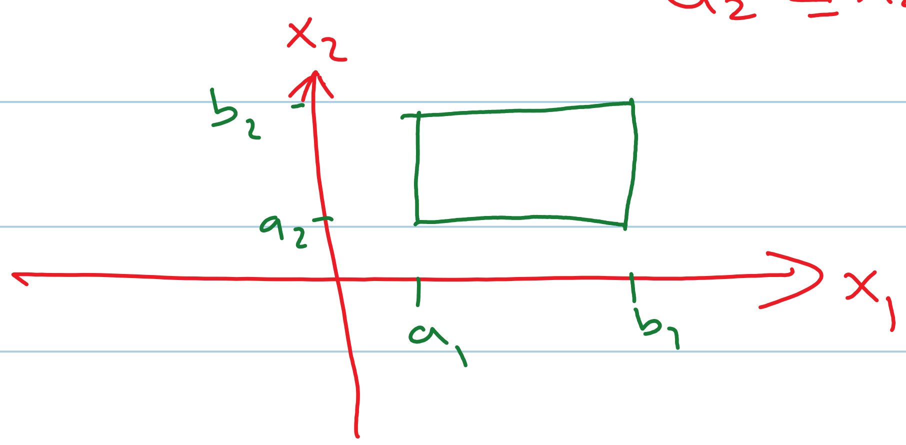
9.34 Remark. *If $a_j \le b_j$ for $j = 1, 2, \ldots, n$, then*

$$R := \{(x_1, \ldots, x_n) : a_j \le x_j \le b_j\}$$

(rectangles in $\mathbb{R}^n$).

is connected.

$\mathbb{R}^2$: $R = \{(x_1, x_2) : a_1 \le x_1 \le b_1 ,\ a_2 \le x_2 \le b_2\}$



<u>H.W's</u> 1, 2, 3, 4, 5, 6, 9.

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CHAPTER 10

Metric Spaces

10.1 INTRODUCTION

rho
$\rho(x,y)$
$x,y \in X$

The following concept shows up in many parts of analysis.

10.1 Definition.

A *metric space* is a set X together with a function $\rho : X \times X \to \mathbf{R}$ (called the *metric* of X) which satisfies the following properties for all $x, y, z \in X$:

POSITIVE DEFINITE $\rho(x, y) \geq 0$ with $\rho(x, y) = 0$ if and only if $x = y$,

SYMMETRIC $\rho(x, y) = \rho(y, x)$,

TRIANGLE INEQUALITY $\rho(x, y) \leq \rho(x, z) + \rho(z, y)$.

[Notice that by definition, $\rho(x, y)$ is finite valued for all $x, y \in X$.]

10.2 ***EXAMPLE.***

Every Euclidean space $\mathbf{R}^n$ is a metric space with metric $\rho(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|$.

Proof. (1) $\rho(x,y) = \|x-y\| = \sqrt{\sum_{i=1}^{n} |x_i - y_i|^2} \geq 0$

$\rho(x,y) = 0 \iff \|x-y\| = 0 \iff x-y=0$

$\iff x=y.$

(positive definite)

(2) $\rho(x,y) = \|x-y\|$

$= \|y-x\| = \rho(y,x)$ (Symmetric)

(3) $\rho(x,y) = \|x-y\|$

$= \|x-z+z-y\|$

$\leq \|x-z\| + \|z-y\|$

$= \rho(x,z) + \rho(z,y).$ (Triangle $\leq$)

$\therefore\ \rho$ is a metric on $\mathbb{R}^n$.

10.3 ***EXAMPLE.***

R is a metric space with metric

$$\rho(x, y) = \begin{cases} 0 & x = y \\ 1 & x \neq y. \end{cases}$$

(This metric is called the *discrete metric*.)

Proof. (1) Positive definite

. $\rho(x,y) \geq 0$

if $x=y \Rightarrow \rho(x,y)=0$
$x \neq y \Rightarrow \rho(x,y)=1$ $\Big\} \Rightarrow \rho(x,y) \geq 0$.

. $\rho(x,y)=0 \Leftrightarrow x=y$ by def'n.

(2) Symmetric

$$\rho(x,y) = \begin{cases} 1, & x \neq y \\ 0, & x=y \end{cases}$$

$$= \begin{cases} 1, & y \neq x \\ 0, & y=x \end{cases}$$

$$= \rho(y,x).$$

(3) Triangle inequality: $\rho(x,y) \leq \rho(x,z)+\rho(z,y)$

. If $x=y$, then $\underbrace{\rho(x,y)}_{0} \leq \rho(x,z)+\rho(z,y)$ is clear.

. If $x \neq y$, then $z \neq x$ or $z \neq y$.

. If $z \neq x$, $\rho(x,y)=1=\rho(x,z) \leq \rho(x,z)+\rho(y,z)$

. If $z \neq y$, then $\rho(x,y)=1=\rho(y,z) \leq \rho(x,z)+\rho(y,z)$

Thus, $\rho(x,z) \le \rho(x,y) + \rho(y,z)$.
$\forall x,y,z \in \mathbb{R}$.

$\therefore$ ρ is a metric on $\mathbb{R}$.

(X,ρ)

(E,ρ)

E

10.4 *EXAMPLE.*

If $E \subseteq X$, then E is a metric space with metric ρ. (We shall call such metric spaces E *subspaces* of X.)

Proof. If the Positive Definite Property, the Symmetric Property, and the Triangle Inequality hold for all $\boxed{x, y \in X}$ then they hold for all $\boxed{x, y \in E}$. ■

10.5 *EXAMPLE.*

Q is a metric space with metric $\rho(x, y) = |x - y|$.

10.6 *EXAMPLE.*

Let $\mathcal{C}[a, b]$ represent the collection of continuous $f : [a, b] \to \mathbf{R}$ and

$$\|f\| := \sup_{x \in [a,b]} |f(x)|.$$

Then $\rho(f, g) := \|f - g\|$ is a metric on $\mathcal{C}[a, b]$.

$f = x^2$, $[1,4]$

$f \in C([1,4])$

$\|f\| = \sup_{x\in[1,4]} |f(x)|$

$= 16$.

$\rho : X \times X \longrightarrow \mathbb{R}$, $X = C[a,b]$

$\rho(f,g) := \|f-g\|$.

proof (i) $\|f\| < \infty$, $\forall f \in C[a,b]$

by the Extreme value thm.

By def'n $\|f\| \geq 0$ and

$\rho(f,g) = \sup_{x\in[a,b]} |f(x) - g(x)| \geq 0$.

$\rho(f,g) = 0$ iff $\|f-g\| = 0$

iff $f = g$.

(ii) $\rho(f,g) = \|f-g\| = \|g-f\| = \rho(g,f)$.

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(iii)

$$\rho(f,g) = \|f-g\| = \sup_{x\in[a,b]} |f(x) - g(x)|$$

$$= \sup_{x\in[a,b]} |f(x) - h(x) + h(x) - g(x)|$$

$$\le \sup_{x\in[a,b]} \left(|f(x) - h(x)| + |h(x) - g(x)|\right)$$

$$\le \sup_{x\in[a,b]} |f(x) - h(x)| + \sup_{x\in[a,b]} |h(x) - g(x)|$$

$$= \|f-h\| + \|h-g\|$$

$$= \rho(f,h) + \rho(h,g)$$

$\therefore \rho(f,g) \le \rho(f,h) + \rho(h,g), \ \forall f,g,h \in C[a,b].$

Thus, ρ is a metric on $C[a,b]$.

$\mathbb{R}^n$

$$B_r(a) := \{x \in \mathbb{R}^n : \|x-a\| < r\}$$

$\mathbb{R}$ $\quad B_r(a) = \{x \in \mathbb{R} : |x-a| < r\}$

X $\quad B_r(a) = \{x \in X : \rho(x,a) < r\}$

$$(\mathbb{R}^n, \|\cdot\|) \longrightarrow (X, \rho)$$

10.7 Definition.

Let $a \in X$ and $r > 0$. The *open ball* (in X) with *center a* and *radius r* is the set

$$B_r(a) := \{x \in X : \rho(x, a) < r\},$$

and the *closed ball* (in X) with *center a* and *radius r* is the set

$$\{x \in X : \rho(x, a) \le r\}.$$

10.8 Definition.

i) A set $V \subseteq X$ is said to be *open* if and only if for every $x \in V$ there is an $\varepsilon > 0$ such that the open ball $B_\varepsilon(x)$ is contained in V.
ii) A set $E \subseteq X$ is said to be *closed* if and only if $E^c := X \setminus E$ is open.

$B_\varepsilon(x) = \{ y \in X : \rho(y,x) < \varepsilon \}$.

10.9 Remark. *Every open ball is open, and every closed ball is closed.*

Proof. Let $B_r(a)$ be an open ball.

We need to show $\forall x \in B_r(a)$, $\exists \varepsilon > 0$ s.t $\boxed{B_\varepsilon(x) \subseteq B_r(a)}$.

Let $x \in B_r(a)$ "i.e., $\rho(x,a) < r$".

Set $\varepsilon := r - \rho(x,a) > 0$

If $\boxed{y \in B_\varepsilon(x)}$ (i.e., $\rho(y,x) < \varepsilon$). Then

$\rho(y,a) \le \boxed{\rho(y,x)} + \rho(x,a)$

$< \varepsilon + \rho(x,a)$ since $y \in B_\varepsilon(x)$.

$= r - \rho(x,a) + \rho(x,a) = r$

$\Rightarrow \rho(y,a) < r$. Thus, $\boxed{y \in B_r(a)}$

$\therefore$ $B_\varepsilon(x) \subseteq B_r(a)$

$\therefore$ $B_r(a)$ is open.

10.10 Remark. *If $a \in X$, then $X \setminus \{a\}$ is open and $\{a\}$ is closed.*

10.11 Remark. *In an arbitrary metric space, the empty set $\emptyset$ and the whole space X are both open and closed.*

Rmk: For some metric spaces (like $\mathbb{R}^n$) there are two sets only which are open and closed (ϕ and $\mathbb{R}^n$). For other metric spaces, there are many such sets

10.12 ***EXAMPLE.***

Every subset of the discrete space **R** is both open and closed.

discrete space $\mathbb{R}$: $\rho(x,y) = \begin{cases} 0, & x = y \\ 1, & x \neq y \end{cases}$ $\forall x,y \in \mathbb{R}$.

proof. Let $E \subseteq \mathbb{R}$. By remark 10.11, $E \neq \phi$.

Let $\boxed{a \in E}$

$B_1(a) = \{x \in \mathbb{R} : \rho(x,a) < 1\}$

by def'n of ρ.

$= \{x \in \mathbb{R} : \rho(x,a) = 0\}$

$= \{x \in \mathbb{R} : x = a\}$

$= \{a\} \subseteq E$

Thus, E is open.

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$\mathbb{R}^n$

تذكير $x_n \to a$, $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t

$n \geq N \Longrightarrow \|x_n - a\| < \varepsilon$

(X, ρ) $n \geq N \Longrightarrow \rho(x_n, a) < \varepsilon$.

Cauchy seq. $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t

$m, n \geq N \Longrightarrow \rho(x_n, x_m) < \varepsilon$.

$\{x_n\}$ is bdd $\exists M > 0$ and $b \in X$

$\rho(x_n, b) \leq M$.

10.13 Definition.

Let $\{x_n\}$ be a sequence in X.

i) $\{x_n\}$ *converges* (in X) if there is a point $a \in X$ (called the *limit* of x_n) such that for every $\varepsilon > 0$ there is an $N \in \mathbf{N}$ such that

$$n \geq N \quad \text{implies} \quad \rho(x_n, a) < \varepsilon.$$

ii) $\{x_n\}$ is *Cauchy* if for every $\varepsilon > 0$ there is an $N \in \mathbf{N}$ such that

$$n, m \geq N \quad \text{implies} \quad \rho(x_n, x_m) < \varepsilon.$$

iii) $\{x_n\}$ is *bounded* if there is an $M > 0$ and a $b \in X$ such that $\rho(x_n, b) \leq M$ for all $n \in \mathbf{N}$.

10.14 Theorem. *Let X be a metric space.*

i) *A sequence in X can have at most one limit.*

ii) *If $x_n \in X$ converges to a and $\{x_{n_k}\}$ is any subsequence of $\{x_n\}$, then x_{n_k} converges to a as $k \to \infty$.*

iii) *Every convergent sequence in X is bounded.*

iv) *Every convergent sequence in X is Cauchy.*

Proof. (i) Spse that $\{x_n\}$ converges to a and b (i.e., $x_n \to a$ and $x_n \to b$). We need to show $a = b$.

Since $x_n \to a$ and $x_n \to b$, as $n \to \infty$, then $\forall \varepsilon > 0$,

$\exists N \in \mathbb{N}$ s.t

$n \geq N \implies \rho(x_n, a) < \frac{\varepsilon}{2}$ and

$\rho(x_n, b) < \frac{\varepsilon}{2}$. thus,

$\rho(a,b) \leq \rho(a, x_n) + \rho(x_n, b)$ by (Triangle inequality)

$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$

$\rho(a,b) < \varepsilon, \ \forall \varepsilon > 0$

$|\rho(a,b)| < \varepsilon, \ \forall \varepsilon > 0$

$\implies \rho(a,b) = 0$ by

$\implies a = b$ since ρ is a metric.

Recall,
$|x| < \varepsilon, \forall \varepsilon > 0 \iff x = 0$

H.W If $a, b \in X$ and $\rho(a, b) < \varepsilon$ for all $\varepsilon > 0$, prove that $a = b$.

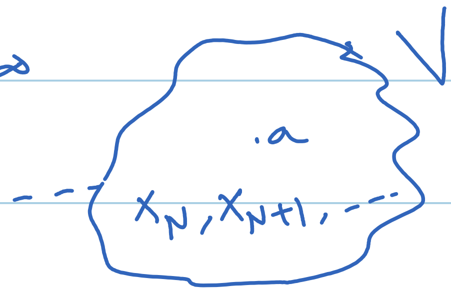
Hint: See Thm 1.9(iii) page 12 (textbook).

10.15 Remark. *Let $x_n \in X$. Then $x_n \to a$ as $n \to \infty$ if and only if for every open set V which contains a there is an $N \in \mathbf{N}$ such that $n \geq N$ implies $x_n \in V$.*

Proof ($\Rightarrow$) Spse that $x_n \to a$ as $n \to \infty$

and V be open set

contains a.



By def 10.8, $\exists \ \varepsilon > 0$ such that $B_\varepsilon(a) \subseteq V$

Since $x_n \to a$ as $n \to \infty$, given this ε

$\exists N \in \mathbb{N}$ s.t $n \geq N \implies x_n \in B_\varepsilon(a) \subseteq V$.

$\therefore x_n \in V, \ \forall n \geq N.$

($\Leftarrow$) Conversely, spse that for every open set $V \ni a$, $\exists N \in \mathbb{N}$ s.t $x_n \in V$.

We need to show $x_n \longrightarrow a$ as $n \to \infty$.

Let $\varepsilon > 0$, and $V = B_\varepsilon(a)$. Then V is open which contains a. By hypothesis, $\exists N \in \mathbb{N}$ s.t

$n \geq N \Longrightarrow x_n \in V = B_\varepsilon(a)$

$B_\varepsilon(a) = \{x \in X : \rho(x,a) < \varepsilon\}$

i.e., $\rho(x_n, a) < \varepsilon, \ \forall n \geq N.$

i.e., $x_n \longrightarrow a$ as $n \to \infty$ $\blacksquare$

10.16 Theorem. *Let $E \subseteq X$. Then E is closed if and only if the limit of every convergent sequence $x_k \in E$ satisfies*

$$\lim_{k\to\infty} x_k \in E.$$

proof. Exercise.

Rmk. The Bolzano–Weierstrass thm is missing here. (X any metric space). Every bdd seq. in $\mathbb{R}^n$ has a convergent subseq.

10.17 Remark. *The discrete space contains bounded sequences which have no convergent subsequences.*

Proof. $X = \mathbb{R}$, $\sigma(x,y) = \begin{cases} 0, & x = y \\ 1, & x \neq y \end{cases}$.

Let $x_n = k$ "constant"

$\sigma(0, x_n) = \sigma(0, k) = 1, \ \forall k \in \mathbb{N}.$

$\therefore \{k\}$ is bounded in X.

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Spse by contradiction, $\{k\}$ has a convergent subseq. i.e., $\exists\, k_1 < k_2 < \cdots$ and $x \in X = \mathbb{R}$ such that $k_j \to x$ as $j \to \infty$.

By def'n, $\exists\, N \in \mathbb{N}$ s.t

$$j \geq N \implies \underbrace{\sigma(k_j, x)}\, < 1$$

i.e., $\sigma(k_j, x) = 0$

$\implies k_j = x \;,\; \forall j \geq N.$

(since σ is metric on X).

a contradiction.

$\therefore \{k\}$ has no convergent subsequences $\blacksquare$

Rmk. Cauchy's thm (Every Cauchy seq. in $\mathbb{R}^n$ is convergent) is missing here (if X is any metric space).

Example Let $X = \mathbb{Q}$, $f(x,y) = |x-y|$.

Let $\{q_k\}$ be a seq. in $X = \mathbb{Q}$

$\left(\forall a \in \mathbb{R} \,,\, \exists \text{ seq. } q_k \in \mathbb{Q} \text{ s.t. } q_k \to a\right)$ تذكر

density of rationals

$q_k \to \sqrt{2}$ (conv.).

$\{q_k\}$ is Cauchy by (thm 10.14 iv) but does not conv. in X ($\sqrt{2} \notin \mathbb{Q}$).

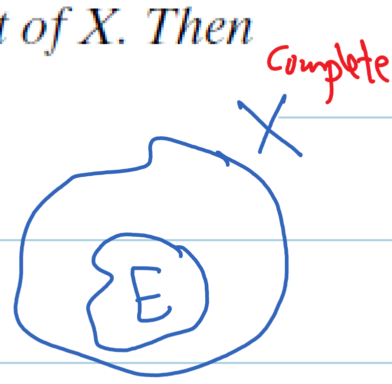
10.19 Definition.

A metric space X is said to be *complete* if and only if every Cauchy sequence $x_n \in X$ converges to some point in X.

ex. $X = \mathbb{R}^n$, $f(x,y) = \|x-y\|$ is Complete.

10.20 Remark. *By Definition 10.19, a complete metric space X satisfies two properties: 1) Every Cauchy sequence in X converges; 2) the limit of every Cauchy sequence in X stays in X.*

10.21 Theorem. *Let X be a complete metric space and E be a subset of X. Then E (as a subspace) is complete if and only if E (as a subset) is closed.*



Proof. Let X be a complete metric space $E \subseteq X$ be complete

($\Rightarrow$) we need to show that E is closed.

(Let $x_n \in E$ converges we need to show $\lim x_n \in E$)

Since $\{x_n\}$ conv, then it is Cauchy.

Since E is complete, by def'n, it follows $\lim_{n\to\infty} x_n \in E$. Thus, E is closed.

($\Leftarrow$) Conversely, spse that E is closed and $x_n \in E$ is Cauchy in E.

Since the metrics on X and E are the same, $\{x_n\}$ is Cauchy in X.

Since X is complete, $x_n \to x$ as $n \to \infty$ for some $x \in X$. But E is closed, $\lim_{n\to\infty} x_n = x \in E$. Thus, E is complete by defn.

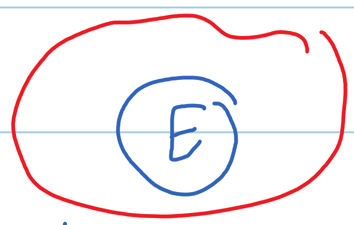
H.W.; 1–9, 12

Discussion (9.2, 9.3)

Section 9.2   1, 4, 7.

9.2.1. Suppose that K is compact in $\mathbf{R}^n$ and $E \subseteq K$. Prove that E is compact if and only if E is closed.

Proof

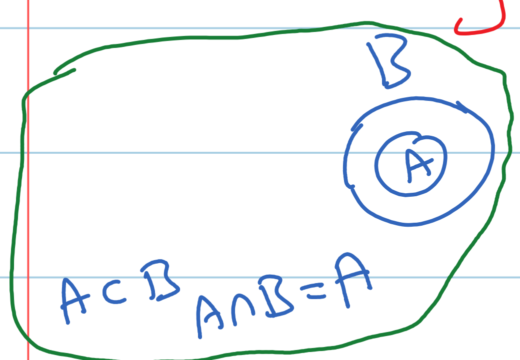


($\Rightarrow$) K is compact $E \subseteq K$ compact.
then by Heine-Borel thm, E is closed (and bounded).

($\Leftarrow$) If E is closed and $E \subseteq K$ "bounded".
$\Rightarrow E$ bdd
Thus, by the Heine-Borel thm, E is compact. ∎

9.2.4. Suppose that K is compact in $\mathbf{R}^n$ and that for every $\mathbf{x} \in K$ there is an $r = r(\mathbf{x}) > 0$ such that $B_r(\mathbf{x}) \cap K = \{\mathbf{x}\}$. Prove that K is a finite set.

Pf: Since K is compact and is covered by $\{B_{r(x)}(x)\}_{x \in K}$, then
$\exists\, x_1, x_2, \dots, x_N$ s.t.



$$K \subset \bigcup_{j=1}^{N} B_{r(x_j)}(x_j) \qquad r(x_j) := r_j$$

thus,
$$K = \bigcup_{j=1}^{N} \Big(\underbrace{B_{r(x_j)}(x_j) \cap K}_{\{x_j\}} \Big) = \{x_1, x_2, \dots, x_N\}$$

That is, K is finite set. ∎

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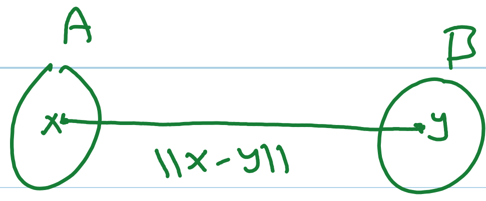


9.2.7. Define the distance between two nonempty subsets A and B of $\mathbf{R}^n$ by

$$\text{dist}(A, B) := \inf\{\|\mathbf{x} - \mathbf{y}\| : \mathbf{x} \in A \quad \text{and} \quad \mathbf{y} \in B\}. \geq 0$$

a) Prove that if A and B are compact sets which satisfy $A \cap B = \emptyset$, then $\text{dist}(A, B) > 0$.

b) Show that there exist nonempty, closed sets A, B in $\mathbf{R}^2$ such that $A \cap B = \emptyset$ but $\text{dist}(A, B) = 0$.

Proof. Recall, $\inf A = \beta < \infty \iff \exists$ a seq. $\{x_k\} \in A$ s.t. $x_k \longrightarrow \inf A = \beta$.

Since $\|x-y\| \geq 0$ and both sets are nonempty ($\|x-y\|$ is bdd below by 0)., then dist(A,B) exists and finite, by the Approximation Property for Infima.

then $\exists$ $x_k \in A$ and $y_k \in B$ such that

$$\|x_k - y_k\| \longrightarrow \text{dist}(A,B) \text{ as } k \to \infty.$$

Since A and B are compact (closed & bdd).

$\Rightarrow x_k, y_k$ are bdd. seq.

by the Bolzano-Weierstrass thm,

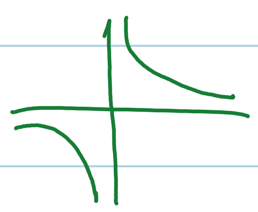
$\exists$ subsequences

$$x_{k_j} \longrightarrow x_0 \in A \text{ and } y_{k_j} \longrightarrow y_0 \in B$$

Since $A \cap B = \phi$, $x_0 \neq y_0$.

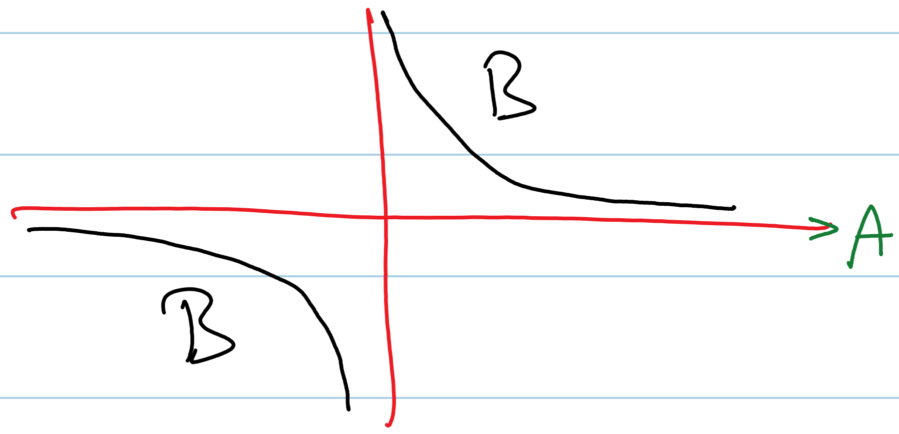
$$\text{dist}(A,B) = \|x_0 - y_0\| > 0 \text{ since } x_0 \neq y_0.$$

(b) $A = \{(x,y) : y = 0\}$ = x-axis. (closed)



$B = \{(x,y) : y = \frac{1}{x}\}$ (closed)

$A \cap B = \phi$ but $dist(A,B) = 0$ since $\frac{1}{x} \to 0$ as $x \to \infty$.



9.3 2(b,c), 5, 7.

9.3.2. Compute the iterated limits at (0,0) of each of the following functions. Determine which of these functions has a limit as $(x, y) \to (0, 0)$ in $\mathbf{R}^2$, and prove that the limit exists.

a) $$f(x, y) = \frac{\sin x \sin y}{x^2 + y^2}$$

Along $x = 0$, $\lim\limits_{(x,y)\to(0,0)} f(x,y) = \lim\limits_{y\to 0} \frac{0}{y^2} = \lim\limits_{y\to 0} 0 = 0$

Along $x = y$, $\lim\limits_{x\to 0} \frac{\sin^2 x}{2x^2} = \frac{1}{2}$

$\therefore \lim\limits_{(x,y)\to(0,0)} f(x,y)$ DNE.

Notice that the iterated limits are 0
ليش؟

b) $$f(x, y) = \frac{x^2 + y^4}{x^2 + 2y^4}$$

$\lim\limits_{x\to 0} \lim\limits_{y\to 0} f(x,y) = \lim\limits_{x\to 0} \frac{x^2}{x^2} = 1$.

$\lim\limits_{y\to 0} \lim\limits_{x\to 0} f(x,y) = \lim\limits_{y\to 0} \frac{y^4}{2y^4} = \frac{1}{2}$

$\Rightarrow \lim\limits_{(x,y)\to(0,0)} f(x,y)$ DNE.

c) $$f(x,y) = \frac{x-y}{(x^2+y^2)^{\alpha}}, \quad \alpha < \frac{1}{2}$$

$x^2+y^2 \geq x^2$ and $x^2+y^2 \geq y^2$

$\sqrt{x^2+y^2} \geq |x|$ $\quad \sqrt{x^2+y^2} \geq |y|$

$$|f(x,y)| = \frac{|x-y|}{(x^2+y^2)^{\alpha}} \leq \frac{|x|+|y|}{(x^2+y^2)^{\alpha}}$$

$$\leq \frac{2\sqrt{x^2+y^2}}{(x^2+y^2)^{\alpha}}$$

$$= 2(x^2+y^2)^{\frac{1}{2}-\alpha}$$

$$|f(x,y)| \leq 2(x^2+y^2)^{\frac{1}{2}-\alpha} \longrightarrow 0$$

as $(x,y) \to (0,0)$, $\alpha < \frac{1}{2}$

$\therefore \lim_{(x,y)\to(0,0)} f(x,y) = 0$, by squeeze thm.

9.3.5. Suppose that $\mathbf{a} \in \mathbf{R}^n$, that $\mathbf{L} \in \mathbf{R}^m$, and that $\mathbf{f} : \mathbf{R}^n \to \mathbf{R}^m$. Prove that if $\mathbf{f}(\mathbf{x}) \to \mathbf{L}$ as $\mathbf{x} \to \mathbf{a}$, then there is an open set V containing $\mathbf{a}$ and a constant $M > 0$ such that $\|\mathbf{f}(\mathbf{x})\| \leq M$ for all $\mathbf{x} \in V$.

Pf. $\lim_{x \to a} f(x) = L$. let $\varepsilon = 1$ and choose

$\delta > 0$ s.t $\boxed{0 < \|x-a\| < \delta} \Rightarrow \|f(x) - L\| < 1$

$x \in B_\delta(a)$

then,

$$\|f(x)\| = \|f(x) - L + L\|$$

$$\leq \|f(x) - L\| + \|L\|$$

$\|f(x)\| < 1 + \|L\|$, $\forall x \in B_\delta(a) \setminus \{a\}$

and $\|f(a)\| \leq \|f(a)\|$

$\|f(x)\| \leq M = \max\{\|L\|+1, \|f(a)\|\}$

for all $x \in V = B_\delta(a)$.

9.3.7. Suppose that $g : \mathbf{R} \to \mathbf{R}$ is differentiable and that $g'(x) > 1$ for all $x \in \mathbf{R}$. Prove that if $g(1) = 0$ and $f(x, y) = (x-1)^2(y+1)/(yg(x))$, then there is an $L \in \mathbf{R}$ such that $f(x, y) \to L$ as $(x, y) \to (1, b)$ for all $b \in \mathbf{R} \setminus \{0\}$.

Sol. $f(x,y) = \dfrac{(x-1)^2(y+1)}{y\,g(x)}$, $g(1) = 0$.

find $\lim\limits_{(x,y)\to(1,b)} f(x,y)$ $b \in \mathbb{R} \setminus \{0\}$.

By the mean value thm, on g $[1, x]$

$g(x) = g(x) - g(1) = g'(c)(x-1)$, c between x and 1.

$|g(x)| = |g'(c)|\,|x-1|$

$= g'(c)\,|x-1| > 1 \cdot |x-1|$

$\dfrac{1}{|g(x)|} < \dfrac{1}{|x-1|}$, $x \neq 1$ $(*)$

$|f(x,y)| = \left| \dfrac{(x-1)^2(y+1)}{y\,g(x)} \right|$

$< \dfrac{|x-1|^2\,|y+1|}{|y|\,|x-1|}$ by $(*)$

$= \dfrac{|x-1|\,|y+1|}{|y|} \longrightarrow 0$ as $(x,y) \to (1,b)$, $b \neq 0$

$\therefore \lim\limits_{(x,y)\to(1,b)} f(x,y) = 0 \quad , \quad b \in \mathbb{R} \setminus \{0\}.$

by squeeze thm.

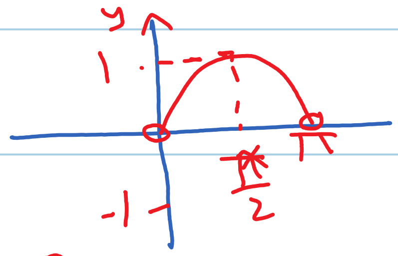
Discussion

9.4 1(a), 3, 6, 9

9.4.1. Define f and g on $\mathbf{R}$ by $f(x) = \sin x$ and $g(x) = x/|x|$ if $x \neq 0$ and $g(0) = 0$.

a) Find $f(E)$ and $g(E)$ for $E = (0, \pi)$, $E = [0, \pi]$, $E = (-1, 1)$, and $E = [-1, 1]$. Compare your answers with what Theorems 9.26, 9.29, and 9.30 predict. Explain any differences you notice.

$f(x) = \sin x$, $E = (0, \pi)$



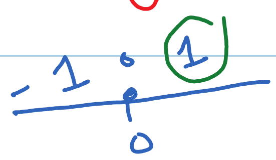
$f((0,\pi)) = (0, 1]$ is not open

$f([0,\pi]) = [0, 1]$ compact and connected.

$f(-1,1) = (-\sin 1, \sin 1)$ open.

$f([-1,1]) = [-\sin 1, \sin 1]$ compact & connected (thm 9.29, thm 9.30).

$$g(x) = \begin{cases} \dfrac{x}{|x|}, & x \neq 0 \\ 0, & x = 0 \end{cases} = \begin{cases} 1, & x > 0 \\ -1, & x < 0 \\ 0, & x = 0 \end{cases}$$



$g((0,\pi)) = \{1\}$ connected

$g([0,\pi]) = \{0, 1\}$ is compact but not connected.

$g((-1,1)) = \{-1, 0, 1\}$ is not open.

$g(\underbrace{[-1,1]}_{\text{compact}}) = \{-1,0,1\}$ compact but not connected

Notice that g is not continuous, so thm 9.29 doesnot apply.

$\boxed{9.4.3}$. **This exercise is used in this section and in Chapter 11.** Suppose that A is open in $\mathbf{R}^n$ and $\mathbf{f} : A \to \mathbf{R}^m$. Prove that $\mathbf{f}$ is continuous on A if and only if $\mathbf{f}^{-1}(V)$ is open in $\mathbf{R}^n$ for every open subset V of $\mathbf{R}^m$.

Pf. f is cont. on A $\Leftrightarrow$ $f^{-1}(V)$ is relatively open for every open subset V in $\mathbb{R}^m$ (Thm)

$\Leftrightarrow$ $f^{-1}(V) = \underset{\text{open}}{O} \cap \underset{\text{open}}{A}$ for some open O in $\mathbb{R}^n$

$\Leftrightarrow$ $f^{-1}(V)$ is open in $\mathbb{R}^n$ for every open subset V of $\mathbb{R}^m$.

9.4.6. Prove that

$$f(x, y) = \begin{cases} e^{-1/|x-y|} & x \neq y \\ 0 & x = y \end{cases}$$

is continuous on $\mathbf{R}^2$.

. f is cont. at every (x,y) which satisfies $x \neq y$.

We need to prove that f is cont. at (x_0,y_0) where $x_0 = y_0$.

Let $g(t) = e^{-\frac{1}{t}} \longrightarrow 0$ as $t \to 0^+$

i.e., $\forall \varepsilon > 0$ $\exists \delta > 0$ s.t

$0 < t < \delta \Rightarrow |g(t) - 0| = e^{-\frac{1}{t}} < \varepsilon$.

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$f:\mathbb{R}^2 \to \mathbb{R}$

If $\boxed{\|(x,y)-(x_0,y_0)\| < \frac{\delta}{2}}$, then

$$|x-y| = |x-x_0+y_0-y| \quad \text{since } x_0=y_0$$
$$\le |x-x_0| + |y-y_0|$$
$$\le 2\sqrt{(x-x_0)^2+(y-y_0)^2}$$
$$= 2\|(x,y)-(x_0,y_0)\|$$

$$0 < \overset{t}{|x-y|} < 2 \cdot \frac{\delta}{2} = \delta .$$

$0<t<\delta \Rightarrow e^{-\frac{1}{t}} < \varepsilon$

. Hence $e^{-\frac{1}{|x-y|}} - 0 < \varepsilon$

$$|f(x,y) - f(x_0,y_0)| < \varepsilon . \quad (f(x_0,y_0)=0 \text{ since } x_0=y_0)$$

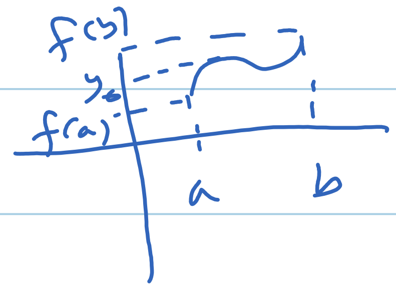
$$\therefore \lim_{(x,y)\to(x_0,y_0)} f(x,y) = 0 = f(x_0,y_0) \quad \text{when } x_0=y_0$$

Thus, f is cont. at (x_0,y_0) where $x_0=y_0$

& we proved f is cont. at (x,y), $x \neq y$.

$\therefore$ f is cont. on $\mathbb{R}^2$

$f : [a,b] \to \mathbb{R}$



$\exists\, c \in (a,b)$
$f(c) = y_0$

9.4.9. [Intermediate Value Theorem]. Let E be a connected subset of $\mathbf{R}^n$. If $f : E \to \mathbf{R}$ is continuous, $f(\mathbf{a}) \neq f(\mathbf{b})$ for some $\mathbf{a}, \mathbf{b} \in E$, and y is a number which lies between $f(\mathbf{a})$ and $f(\mathbf{b})$, then prove that there is an $\mathbf{x} \in E$ such that $f(\mathbf{x}) = y$. (You may use Theorem 8.30.)

Pf. Spse $f(a) < f(b)$. Since E is connected, then $f(E)$ is connected in $\mathbb{R}$ by Thm 8.30 means $f(E)$ is an interval.

Since $f(a), f(b) \in f(E) \Rightarrow [f(a), f(b)] \subset f(E)$

Since $y \in [f(a), f(b)]$, then $y \in f(E)$

i.e, $y = f(x)$, for some $x \in E$ ∎

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Discussion.

10.1 4−7

$x_n = a$

$|x_n - a| < \varepsilon$ $\quad x_n \to a$ $\quad |x_n - a| < \varepsilon$ $\quad x_n = a$

10.1.4. a) Let $a \in X$. Prove that if $\boxed{x_n = a \text{ for every } n \in \mathbf{N},}$ then $\boxed{x_n \text{ converges.}}$ What does it converge to?

b) Let $X = \mathbf{R}$ with the discrete metric. Prove that $x_n \to a$ as $n \to \infty$ iff and only if $x_n = a$ for large n.

Pf (a) claim $x_n \longrightarrow a$ as $n \to \infty$

We need to prove $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t

$\rho(x_n, a) < \varepsilon$ for large n. $\quad$ $n \geq N \Rightarrow \rho(x_n, a) < \varepsilon$

Let $\varepsilon > 0$. Then $\rho(x_n, a) = 0 < \varepsilon$, since ρ is a metric for all $n \in \mathbb{N}$.

(b) $X = \mathbb{R}$ with discrete metric

$$\rho(x,y) = \begin{cases} 1, & x \neq y \\ 0, & x = y \end{cases}$$

($\Leftarrow$) done part (a).

($\Rightarrow$) Spse $x_n \longrightarrow a$ as $n \longrightarrow \infty$

i.e, $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t

$n \geq N \Longrightarrow \rho(x_n, a) < \varepsilon$ $\qquad \rho(x,y) = \begin{cases} 1, & x \neq y \\ 0, & x = y \end{cases}$

In particular, ($\varepsilon = 1$) $\rho(x_n, a) < 1$. for large n.

$\rho(x_n, a) = 0$ (since ρ is discrete metric)

$\Rightarrow x_n = a$ since ρ is a metric

$x_n \to a$ and $y_n \to a$ as $n \to \infty$.

10.1.5. a) Let $\{x_n\}$ and $\{y_n\}$ be sequences in X which converge to the same point. Prove that $\rho(x_n, y_n) \to 0$ as $n \to \infty$.

b) Show that the converse of part a) is false.

Pf. (a) spse $x_n \to a$ and $y_n \to a$ as $n \to \infty$.

i.e., $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t

$n \geq N \implies \rho(x_n, a) < \frac{\varepsilon}{2}$ and $\rho(y_n, a) < \frac{\varepsilon}{2}$

Then, $\rho(x_n, y_n) \leq \rho(x_n, a) + \rho(a, y_n)$

$= \rho(x_n, a) + \rho(y_n, a)$

$|\rho(x_n, y_n) - 0| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$, $\forall n \geq N$

$\therefore \lim_{n\to\infty} \rho(x_n, y_n) = 0$

(b) If $\rho(x_n, y_n) \to 0$ as $n \to \infty$,

False. then x_n & $y_n \to a$ as $n \to \infty$

ex. $x_n = n$, $y_n = n + \frac{1}{n}$, $X = \mathbb{R}$.

$\rho(x, y) = |x - y|$

$\rho(x_n, y_n) = |x_n - y_n| = \frac{1}{n} \to 0$ as $n \to \infty$.

but neither x_n nor y_n conv.

$x_n = (-1)^n$ in $\mathbb{R}$ $\{-1, 1, -1, 1, \ldots$

$\Leftrightarrow$ $\{-1, -1, \ldots$ conv.
$\{1, 1, \ldots$ conv.

10.1.6. Let $\{x_n\}$ be Cauchy in X. Prove that $\{x_n\}$ converges if and only if at least one of its subsequences converges.

Proof. $(\Rightarrow)$ By thm 10.14, if $x_n \to a$, then $x_{n_k} \to a$.

$(\Leftarrow)$ If x_n is Cauchy and $x_{n_k} \to a$, then $\forall \varepsilon > 0$, $\exists$ $m, n \in \mathbb{N}$ s.t

$m, n \geq N \Rightarrow \rho(x_n, x_m) < \frac{\varepsilon}{2}$

In particular, $k, n \geq N \Rightarrow \boxed{\rho(x_n, x_{n_k}) < \frac{\varepsilon}{2}}$

Then $\rho(x_n, a) \leq \underbrace{\rho(x_n, x_{n_k})}_{\text{Cauchy}} + \underbrace{\rho(x_{n_k}, a)}_{\text{conv. of subseq.}}$

$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.

By def'n, $x_n \to a$ as $n \to \infty$ ∎

10.1.7. Prove that the discrete space $\mathbf{R}$ is complete.

Pf. $X = \mathbb{R}$, $\rho(x,y) = \begin{cases} 1, & x \neq y \\ 0, & x = y \end{cases}$.

Let $\{x_n\}$ be a Cauchy seq. in $\mathbb{R}$.
$\exists$ $N \in \mathbb{N}$ s.t $m, n \geq N \Rightarrow \rho(x_n, x_m) < \varepsilon$.

$n \geq N \Rightarrow \rho(x_N, x_n) < 1$

$\Rightarrow \rho(x_N, x_n) = 0$ (discrete space)

$\Rightarrow x_N = x_n$ by defn of ρ.

$\Rightarrow x_n = x_N := a$ for large n.

$\therefore\ x_n \longrightarrow a \in \mathbb{R}$ as $n \longrightarrow \infty$.

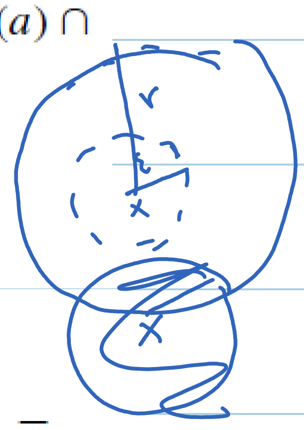
Hence $X = \mathbb{R}$ " the discrete space " is Complete

10.1.9. a) Show that if $x \in B_r(a)$, then there is an $\varepsilon > 0$ such that the closed ball centered at x of radius ε is a subset of $B_r(a)$.

b) If $a \neq b$ are distinct points in X, prove that there is an $r > 0$ such that $B_r(a) \cap B_r(b) = \emptyset$.

c) Show that given two balls $B_r(a)$ and $B_s(b)$, and a point $x \in B_r(a) \cap B_s(b)$, there are radii c and d such that

$$B_c(x) \subseteq B_r(a) \cap B_s(b) \quad \text{and} \quad B_d(x) \supseteq B_r(a) \cup B_s(b).$$



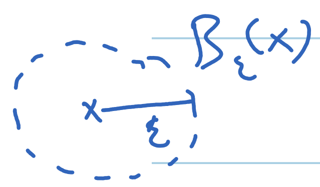
8.21 Remark. *For every* $\mathbf{x} \in B_r(\mathbf{a})$ *there is an* $\varepsilon > 0$ *such that* $B_\varepsilon(\mathbf{x}) \subseteq B_r(\mathbf{a})$.

Proof. Let $\mathbf{x} \in B_r(\mathbf{a})$. Using Figure 8.5 for guidance, we set $\varepsilon = r - \|\mathbf{x} - \mathbf{a}\|$. If $\mathbf{y} \in B_\varepsilon(\mathbf{x})$, then by the Triangle Inequality, assumption, and the choice of ε,

$\rho(x,a)$

$\rho(y,a) \leq \rho(y,x) + \rho(x,a) < \varepsilon + \rho(x,a) = r$

$$\|\mathbf{y} - \mathbf{a}\| \leq \|\mathbf{y} - \mathbf{x}\| + \|\mathbf{x} - \mathbf{a}\| < \varepsilon + \|\mathbf{x} - \mathbf{a}\| = r.$$



Thus, by definition, $\mathbf{y} \in B_r(\mathbf{a})$. In particular, $B_\varepsilon(\mathbf{x}) \subseteq B_r(\mathbf{a})$. ■

$\overline{B_{\frac{\varepsilon}{2}}(x)} \subseteq B_\varepsilon(x)$.

$\overline{B_{\frac{\varepsilon}{2}}(x)} \subseteq B_\varepsilon(x) \subseteq B_r(a)$

9.4.5. Suppose that $E \subseteq \mathbf{R}^n$ and that $\mathbf{f} : E \to \mathbf{R}^m$.

a) Prove that $\mathbf{f}$ is continuous on E if and only if $\mathbf{f}^{-1}(B)$ is relatively closed in E for every closed subset B of $\mathbf{R}^m$.

b) Suppose that $\mathbf{f}$ is continuous on E. Prove that if V is relatively open in $\mathbf{f}(E)$, then $\mathbf{f}^{-1}(V)$ is relatively open in E, and if B is relatively closed in $\mathbf{f}(E)$, then $\mathbf{f}^{-1}(B)$ is relatively closed in E.

A, B are relativey open in E $\Rightarrow A \cap B$ open

$A \cap B = (O_1 \cap E) \cap (O_2 \cap E)$, O_1, O_2 open

$= \underbrace{(O_1 \cap O_2)}_{\text{open}} \cap E$

✓

$= O \cap E$ relatively open

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9.3.3. Prove that each of the following functions has a limit as $\boxed{(x, y) \to (0, 0)}$.

a) $$f(x, y) = \frac{x^3 - y^3}{x^2 + y^2}, \qquad (x, y) \neq (0, 0)$$

b) $$\boxed{f(x, y) = \frac{|x|^\alpha y^4}{x^2 + y^4}, \qquad (x, y) \neq (0, 0),}$$

where α is ANY positive number.

$$|f(x,y)| = |x|^\alpha \cdot \frac{y^4}{x^2+y^4} \leq 1 \qquad \frac{a}{a+b} \leq 1$$

$$|f(x,y)| \leq |x|^\alpha \longrightarrow 0 \text{ as } (x,y) \to (0,0), \ \alpha > 0$$

$$\lim_{(x,y)\to(0,0)} f(x,y) = 0$$

$(\varepsilon\text{-}\delta)$ def'n.

If $\|(x,y) - (0,0)\| < \delta \Rightarrow |f(x,y) - 0| < \varepsilon$

$|x| < \sqrt{x^2+y^2} < \delta$

$$|f(x,y)| \leq |x|^\alpha < \delta^\alpha < \varepsilon$$

choose $\delta = \varepsilon^{\frac{1}{\alpha}}$ $\qquad \alpha > 0$

Let $\varepsilon > 0$, choose $\delta = \varepsilon^{\frac{1}{\alpha}}$. Then

$|x| < \|(x,y) - (0,0)\| < \delta$, then

$$|f(x,y) - 0| \leq |x|^\alpha < \delta^\alpha = \left(\varepsilon^{\frac{1}{\alpha}}\right)^\alpha = \varepsilon.$$

Analysis I (Review of ch4).
on $\mathbb{R}$.

f'

CHAPTER 11

Differentiability on $\mathbf{R}^n$

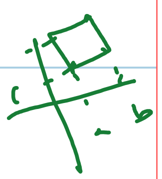
11.1 PARTIAL DERIVATIVES AND PARTIAL INTEGRALS

$\mathbb{R}\times\mathbb{N} = \{(x_1,x_2) : x_1\in\mathbb{R},\ x_2\in\mathbb{N}$

We begin with some notation. The *Cartesian product* of a finite collection of sets $E_1, E_2, \ldots, E_n$ is the set of ordered n-tuples defined by

$$\boxed{E_1 \times E_2 \times \cdots \times E_n := \{(x_1, x_2, \ldots, x_n) : x_j \in E_j \text{ for } j = 1, 2, \ldots, n\}.}$$

$x_1\in E_1$
$x_2\in E_2$
$\vdots$

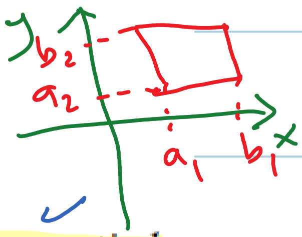


Thus the Cartesian product of n subsets of $\mathbf{R}$ is a subset of $\mathbf{R}^n$. By a *rectangle in* $\mathbf{R}^n$ (or an *n-dimensional rectangle*) we mean a Cartesian product of n closed, nondegenerate, bounded intervals. An n-dimensional rectangle $H = [a_1, b_1] \times \cdots \times [a_n, b_n]$ is called an *n-dimensional cube with side s* if $|b_j - a_j| = s$ for $j = 1, \ldots, n$.

$\mathbb{R}^2$ rectangle $[a_1,b_1]\times[a_2,b_2]$

Let $f : \{x_1\} \times \cdots \times \{x_{j-1}\} \times \underset{x_j}{[a, b]} \times \{x_{j+1}\} \times \cdots \times \{x_n\} \to \mathbf{R}$. We shall denote the function

$$g(t) := f(x_1, \ldots, x_{j-1}, \overset{x_j}{t}, x_{j+1}, \ldots, x_n), \qquad t \in [a, b],$$



by $f(x_1, \ldots, x_{j-1}, \cdot, x_{j+1}, \ldots, x_n)$. If g is integrable on $[a, b]$, then the *partial integral* of f on $[a, b]$ with respect to x_j is defined by

$\int_a^b f(x,y)\,dx = F(y)$

$$\int_a^b f(x_1, \overset{x_j=t}{\ldots}, x_n)\, dx_j := \int_a^b g(t)\, dt.$$

If g is differentiable at some $t_0 \in (a, b)$, then the *partial derivative* (or *first-order partial derivative*) of f at $(x_1, \ldots, x_{j-1}, t_0, x_{j+1}, \ldots x_n)$ with respect to x_j is defined by

$$\boxed{\frac{\partial f}{\partial x_j}(x_1, \ldots, x_{j-1}, t_0, x_{j+1}, \ldots, x_n) := g'(t_0).} = f_{x_j}(x_1,\ldots,x_{j-1},$$

We will also denote this partial derivative by $f_{x_j}(x_1, \ldots, x_{j-1}, t_0, x_{j+1}, \ldots, x_n)$. Thus the partial derivative f_{x_j} exists at a point $\mathbf{a}$ if and only if the limit

$$\frac{\partial f}{\partial x_j}(\mathbf{a}) := \lim_{h\to 0} \frac{f(\mathbf{a} + h\mathbf{e}_j) - f(\mathbf{a})}{h}$$ exists.

$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$

$f = f(x,y)$, $f_x(a,b) = \lim_{h\to 0} \frac{f(a+h,b) - f(a,b)}{h}$

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$$f_y(a,b) = \lim_{k\to 0} \frac{f(a, b+k) - f(a,b)}{k}$$

$f : \mathbb{R}^2 \to \mathbb{R}$

$f = f(x,y)$.

$$f_{xy}(a,b) = \frac{\partial^2 f}{\partial y \partial x}(a,b)$$

$$= \frac{\partial}{\partial y}\left(f_x(a,b) \right)$$

$$= \lim_{k\to 0} \frac{f_x(a, b+k) - f_x(a,b)}{k}$$

$$= \lim_{k\to 0} \frac{1}{k}\left[\lim_{h\to 0} \frac{f(a+h, b+k) - f(a, b+k)}{h} - \lim_{h\to 0} \frac{f(a+h,b) - f(a,b)}{h} \right]$$

$$= \lim_{k\to 0} \lim_{h\to 0} \left[\frac{f(a+h, b+k) - f(a,b+k) - f(a+h,b) + f(a,b)}{hk} \right]$$

As a notation $\displaystyle \lim_{k\to 0} \lim_{h\to 0} \frac{\Delta(h,k)}{hk}$

We extend partial derivatives to vector-valued functions in the following way. Suppose that $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbf{R}^n$ and $\mathbf{f} = (f_1, f_2, \ldots, f_m) : \{a_1\} \times \cdots \times \{a_{j-1}\} \times I \times \{a_{j+1}\} \times \cdots \times \{a_n\} \to \mathbf{R}^m$, where $j \in \{1, 2, \ldots, n\}$ is fixed and I is an open interval containing a_j. If for each $k = 1, 2, \ldots, m$ the first-order partial derivative $\partial f_k / \partial x_j$ exists at $\mathbf{a}$, then we define the *first-order partial derivative* of $\mathbf{f}$ with respect to x_j to be the vector-valued function

$f = (f_1, \ldots, f_n)$

$$\mathbf{f}_{x_j}(\mathbf{a}) := \frac{\partial \mathbf{f}}{\partial x_j}(\mathbf{a}) := \left(\frac{\partial f_1}{\partial x_j}(\mathbf{a}), \ldots, \frac{\partial f_m}{\partial x_j}(\mathbf{a}) \right).$$

ex. $f = (xy, x^2+y^2)$

$$\frac{\partial f}{\partial x} = \left(\frac{\partial (xy)}{\partial x}, \frac{\partial}{\partial x}(x^2+y^2) \right)$$

$$= (y, 2x).$$

Higher-order partial derivatives are defined by iteration. For example, the *second-order partial derivative* of $\mathbf{f}$ with respect to x_j and x_k is defined by

$f_{xy} = \frac{\partial^2 f}{\partial y \partial x}$

$$\mathbf{f}_{x_j x_k} := \frac{\partial^2 \mathbf{f}}{\partial x_k \, \partial x_j} := \frac{\partial}{\partial x_k}\left(\frac{\partial \mathbf{f}}{\partial x_j}\right)$$

when it exists. Second-order partial derivatives are called *mixed* when $j \neq k$.

Note. $\mathbb{R}^2$ f_{xy}, f_{yx} mixed

f_{xx}, f_{yy} not mixed

$\mathbb{R}^3$: $f_{xy}, f_{yx}, f_{xz}, f_{zx}, f_{yz}, f_{zy}$.

11.1 Definition. $V \subseteq \mathbb{R}^n$

Let V be a nonempty, open subset of $\mathbf{R}^n$, let $\mathbf{f}: V \to \mathbf{R}^m$, and let $p \in \mathbf{N}$.

i) $\mathbf{f}$ is said to be $\mathcal{C}^p$ on V if and only if each partial derivative of $\mathbf{f}$ of order $k \le p$ exists and is continuous on V.

ii) $\mathbf{f}$ is said to be $\mathcal{C}^\infty$ on V if and only if $\mathbf{f}$ is $\mathcal{C}^p$ on V for all $p \in \mathbf{N}$.

ex. $V \subseteq \mathbb{R}^2$ f is $C^1(V)$ if f_x and f_y exists and is cont. on V.

$V \subseteq \mathbb{R}^2$, f is $C^2(V)$. f_x, f_y, f_{xy}, f_{yx} exists and is cont. on V.

ex. $f(x,y) = e^{x+y}$ is C^∞

ex. $f(x,y) = \sin(x-y)$ is C^∞.

Rule. If $f \in C^p(V)$ and $q < p$, then $f \in C^q(V)$.

In this section we focus $f: V \subseteq \mathbb{R}^2 \to \mathbb{R}$. ($n=2$, $m=1$)

$x_1 = x$, $x_2 = y$.

Take. $f(x_1, x_2) = f(x,y)$, for simplicity.

$$(fg)' = f'g + fg'$$

$$\frac{\partial}{\partial x}(fg) = f_x g + f g_x.$$

MVT $\quad f'(c) = \frac{f(b) - f(a)}{b-a}, \quad c \text{ in } a<c<b$

أ

$$f_x(c,y) = \frac{f(b,y) - f(a,y)}{b-a}$$

c in (a,b)

$$\frac{\partial}{\partial x}\int_a^x f(t)\,dt = f(x)$$

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$$\frac{\partial}{\partial x}\int_a^x f(t,y)\,dt = f(x,y)$$

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1) By the Product Rule (Theorem 4.10), if f_x and g_x exist, then

$$\boxed{\frac{\partial}{\partial x}(fg) = f\frac{\partial g}{\partial x} + g\frac{\partial f}{\partial x}.}$$

2) By the Mean Value Theorem (Theorem 4.15), if $f(\cdot, y)$ is continuous on $[a, b]$ and the partial derivative $f_x(\cdot, y)$ exists on (a, b), then there is a point $c \in (a, b)$ (which may depend on y as well as a and b) such that

$$f(b, y) - f(a, y) = (b - a)\frac{\partial f}{\partial x}(c, y).$$

5.3

3) By the Fundamental Theorem of Calculus (Theorem 5.28), if $f(\cdot, y)$ is continuous on $[a, b]$, then

$$\boxed{\frac{\partial}{\partial x}\int_a^x f(t, y)\,dt = f(x, y),}$$

$\int_a^b f'(t)\,dt$
$= f(b) - f(a)$

and if the partial derivative $f_x(\cdot, y)$ exists and is integrable on $[a, b]$, then

$$\int_a^b \boxed{\frac{\partial f}{\partial x}(x, y)\,dx} = f(b, y) - f(a, y).$$

$$f(x,y)\Big|_{x=a}^{x=b}$$

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$\lim_{(x,y)\to(a,b)} f_{xy}(x,y) = f_{xy}(a,b)$. or $\lim_{(x,y)\to(a,b)} f_{yx}(x,y) = f_{yx}(a,b)$

$f : V \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}$

11.2 Theorem. *Suppose that V is open in $\mathbf{R}^2$, that $(a,b) \in V$, and that $f : V \to \mathbf{R}$. If f is $\mathcal{C}^1$ on V, and if one of the mixed second partial derivatives of f exists on V and is continuous at the point (a,b), then the other mixed second partial derivative exists at (a,b) and*

$$\boxed{\frac{\partial^2 f}{\partial y\,\partial x}(a,b) = \frac{\partial^2 f}{\partial x\,\partial y}(a,b).}$$

or $f_{xy}(a,b) = f_{yx}(a,b)$

NOTE: *These hypotheses are met if $f \in \mathcal{C}^2(V)$.*

Proof. Spse that f_{yx} exists on V and is continuous at (a,b). $(*)$

Consider $\boxed{\Delta(h,k) := \underbrace{f(a+h,b+k) - f(a+h,b)} - f(a,b+k) + f(a,b)}$

defind for $|h|, |k| < \frac{r}{\sqrt{2}}$, $r>0$ so small that $B_r(a,b) \subset V$.

Apply the Mean value Thm $f(x,y)$. $\exists$ $\boxed{t \in (0,1)}$ s.t

$$\frac{f(a+h,b+k) - f(a+h,b)}{\cancel{b}+k-\cancel{b}} = \underbrace{\frac{\partial f}{\partial y}}(a+h,c)$$

c is between b and $b+k$.

$b < c < b+k$

$\cancel{b} < \cancel{b}+tk < \cancel{b}+k$

$0 < tk < k$

$0 < t < 1$

$$\boxed{f(a+h,b+k) - f(a+h,b) = k\,\frac{\partial f}{\partial y}(a+h,b+tk)} \quad (1)$$

Similarly,

$$\boxed{f(a,b+k) - f(a,b) = k\,\frac{\partial f}{\partial y}(a,b+tk)} \quad (2)$$

put (1) and (2) into $(*)$:

$$D(h,k) = k\frac{\partial f}{\partial y}(a+h, b+tk) - k\frac{\partial f}{\partial y}(a, b+tk)$$

$$= k\left[\frac{\partial f}{\partial y}(a+h, b+tk) - \frac{\partial f}{\partial y}(a, b+tk)\right]$$

Apply the MVT again on f_y , $[a, a+h]$

$\exists\ s \in (0,1)$ s.t,

$$D(h,k) = kh\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}(a+sh, b+tk)\right)$$

$$\boxed{D(h,k) = kh\frac{\partial^2 f}{\partial x\,\partial y}(a+sh, b+tk).}$$

$$\Rightarrow \lim_{k\to 0}\lim_{h\to 0}\frac{D(h,k)}{hk} = \lim_{k\to 0}\lim_{h\to 0} f_{yx}(a+sh, b+tk)$$

$$= f_{yx}(a,b), \text{----} (3)$$

since f_{yx} is continuous at (a,b).

on the otherhand, Apply the mean value thm,

$\exists\ \boxed{u\in(0,1)}$ s.t

$$D(h,k) = \underbrace{[f(a+h, b+k) - f(a, b+k)]}\,-[f(a+h,b) - f(a,b)]$$

$a < a+uh < a+h$

$0 < u < 1$

$$= h\frac{\partial f}{\partial x}(a+uh, b+k) - h\frac{\partial f}{\partial x}(a+uh, b)$$

$$\boxed{\Delta(h,k) = h\left[\frac{\partial f}{\partial x}(a+uh, b+k) - \frac{\partial f}{\partial x}(a+uh, b)\right]} \quad (4)$$

Back to (3) :

$$\underbrace{\lim_{k\to 0}\lim_{h\to 0}\frac{\Delta(h,k)}{hk}}_{\text{from } (4)} = f_{yx}(a,b)$$

$$\Rightarrow \lim_{k\to 0}\lim_{h\to 0}\frac{1}{k}\left(\frac{\partial f}{\partial x}(a+uh, b+k) - \frac{\partial f}{\partial x}(a+uh, b)\right) = f_{yx}(a,b)$$

$$\Rightarrow \lim_{k\to 0}\frac{1}{k}\left[\lim_{h\to 0} f_x(a+uh, b+k) - f_x(a+uh, b)\right] = f_{yx}(a,b)$$

since f_x exists & cont. at (a,b) :

$$\Rightarrow \lim_{k\to 0}\frac{f_x(a,b+k) - f_x(a,b)}{k} = f_{yx}(a,b)$$

$$\xRightarrow{\text{by def'n.}} \underbrace{\frac{\partial}{\partial y}\left(f_x(a,b)\right)}_{} = f_{yx}(a,b)$$

$$f_{xy}(a,b) = f_{yx}(a,b)$$

$f : V \subseteq \mathbb{R}^2 \to \mathbb{R}$

$f_{xy} = f_{yx}$

Rules.

(1) We shall refer to the conclusion of Theorem 11.2 by saying the first partial derivatives of f *commute*. Thus, if f is $\mathcal{C}^2$ on an open subset V of $\mathbf{R}^n$, if $\mathbf{a} \in V$, and if $j \neq k$, then

$f : V \subseteq \mathbb{R}^n \to \mathbb{R}$

$$\frac{\partial^2 f}{\partial x_j \partial x_k}(\mathbf{a}) = \frac{\partial^2 f}{\partial x_k \partial x_j}(\mathbf{a}).$$

(2) The following example shows that Theorem 11.2 is false if the assumption about continuity of the second-order partial derivative is dropped.

$f : V \subseteq \mathbb{R}^3 \to \mathbb{R}$ $\quad f_{xy} = f_{yx}, \ f_{xz} = f_{zx}, \ f_{yz} = f_{zy}$

(x_1, x_2, x_3) $\quad j \neq k$

$f_{x_1 x_2}, \ f_{x_1 x_3}, \ f_{x_2 x_3}$

11.3 *EXAMPLE.*

Prove that

$$f(x, y) = \begin{cases} xy\left(\dfrac{x^2 - y^2}{x^2 + y^2}\right) & (x, y) \neq \mathbf{0} = (0,0) \\ 0 & (x, y) = \mathbf{0} = (0,0) \end{cases}$$

(1) is $\mathcal{C}^1$ on $\mathbf{R}^2$, (2) both mixed second partial derivatives of f exist on $\mathbf{R}^2$, but the first (3) partial derivatives of f do not commute at $(0, 0)$; that is, $f_{xy}(0, 0) \neq f_{yx}(0, 0)$.

(1) f_x is Cont. on $\mathbb{R}^2$

$\lim_{(x,y)\to(a,b)} f_x(x,y) = f_x(a,b)$. exists.

For $(x,y) \neq (0,0)$:

$$\frac{\partial f}{\partial x} = (xy) \frac{\partial}{\partial x}\left(\frac{x^2-y^2}{x^2+y^2}\right) + \left(\frac{x^2-y^2}{x^2+y^2}\right) \frac{\partial}{\partial x}(xy)$$

$$\frac{\partial f}{\partial x}(x,y) = (xy) \frac{4xy^2}{(x^2+y^2)^2} + \left(\frac{x^2-y^2}{x^2+y^2}\right) \cdot y$$

For $(x,y) \neq (0,0)$

clearly, f_x is Cont. $\forall (x,y) \neq (0,0)$.

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If $(x,y) = (0,0)$:

$$f_x(0,y) = \lim_{h\to 0} \frac{f(0+h,y) - \overset{\nearrow 0}{f(0,y)}}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{h}\, y\left(\frac{h^2-y^2}{h^2+y^2}\right)}{\cancel{h}} = y\left(\frac{-y^2}{y^2}\right)$$

$$= -y .$$

$$\boxed{f_x(0,y) = -y}$$

$f_x(0,0) = 0$ exists.

. $\lim_{(x,y)\to(0,0)} f_x(x,y) = ??$

$$|f_x| = \left| xy \cdot \frac{4xy^2}{(x^2+y^2)^2} + y\left(\frac{x^2-y^2}{x^2+y^2}\right)\right|$$

$$\le \frac{(2|xy|)^2 \, |y|}{(x^2+y^2)^2} + |y| \cdot 1 .$$

$$\le \frac{\cancel{(x^2+y^2)^2}\, |y|}{\cancel{(x^2+y^2)^2}} + |y| , \quad \boxed{2|xy| \le x^2+y^2}$$

$$= 2|y|$$

$\therefore |f_x(x,y)| \le 2|y| \longrightarrow 0$ as $(x,y)\to(0,0)$

$\therefore \lim_{(x,y)\to(0,0)} f_x(x,y) = 0 = f_x(0,0)$

$\therefore f_x$ is cont. on $\mathbb{R}^2$ ($f \in C^1$ on $\mathbb{R}^2$).

Now, $\boxed{f_x(0,y) = -y}$, $\forall y \in \mathbb{R}$ (proved).

$f_y(x,0) = x$, $\forall x \in \mathbb{R}$ (Exercise)

$$\left.\frac{\partial^2 f}{\partial y \partial x}\right|_{(0,0)} = f_{xy}(0,0) = \left.\frac{\partial}{\partial y}(-y)\right|_{(0,0)}$$

$$= -1.$$

$$\left.\frac{\partial^2 f}{\partial x \partial y}\right|_{(0,0)} = f_{yx}\Big|_{(0,0)} = \left.\frac{\partial}{\partial x}(x)\right|_{(0,0)}$$

$$= 1$$

$$\Longrightarrow \quad f_{xy}(0,0) \neq f_{yx}(0,0).$$

<u>ex.</u> $f(x,y) = x^2+y^2 \in C^\infty \Rightarrow f_{xy} = f_{yx} = 0$

$\forall (x,y) \in \mathbb{R}^2$.

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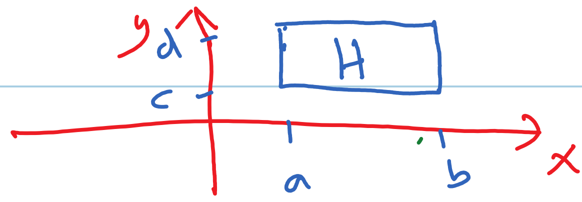

compact.

11.4 Theorem. *Let $H = [a, b] \times [c, d]$ be a rectangle and let $f : H \to \mathbf{R}$ be continuous. If*

$$F(y) = \int_a^b f(x, y)\, dx,$$

integrand

then F is continuous on $[c, d]$; that is,

$$\lim_{y \to y_0} F(y) = \lim_{\substack{y \to y_0 \\ y \in [c,d]}} \underbrace{\int_a^b f(x, y)\, dx}_{F(y)} = \int_a^b \lim_{\substack{y \to y_0 \\ y \in [c,d]}} f(x, y)\, dx = \int_a^b f(x, y_0)\, dx = F(y_0)$$

for all $y_0 \in [c, d]$.

Application. Find $\displaystyle\lim_{y \to 0} \int_0^1 e^{(x^3 y^2 + x)}\, dx$

$f(x,y) = e^{x^3 y^2 + x}$ is cont. on $H = [0,1] \times [-1,1]$

f is cont on $\mathbb{R}^2$ and hence on H.

By Thm 11.4,

$$\lim_{y \to 0} \int_0^1 e^{x^3 y^2 + x}\, dx = \int_0^1 \left[\lim_{y \to 0} e^{x^3 y^2 + x} \right] dx$$

$$= \int_0^1 e^x\, dx$$

$$= e^x \Big|_0^1 = e - 1.$$

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Thm 5.10 page 134

Recall, spse $a, b \in \mathbb{R}$, $a < b$. If $f: [a,b] \to \mathbb{R}$. f is cont. on $[a,b]$, then f is integrable on $[a,b]$ (i.e, $\int_a^b f(x)\,dx$ exists).

Proof (Thm 11.4).

$\forall y \in [c,d]$, $f(\cdot, y)$ is cont. on $[a,b]$.

by the above recalling: $\int_a^b f(x,y)\,dx$ exists

i.e., $F(y)$ exists, $\forall y \in [c,d]$.

Fix $y_0 \in [c,d]$ and let $\varepsilon > 0$. Since H is compact, f is uniformly cont. on H.

Then $\exists\, \delta > 0$ s.t

$$\|(x,y) - (z,w)\| < \delta, \quad (x,y), (z,w) \in H \implies |f(x,y) - f(z,w)| < \frac{\varepsilon}{b-a}. \quad (I)$$

we need to show F is cont. at $y_0 \in [c,d]$

($\forall \varepsilon > 0, \exists\, \delta > 0$ s.t if $|y - y_0| < \delta$, then $|F(y) - F(y_0)| < \varepsilon$

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$|y-y_0| = \sqrt{(x-x)^2+(y-y_0)^2}$

If $|y-y_0| = \underbrace{\|(x,y)-(x,y_0)\|}$ $< \frac{\delta}{2}$, then

$$|F(y)-F(y_0)| = \left|\int_a^b f(x,y)\,dx - \int_a^b f(x,y_0)\,dx\right|$$

$$\le \int_a^b \underbrace{|f(x,y)-f(x,y_0)|}\,dx$$

$$< \int_a^b \frac{\varepsilon}{b-a}\,dx \quad \text{by (I)}.$$

$$= \varepsilon \;, \quad \forall y \in [c,d]$$

which satisfy $|y-y_0| < \delta$. We conclude that F is cont. on $[c,d]$

11.5 Theorem. *Let $H = [a,b] \times [c,d]$ be a rectangle in $\mathbf{R}^2$ and let $f : H \to \mathbf{R}$. Suppose that $f(\cdot, y)$ is integrable on $[a,b]$ for each $y \in [c,d]$ and that the partial derivative $f_y(x,\cdot)$ exists on $[c,d]$ for each $x \in [a,b]$. If the two-variable function $f_y(x,y)$ is continuous on H, then*

$F(y)$ $\qquad f(x,y)$.

$$\frac{d}{dy}\int_a^b f(x,y)\,dx = \int_a^b \frac{\partial f}{\partial y}(x,y)\,dx$$

for all $y \in [c,d]$.

NOTE: *These hypotheses are met if $f \in C^1(H)$.*

Proof. Exercise.

Rmk. If $H = [a_1,b_1] \times \cdots \times [a_n,b_n]$ is an n-dimensional rectangle, if f is C^1 on H, and if $k \ne j$, then

$$\frac{\partial}{\partial x_k}\int_{a_j}^{b_j} f(x_1,\ldots,x_n)\,dx_j = \int_{a_j}^{b_j} \frac{\partial f}{\partial x_k}(x_1,\ldots,x_n)\,dx_j.$$

11.1.5. b) $\dfrac{d}{dy}\displaystyle\int_0^1 \sin(e^x y - y^3 + \pi - e^x)\,dx$ at $y = 1$

Sol. $f(x,y) = \sin(e^x y - y^3 + \pi - e^x)$

is C^∞ on $\mathbb{R}^2$, and hence on $[0,1]\times[-1,1]$

by thm 11.5,

$$\frac{d}{dy}\int_0^1 \sin(e^x y - y^3 + \pi - e^x)\,dx$$

$$= \int_0^1 \left[\frac{\partial}{\partial y}\sin(e^x y - y^3 + \pi - e^x)\right] dx$$

$$= \int_0^1 \cos(e^x y - y^3 + \pi - e^x)\cdot(e^x - 3y^2)\,dx$$

At $y=1$ $$= \int_0^1 \cos(\cancel{e^x} - 1 + \pi - \cancel{e^x})(e^x - 3)\,dx$$

$$= \cos(\pi - 1)\int_0^1 (e^x - 3)\,dx$$

$$= \cos(\pi - 1)\left(e^x - 3x\right)\Big]_0^1$$

$$= \cos(\pi - 1)\left(e - 3 - 1\right)$$

$$= (e - 4)\cos(\pi - 1).$$

H.W's 1, 2, 4, 5, 6.

11.2 THE DEFINITION OF DIFFERENTIABILITY

$f = (f_1, f_2, \ldots, f_m)$

In this section we define what it means for a vector function to be differentiable at a point. Whatever our definition, we expect two things: If **f** is differentiable at **a**, then **f** will be continuous at **a** and all first-order partial derivatives of **f** will exist at **a**.

$(a_1, a_2, \ldots, a_n)$

Working by analogy with the one-variable case, we guess that **f** is differentiable at **a** if and only if all its first-order partial derivatives exist at **a**. The following example shows that this guess is wrong even when the range of **f** is one dimensional.

$\times$

11.11 ***EXAMPLE.***

f_x and f_y

Prove that the first-order partial derivatives of

$$f(x, y) = \begin{cases} x + y & x = 0 \text{ or } y = 0 \\ 1 & \text{otherwise} \end{cases}$$

$x \neq 0$ and $y \neq 0$

exist at $(0, 0)$, but f is not continuous at $(0, 0)$. (hence f is not diff'ble)

Pf. $\lim_{(x,y)\to(0,0)} f(x,y) = 1$, $f(0,0) = 0 + 0 = 0$

$\Rightarrow f$ is not cont. at $(0,0)$.

But

$$f_x(0,0) = \lim_{h\to 0} \frac{f(0+h, 0) - f(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{(h+0) - (0+0)}{h} = 1 \text{ exists}$$

Similarly, $$f_y(0,0) = \lim_{k\to 0} \frac{f(0, 0+k) - f(0,0)}{k}$$

$$= \lim_{k\to 0} \frac{k}{k} = 1 \text{ exists.}$$

$f : V \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^m$

11.12 Definition.

Suppose that $\mathbf{a} \in \mathbf{R}^n$, that V is an open set containing $\mathbf{a}$, and that $\mathbf{f} : V \to \mathbf{R}^m$.

i) $\mathbf{f}$ is said to be differentiable at $\mathbf{a}$ if and only if there is a $\mathbf{T} \in \mathcal{L}(\mathbf{R}^n; \mathbf{R}^m)$ such that the function

$T : \mathbb{R}^n \to \mathbb{R}^m$

$$\varepsilon(\mathbf{h}) := \mathbf{f}(\mathbf{a}+\mathbf{h}) - \mathbf{f}(\mathbf{a}) - \mathbf{T}(\mathbf{h})$$

(defined for $\|\mathbf{h}\|$ sufficiently small) satisfies $\varepsilon(\mathbf{h})/\|\mathbf{h}\| \to \mathbf{0}$ as $\mathbf{h} \to \mathbf{0}$.

ii) $\mathbf{f}$ is said to be differentiable on a set E if and only if E is nonempty and $\mathbf{f}$ is differentiable at every point in E.

$$\lim_{h \to 0} \frac{f(a+h) - f(a) - T(h)}{\|h\|} = 0.$$

or $\forall \varepsilon > 0$, $\exists$ a $\delta > 0$ s.t

$$\|h\| < \delta \Longrightarrow \frac{\| f(a+h) - f(a) - T(h) \|}{\|h\|} < \varepsilon$$

Rmk. If f is diffble at $a \in \mathbb{R}^n$, then $\exists$ on T satisfies Def 11.12

It representing $m \times n$ matrix is called the total derivative of f, denoted by $Df(a)$.

$$Df(a) = \left[\frac{\partial f_i}{\partial x_j}(a) \right]_{m \times n}$$

(next thm 11.14)

$$= \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(a) & \cdots & \frac{\partial f_1}{\partial x_n}(a) \\ \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_1}(a) & \cdots & \frac{\partial f_m}{\partial x_n}(a) \end{bmatrix}_{m \times n}$$

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11.13 Theorem. *If a vector function* **f** *is differentiable at* **a**, *then* **f** *is continuous at* **a**.

Proof. Spse that f is diff'ble at a. Then by Def'n 11.12, $\exists$ a $T \in \mathcal{L}(\mathbb{R}^n; \mathbb{R}^m)$ and a $\delta > 0$ such that

$$\|f(a+h) - f(a) - T(h)\| \le \|h\|,$$

for all $\|h\| < \delta$. It follows,

$$\|f(a+h) - f(a)\| = \|f(a+h) - f(a) - T(h) + T(h)\|$$

$$\le \|f(a+h) - f(a) - T(h)\| + \|T(h)\|$$

diff'blity

$$\le \|h\| + \|T(h)\|$$

$$\|T(x)\| \le \|T\|\,\|x\|$$

$$\le \|h\| + \|T\|\,\|h\|$$

$$= \|h\|(1 + \|T\|) \to 0$$

as $h \to 0$ since $\|T\| < +\infty$

$\therefore \|f(a+h) - f(a)\| \to 0$

$$\lim_{h \to 0} f(a+h) = f(a)$$ by Squeeze thm.

i.e, f is continuous at a ∎

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$Df(a)$

If **f** is differentiable at **a**, is there an easy way to compute the total derivative $D\mathbf{f}(\mathbf{a})$? The following result shows that the answer to this question is yes.

11.14 Theorem. *Let* **f** *be a vector function. If* **f** *is differentiable at* **a**, *then all first-order partial derivatives of* **f** *exist at* **a**. *Moreover, the total derivative of* **f** *at* **a** *is unique and can be computed by*

$$D\mathbf{f}(\mathbf{a}) = \left[\frac{\partial f_i}{\partial x_j}(\mathbf{a})\right]_{m\times n} := \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(\mathbf{a}) & \dots & \frac{\partial f_1}{\partial x_n}(\mathbf{a}) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1}(\mathbf{a}) & \dots & \frac{\partial f_m}{\partial x_n}(\mathbf{a}) \end{bmatrix}_{m\times n}.$$

$f = (f_1, f_2, \dots, f_m)$

ex. $(Df(a))$.

$f : \mathbb{R}^{2\,n} \longrightarrow \mathbb{R}^{3\,m}$

$f(x,y) = (\underbrace{x+2y}_{f_1(x,y)}, \underbrace{\sin x}_{f_2(x,y)}, \underbrace{e^y}_{f_3(x,y)})$

find $Df(a)$, $a \in \mathbb{R}^2$

$$Df(x,y) = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} \end{bmatrix}_{3\times 2} = \begin{bmatrix} 1 & 2 \\ \cos x & 0 \\ 0 & e^y \end{bmatrix}_{3\times 2}$$

is called the total derivative of f at $a = (x,y)$.

$$Df(0,0) = \begin{bmatrix} 1 & 2 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Proof. (Thm 11.14)

Since f is diff'ble, $\exists$ an $m\times n$ matrix $B=[b_{ij}]_{m\times n}$ such that

$$\lim_{h\to 0}\frac{f(a+h)-f(a)-Bh}{\|h\|}=0 \quad (*)$$

$e_j=(0,0,\dots,\underset{j\text{th}}{1},0,\dots,0)$

Fix $1\le j\le n$ and set $h=te_j=(0,0,\dots,0,t,0,\dots,0)$ $t>0$

$\|h\|=\|te_j\|$
$=|t|\,\cancel{\|e_j\|}^{\,1}=|t|=t,\ t>0$

Back to $(*)$:

$$\frac{f(a+h)-f(a)-Bh}{\|h\|}:=\frac{f(a+te_j)-f(a)-tBe_j}{t}$$

$$\therefore \lim_{t\to 0^+}\frac{f(a+te_j)-f(a)}{t}=Be_j$$

$$=\begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n}\\ \vdots & & & \\ b_{m1} & b_{m2} & \cdots & b_{mn}\end{bmatrix}\begin{bmatrix}0\\0\\ \vdots\\ 1\\ \vdots\\ 0\end{bmatrix}$$

$$=(b_{1j},b_{2j},\dots,b_{mj}).$$

Similarly, $\lim_{t\to 0^-}\frac{f(a+te_j)-f(a)}{t}=(b_{1j},b_{2j},\dots,b_{mj})$

$$\frac{\partial f_i(a)}{\partial x_j}=\lim_{t\to 0}\frac{f(a+te_j)-f(a)}{t}=(b_{ij}),$$

for $i=1,2,\dots,m$.

$$Df(a)=[b_{ij}]_{m\times n}=\left[\frac{\partial f_i}{\partial x_j}(a)\right]_{m\times n}.$$

$\blacksquare$

Summary. f is diffble at a iff $\exists$ an $m \times n$ matrix B such that

$$\lim_{\mathbf{h}\to\mathbf{0}} \frac{\mathbf{f}(\mathbf{a}+\mathbf{h}) - \mathbf{f}(\mathbf{a}) - B\mathbf{h}}{\|\mathbf{h}\|} = \mathbf{0},$$

if and only if

$$\lim_{\mathbf{h}\to\mathbf{0}} \frac{\|\mathbf{f}(\mathbf{a}+\mathbf{h}) - \mathbf{f}(\mathbf{a}) - B\mathbf{h}\|}{\|\mathbf{h}\|} = 0,$$

or if and only if

$$\lim_{\mathbf{h}\to\mathbf{0}} \frac{\mathbf{f}(\mathbf{a}+\mathbf{h}) - \mathbf{f}(\mathbf{a}) - D\mathbf{f}(\mathbf{a})(\mathbf{h})}{\|\mathbf{h}\|} = \mathbf{0}.$$

$Df(a) = [b_{ij}]$.

Rmk. 1) If all first-order derivatives of f exists at a, we call $Df(a)$ " the "Jacobian matrix"

2) If f is diffble at a, we call $Df(a)$ " the total derivative of f at a.

If $n=1$, $m=1$, the Jacobian matrix Df is an $m\times 1$ or $1\times n$ matrix.

For $n=1$, $Df(a) = \begin{bmatrix} f_1'(a) \\ \vdots \\ f_m'(a) \end{bmatrix}_{m\times 1}$

$f: V \subseteq \mathbb{R} \to \mathbb{R}^m$

$f = (f_1, \dots, f_m)$

$f_1(x)$

$\boxed{m=1}$ $f: V \subseteq \mathbb{R}^n \to \mathbb{R}$

$f = f(x_1, \dots, x_n)$

$Df(a) = \left[\frac{\partial f}{\partial x_1}(a) \; \dots \; \frac{\partial f}{\partial x_n}(a) \right]$

gradian

$\nabla f(a) := \left(\frac{\partial f}{\partial x_1}(a), \dots, \frac{\partial f}{\partial x_n}(a) \right).$

11.15 Theorem. *Let V be open in $\mathbf{R}^n$, let $\mathbf{a} \in V$, and suppose that $\mathbf{f}: V \to \mathbf{R}^m$. If all first-order partial derivatives of $\mathbf{f}$ exist in V and are continuous at $\mathbf{a}$, then $\mathbf{f}$ is differentiable at $\mathbf{a}$.*

Proof. Exercise (P.398)

$f(x,y)$ ✓

Rmk. the procedure to determine whether a vector function f is diff'ble at a point a.

1) Compute all first-order partial derivatives of $\mathbf{f}$ at $\mathbf{a}$. If one of these does not exist, then $\mathbf{f}$ is not differentiable at $\mathbf{a}$ (Theorem 11.14).
2) If all first-order partial derivatives exist and are continuous at $\mathbf{a}$, then $\mathbf{f}$ is differentiable at $\mathbf{a}$ (Theorem 11.15).
3) If the first-order partial derivatives of $\mathbf{f}$ exist but one of them fails to be continuous at $\mathbf{a}$, then use the definition of differentiability directly. By the

$x>0$.

11.16 *EXAMPLE.*

$f: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

Is $f(x, y) = (\cos(xy), \ln x - e^y)$ differentiable at $(1, 1)$?

$$f_x = \left(\frac{\partial}{\partial x}(\cos xy), \frac{\partial}{\partial x}(\ln x - e^y) \right)$$

$$= \left(-y \sin(xy), \frac{1}{x} \right)$$

$$f_y = \left(-x \sin(xy), -e^y \right)$$

f_x and f_y both exist and are continuous at any $(x,y) \in \mathbb{R}^2$ with $x>0$.

$\left(\{ (x,y) \in \mathbb{R}^2 : x>0 \} \right.$

In particular, f_x and f_y are exist and continuous at $(1,1)$

$\cdot (1,1)$.

by Thm 11.15, f is diff'ble at $(1,1)$.

11.17 *EXAMPLE.*

Is

$$f(x, y) = \begin{cases} \dfrac{y^2}{x^2 + y^2} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$$

$\mathbb{R}$.

$(h,0) \neq (0,0)$

differentiable at $(0, 0)$?

Sol.

$$f_x(x,y) = \frac{(x^2+y^2)(0) - y^2(2x)}{(x^2+y^2)^2} = \frac{-2xy^2}{(x^2+y^2)^2}$$

$\forall (x,y) \neq (0,0)$

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$$f_x(0,0) \overset{\text{by def'n.}}{=} \lim_{h\to 0} \frac{f(0+h,0)-f(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{f(h,0) - \cancel{f(0,0)}^{\,0}}{h}$$

$$= \lim_{h\to 0} \frac{\frac{0}{h^2+0} - 0}{h} = \lim_{h\to 0} \frac{0}{h^3}$$

$$= \lim_{h\to 0} 0 = 0 \text{ exists}$$

$f_x(0,0)$ exists

$$\lim_{(x,y)\to(0,0)} f_x(x,y) = \lim_{(x,y)\to(0,0)} \frac{-2xy^2}{(x^2+y^2)^2} \text{ DNE}$$

(check)

f_x exist but not cont. at $(0,0)$

لا نستنتج شيء .

$$f_y(0,0) = \lim_{k\to 0} \frac{f(0, 0+k) - f(0,0)}{k}$$

$$= \lim_{k\to 0} \frac{f(0,k) - f(0,0)}{k}$$

$$= \lim_{k\to 0} \frac{\frac{k^2}{0+k^2} - 0}{k}$$

$$= \lim_{k\to 0} \frac{1}{k} \text{ DNE.}$$

$\Rightarrow f_y$ DNE at $(0,0)$

$\Rightarrow f$ is not diff'ble at $(0,0)$. (Thm 11.14).

11.18 *EXAMPLE.*

Prove that

$$f(x, y) = \begin{cases} (x^2 + y^2) \sin \dfrac{1}{\sqrt{x^2 + y^2}} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$$

is differentiable on $\mathbf{R}^2$ but not continuously differentiable at $(0, 0)$.

Sol. If $(x,y) \neq (0,0)$,

$$f_x(x,y) = -\frac{x}{\sqrt{x^2+y^2}} \cos\left(x^2+y^2\right) + 2x \sin\left(\frac{1}{\sqrt{x^2+y^2}}\right)$$

Thus, f is diff'ble on $\mathbb{R}^2 \setminus \{(0,0)\}$.

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{h^2 \sin\left(\frac{1}{|h|}\right) - 0}{h}$$

$$= \lim_{h\to 0} h \sin\left(\frac{1}{|h|}\right) = 0 \text{ by Squeeze thm}$$

$$\left(\left| h \sin\left(\frac{1}{|h|}\right) \right| \leq |h| \to 0 \text{ as } h \to 0 \right.$$

$\therefore f_x(0,0) = 0$ exists.

Similarly, $f_y(0,0) = 0$ (Exercise).

but $\lim_{(x,y)\to(0,0)} f_x(x,y)$ DNE

$\therefore f_x$ is not continuous at $(0,0)$

$\Rightarrow f$ is not continuously diff'ble.

$$Df(0,0) = \nabla f(0,0) = (f_x(0,0), f_y(0,0)) = (0,0).$$

To prove the differentiability, we use the def'n.

$$\frac{f(a+h) - f(a) - \nabla f(a)\cdot h}{\|h\|}$$

$a = (0,0)$

$h = (h,k)$

$\|h\| = \sqrt{h^2+k^2}$

$$= \frac{f((0,0)+(h,k)) - f(0,0) - \nabla f(0,0)\cdot(h,k)}{\sqrt{h^2+k^2}}$$

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$$= \frac{f(h,k) - f(0,0) - \nabla f(0,0) \cdot (h,k)}{\sqrt{h^2+k^2}}$$

$(f_x(0,0), f_y(0,0))$

$$= \frac{(h^2+k^2)\sin\left(\frac{1}{\sqrt{h^2+k^2}}\right) - (0,0)\cdot(h,k)}{\sqrt{h^2+k^2}}$$

$$= \sqrt{h^2+k^2}\,\sin\left(\frac{1}{\sqrt{h^2+k^2}}\right) \longrightarrow 0 \text{ exists}$$

as $(h,k) \to (0,0)$

by squeeze thm

$\therefore$

$\therefore$ f is diffble at $(0,0)$ by defn.

HWs 1, 2, 3, 4, 5, 6, 7, 8, 9, 10

H.W's Exercise # 1 (a–d)

11.20 Theorem. *Let* $\alpha \in \mathbf{R}$, $\mathbf{a} \in \mathbf{R}^n$, *and suppose that* $\mathbf{f}$ *and* $\mathbf{g}$ *are vector functions. If* $\mathbf{f}$ *and* $\mathbf{g}$ *are differentiable at* $\mathbf{a}$, *then* $\mathbf{f}+\mathbf{g}$, $\alpha\mathbf{f}$, *and* $\mathbf{f}\cdot\mathbf{g}$ *are all differentiable at* $\mathbf{a}$. *In fact,*

$$D(\mathbf{f}+\mathbf{g})(\mathbf{a}) = D\mathbf{f}(\mathbf{a}) + D\mathbf{g}(\mathbf{a}), \tag{7}$$
$$D(\alpha\mathbf{f})(\mathbf{a}) = \alpha D\mathbf{f}(\mathbf{a}), \tag{8}$$

and

$$\overset{1\times n}{D(\mathbf{f}\cdot\mathbf{g})(\mathbf{a})} = \underbrace{\overset{1\times m}{\mathbf{g}(\mathbf{a})}\overset{m\times n}{D\mathbf{f}(\mathbf{a})}}_{1\times n} + \underbrace{\underset{1\times m}{\mathbf{f}(\mathbf{a})}\overset{m\times n}{D\mathbf{g}(\mathbf{a})}}_{1\times n}. \tag{9}$$

[The sums which appear on the right side of (7) and (9) represent matrix addition, and the products which appear on the right side of (9) represent matrix multiplication.]

Proof. Exercise.

f is diff'ble if $\exists\ T \in \mathcal{L}\{\mathbb{R}^n, \mathbb{R}^m\}$

s.t $\displaystyle\lim_{h\to 0} \frac{\overbrace{f(a+h) - f(a) - T(h)}^{\varepsilon(h)}}{\|h\|} = 0$

$T(h)$: $Df(a)$ ↳ total derivative of f at a.

or $\dfrac{\varepsilon(h)}{\|h\|} \to 0$ as $h \to 0$ (for $\|h\|$ sufficiently small).

$$Df(a) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}_{m\times n}.$$

$f:\mathbb{R}^2 \to \mathbb{R}$ $\quad g:\mathbb{R}^2 \to \mathbb{R}$.

Ex. 1 ⓐ $f(x,y) = x-y$, $g(x,y) = x^2+y^2$

$$D(f+g)(x,y) = (Df)(x,y) + (Dg)(x,y)$$
$$= \begin{bmatrix} f_x & f_y \end{bmatrix} + \begin{bmatrix} g_x & g_y \end{bmatrix}$$
$$= \begin{bmatrix} 1 & -1 \end{bmatrix} + \begin{bmatrix} 2x & 2y \end{bmatrix}$$
$$= \begin{bmatrix} 1+2x & -1+2y \end{bmatrix}_{1\times 2}$$

scalar $g:\mathbb{R}^2 \to \mathbb{R}$

$$D(f\cdot g)(x,y) = g(x,y)\, Df(x,y) + f(x,y)\, Dg(x,y)$$
$$= (x^2+y^2)\begin{bmatrix} 1 & -1 \end{bmatrix} + (x-y)\begin{bmatrix} 2x & 2y \end{bmatrix}$$
$$= \begin{bmatrix} x^2+y^2 & -x^2-y^2 \end{bmatrix} + \begin{bmatrix} 2x^2-2xy & 2xy-2y^2 \end{bmatrix}$$
$$= \begin{bmatrix} 3x^2+y^2-2xy & -x^2+2xy-3y^2 \end{bmatrix}_{1\times 2}$$

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11.4 THE CHAIN RULE

Here is the Chain Rule for vector functions.

in 1-dim., $(f\circ g)'(a) = f'(g(a))\, g'(a)$

11.28 Theorem. [Chain Rule]. *Suppose that* **f** *and* **g** *are vector functions. If* **g** *is differentiable at* **a** *and* **f** *is differentiable at* **g(a)**, *then* **f** ∘ **g** *is differentiable at* **a** *and*

$$D(\mathbf{f}\circ\mathbf{g})(\mathbf{a}) = D\mathbf{f}(\mathbf{g}(\mathbf{a}))D\mathbf{g}(\mathbf{a}). \tag{20}$$

[The product $D\mathbf{f}(\mathbf{g}(\mathbf{a}))D\mathbf{g}(\mathbf{a})$ is matrix multiplication.]

Proof. $a \in \mathbb{R}^n$, set $b := g(a) \in \mathbb{R}^m$, $f(b) \in \mathbb{R}^p$

Set $\boxed{\underset{p\times n}{T} = \underbrace{Df(g(a))}_{p\times m}\ \underbrace{Dg(a)}_{m\times n}.}$

We need to show

$$\lim_{h\to 0} \frac{f(g(a+h)) - f(g(a)) - T(h)}{\|h\|} = 0$$

Set $\boxed{\varepsilon(h) = \overbrace{g(a+h) - g(a)}^{k} - Dg(a)(h).}$ (1)

and $\boxed{\delta(k) = f(b+k) - f(b) - Df(b)(k).}$ (2)

for $\|h\|$ and $\|k\|$ sufficiently small.

Since g is diffble at a, then by def'n

$$\frac{\varepsilon(h)}{\|h\|} \to 0 \text{ in } \mathbb{R}^m \text{ as } h \to 0 \text{ in } \mathbb{R}^n$$

Since f is diffble at $g(a) := b$, then by

by def'n, $\dfrac{\delta(k)}{\|k\|} \to 0$ in $\mathbb{R}^p$ as $k \to 0$ in $\mathbb{R}^m$.

Fix h small and set $\boxed{k = g(a+h) - g(a)}$ (3)

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Then (1) and (2) imply تعويض

$f(g(a+h)) - f(g(a))$

$= f(b+k) - f(b)$

set $g(a+h) - \underbrace{g(a)}_{b} = k$

$g(a+h) = b+k$

$\overset{eq(2)}{=} Df(b)(k) + \delta(k)$

$\overset{eq(1)+(3)}{=} Df(b)[Dg(a)(h) + \varepsilon(h)] + \delta(k)$

$= T(h) + Df(b)(\varepsilon(h)) + \delta(k)$

We have

$$\boxed{f(g(a+h)) - f(g(a)) - T(h) = Df(b)(\varepsilon(h)) + \delta(k)}$$

$$\boxed{\frac{(f\circ g)(a+h) - (f\circ g)(a) - T(h)}{\|h\|} = \underbrace{Df(b)}_{\text{matrix}} \frac{\varepsilon(h)}{\|h\|} + \frac{\delta(k)}{\|k\|}} \quad (*)$$

Now, $\frac{\varepsilon(h)}{\|h\|} \to 0$ as $h \to 0$ since g is diff'ble at a.

~~Ef~~ If $k = 0 \Rightarrow \delta(k) = 0$

If $k \neq 0$,

$$\frac{\delta(k)}{\|h\|} = \frac{\|k\|}{\|h\|} \frac{\delta(k)}{\|k\|}$$

$$= \frac{\|g(a+h) - g(a)\|}{\|h\|} \frac{\delta(k)}{\|k\|}$$

$\|Tx\| \le \|T\| \|x\|$

$$= \frac{\|Dg(a)(h) + \varepsilon(h)\|}{\|h\|} \frac{\delta(k)}{\|k\|}$$

$$\le \frac{\|Dg(a)\| \|h\| + \|\varepsilon(h)\|}{\|h\|} \frac{\delta(k)}{\|k\|}$$

$$\frac{\delta(k)}{\|h\|} \le \left(\underbrace{\|Dg(a)\| + \frac{\|\varepsilon(h)\|}{\|h\|}}_{} \right) \frac{\delta(k)}{\|k\|}$$

$\|Dg(a)\|$ exist as $h \to 0$

$\frac{\delta(k)}{\|k\|} \to 0$ as $h \to 0$

since $k \to 0$ implies $h \to 0$

why?

$k = g(a+h) - g(a) \to 0$

$g(a+h) \xrightarrow[h \to 0]{} g(a)$

$$\therefore \frac{\delta(k)}{\|h\|} \to 0 \quad \text{as } h \to 0.$$

Back to $(*)$:

$$\frac{(f \circ g)(a+h) - (f \circ g)(a) - T(h)}{\|h\|} \to 0 \text{ as } h \to 0.$$

We conclude that $f \circ g$ is diff'ble at a. and the derivative is

$$T = Df(g(a))\, Dg(a).$$

$f \circ g$

$g: \mathbb{R}^n \to \mathbb{R}^m$, $f: \mathbb{R}^m \to \mathbb{R}$

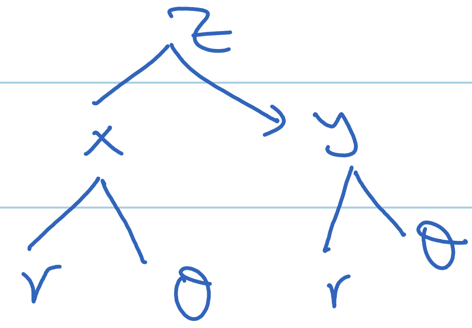
Rmk. $z = f\left(g(x_1, \ldots, x_n)\right) : \mathbb{R}^n \to \mathbb{R}$

Let $\vec{u} = g(x_1, \ldots, x_n) \in \mathbb{R}^m = (u_1, u_2, \ldots, u_m)$, $z = f(u)$.

$$\frac{\partial z}{\partial x_j} = \frac{\partial f}{\partial u_1}\frac{\partial u_1}{\partial x_j} + \frac{\partial f}{\partial u_2}\frac{\partial u_2}{\partial x_j} + \cdots + \frac{\partial f}{\partial u_m}\frac{\partial u_m}{\partial x_j}$$

11.29 *EXAMPLES.*

i) If $F, G, H : \mathbf{R}^2 \to \mathbf{R}$ are differentiable and $z = F(x, y)$, where $x = G(r, \theta)$, and $y = H(r, \theta)$, then

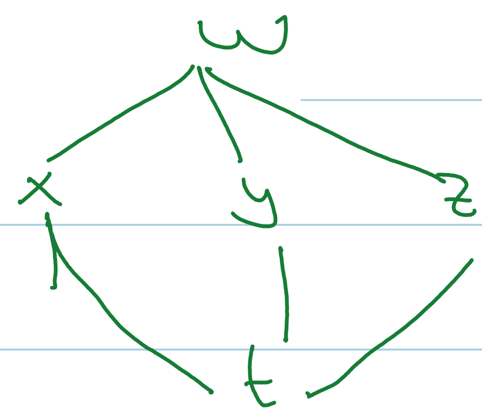


$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r}$$

$$\frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial \theta}.$$

ii) If $f : \mathbf{R}^3 \to \mathbf{R}$ and $\phi, \psi, \sigma : \mathbf{R} \to \mathbf{R}$ are differentiable and $w = f(x, y, z)$, $= w(t)$ where $\boxed{x = \phi(t)}$, $\boxed{y = \psi(t)}$, and $\boxed{z = \sigma(t)}$, then

$$\boxed{\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}}.$$



H.W's 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11.

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Discussion 11.1, 11.2, 11.4

Section 11.1

1, 2, 4, 5, 6

$|f(x,y) - f(x,w)| < \varepsilon$

11.1.4. Suppose that $H = [a,b] \times [c,d]$ is a rectangle, that $f : H \to \mathbf{R}$ is continuous, and that $g : [a,b] \to \mathbf{R}$ is integrable. Prove that

$$\boxed{F(y) = \int_a^b g(x) f(x,y)\,dx}$$

is uniformly continuous on $[c,d]$.

$\Rightarrow g$ is bdd

$|g(x)| \le M, \ \forall x \in [a,b]$

proof. We want to show that $\forall \varepsilon > 0, \exists$ a $\delta > 0$ s.t. if $|y-w| < \delta$, $y, w \in [c,d] \Rightarrow$

$|F(y) - F(w)| < \varepsilon$.

Let $\varepsilon > 0$, Since f is uniformly cont. on H, $\exists\, \delta > 0$ s.t. $y, w \in [c,d]$, $|y-w| < \delta \Rightarrow |f(x,y) - f(x,w)| < \frac{\varepsilon}{M(b-a)}$

Therefore, $|F(y) - F(w)| = \left| \int_a^b f(x,y) g(x)\,dx - \int_a^b f(x,w) g(x)\,dx \right|$

$$= \left| \int_a^b g(x) [f(x,y) - f(x,w)]\,dx \right|$$

$$\le \int_a^b \underbrace{|g(x)|}_{\le M} |f(x,y) - f(x,w)|\,dx$$

Since g is integrable on $[a,b]$ $\Rightarrow g$ is bdd on $[a,b]$

$$\le M \int_a^b |f(x,y) - f(x,w)|\,dx$$

f is cont. on "H" compact $\Rightarrow f$ is uniformly cont. on H.

$$\le M \int_a^b \frac{\varepsilon}{M(b-a)}\,dx = \varepsilon.$$

$\therefore F$ is uniformly cont. on $[c,d]$.

11.1.5. Evaluate each of the following expressions.

a) $$\lim_{y\to 0}\int_0^1 e^{x^3y^2+x}\,dx$$

b) $$\frac{d}{dy}\int_0^1 \sin(e^x y - y^3 + \pi - e^x)\,dx \quad \text{at } y = 1$$

c) $$\frac{\partial}{\partial x}\int_1^3 \underbrace{\sqrt{x^3+y^3+z^3-2}}_{f(x,y,z)}\,dz \quad \text{at } (x,y) = (1,1)$$

$H = [1,3]\times[1,3]\times[1,3]$ (x, y, z)

$f = \sqrt{x^3+y^3+z^3-2} \geq \sqrt{1+1+1-2} = 1 > 0$

$$f_x = \frac{3x^2}{2\sqrt{x^3+y^3+z^3-2}}$$

$$f_y = \frac{3y^2}{2\sqrt{x^3+y^3+z^3-2}}, \quad f_z = \cdots\cdots$$

f_x, f_y, f_z exist & cont. on H

Thm 11.5 $\Longrightarrow$ At $(x,y) = (1,1)$

$$\frac{\partial}{\partial x}\int_1^3 f(x,y,z)\,dz = \int_1^3 \left(\frac{\partial f}{\partial x}\right) dz$$

$$= \int_1^3 \frac{3x^2}{2\sqrt{x^3+y^3+z^3-2}}\,dz \Bigg|_{(x,y)=(1,1)}$$

$$= \int_1^3 \frac{3}{2\sqrt{z^3}}\,dz$$

$$= \frac{3}{2}\int_1^3 z^{-\frac{3}{2}}\,dz = \cdots\cdots$$

11.1.6. Suppose that f is a continuous real function.

a) If $\int_0^1 f(x)\,dx = 1$, find the exact value of

$$\lim_{y\to 0}\int_0^2 f(|x-1|)e^{x^2y+xy^2}\,dx.$$

cont. on $\mathbb{R}$ & hence on $[0,2]$

cont. on $\mathbb{R}^2$

$\therefore$ the integrand is cont. on $\mathbb{R}^2$

by thm 11.4

$$= \int_0^2 \lim_{y\to 0}\left[f(|x-1|)\,e^{x^2y+xy^2}\right]dx$$

$$= \int_0^2 f(|x-1|)\,dx$$

$1-x$, $x-1$; 0, 1, 2

$$= \int_0^1 f(1-x)\,dx + \int_1^2 f(x-1)\,dx$$

$y = 1-x$ $\qquad$ $w = x-1$

$dy = -dx$ $\qquad$ $dw = dx$

$$= \int_1^0 f(y)\,(-dy) + \int_0^1 f(w)\,dw$$

$$= \int_0^1 f(y)\,dy + \int_0^1 f(w)\,dw$$

$$= 1+1 = 2.$$

Section 11.2 (1 - 10)

11.2.6. Prove that if $\alpha > 1/2$, then

$$f(x,y) = \begin{cases} |xy|^{\alpha} \log(x^2+y^2) & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

$|h.0|^{\alpha} \log(h^2)$

is differentiable at (0, 0).

Sol. we want

$$\lim_{(h,k)\to(0,0)} \frac{f(h,k) - f(0,0) - \nabla f(0,0)\cdot(h,k)}{\|(h,k)\|} = 0 \qquad g(h,k)$$

$$f_x(0,0) = \lim_{h\to 0} \frac{f(0+h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h}$$

$$= \lim_{h\to 0} 0 = 0$$

Similarly, $f_y(0,0) = 0$

$$\nabla f(0,0) = (f_x(0,0), f_y(0,0)) = (0,0)$$

$$= \lim_{(h,k)\to(0,0)} \frac{|hk| \log(h^2+k^2) - 0 - (0,0)\cdot(h,k)}{\sqrt{h^2+k^2}}$$

$$= \lim_{(h,k)\to(0,0)} \frac{|hk| \log(h^2+k^2)}{\sqrt{h^2+k^2}} \qquad g(h,k)$$

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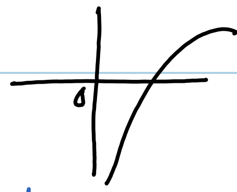

$$|g(h,k)| = \left| \frac{(hk)^\alpha \log(h^2+k^2)}{\sqrt{h^2+k^2}} \right|$$

$|\log x| = -\log x$

$$\le \frac{\left(\frac{h^2+k^2}{2}\right)^\alpha}{(h^2+k^2)^{\frac{1}{2}}}\left(-\log(h^2+k^2)\right)$$

$h^2+k^2-2hk \geq 0$

$h^2+k^2 \geq 2hk$

$\boxed{hk \leq \frac{h^2+k^2}{2}}$

$\alpha > \frac{1}{2}$

$$= \frac{(h^2+k^2)^{\alpha-\frac{1}{2}}}{2^\alpha} \log\left(\frac{1}{h^2+k^2}\right) \longrightarrow 0$$

as $(h,k) \to (0,0)$.

Since $\boxed{\lim_{u\to 0} u^\varepsilon \log\left(\frac{1}{u}\right) = 0, \quad \forall \varepsilon > 0}$ (*)

$(0 \cdot \infty)$

(14.w)

$\therefore \lim_{(h,k)\to(0,0)} g(h,k) = 0$ by Squeeze thm.

Section 11.4 All (1 – 11).

11.4.4. Let $f, g : \mathbf{R} \to \mathbf{R}$ be twice differentiable. Prove that $\boxed{u(x, y) := f(xy)}$ satisfies

$$\boxed{x\frac{\partial u}{\partial x} - y\frac{\partial u}{\partial y} = 0,}$$

and $\boxed{v(x, y) := f(x - y) + g(x + y)}$ satisfies the *wave equation*; that is,

$$\frac{\partial^2 v}{\partial x^2} - \frac{\partial^2 v}{\partial y^2} = 0.$$

$V_x = f'(x-y)\cdot 1 + g'(x+y)\cdot 1$

$V_{xx} = f''(x-y) + g''(x+y)$

Sol.
$$\frac{\partial u}{\partial x} = f'(xy)\left[\frac{\partial}{\partial x}(xy)\right] = y f'(xy).$$

$$\frac{\partial u}{\partial y} = f'(xy) \frac{\partial}{\partial y}(xy) = x f'(xy)$$

$$\therefore\ x\frac{\partial u}{\partial x} - y\frac{\partial u}{\partial y} = xyf'(xy) - yxf'(xy) = 0.$$

11.4.7. Let

$$\boxed{u(x, t) = \frac{e^{-x^2/4t}}{\sqrt{4\pi t}},} \qquad t > 0,\ x \in \mathbf{R}.$$

a) Prove that u satisfies the *heat equation* (i.e., $\boxed{u_{xx} - u_t = 0}$ for all $t > 0$ and $x \in \mathbf{R}$).

$$U_x = \frac{e^{-\frac{x^2}{4t}}}{\sqrt{4\pi t}} \cdot \frac{-2x}{4t} = \left(\frac{-x}{2t\sqrt{4\pi t}}\right) e^{-\frac{x^2}{4t}}$$

$$U_{xx} = \frac{-1}{2t\sqrt{4\pi t}} e^{-\frac{x^2}{4t}} - \frac{x}{2t\sqrt{4\pi t}} e^{-\frac{x^2}{4t}}\left(\frac{-x}{2t}\right)$$

$$\boxed{U_{xx} = \frac{-1}{2t\sqrt{4\pi t}} e^{-\frac{x^2}{4t}} + \frac{x^2}{4t^2\sqrt{4\pi t}} e^{-\frac{x^2}{4t}}}$$

$U_t = \cdots\cdots$

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$u \to L$ $\qquad$ $x \geq a$.

b) If $a > 0$, prove that $u(x, t) \to 0$, as $t \to 0+$, uniformly for $x \in [a, \infty)$.

Sol.

$$u(x, t) = \frac{e^{-x^2/4t}}{\sqrt{4\pi t}},$$

$x \geq a \quad x^2 \geq a^2$

$-x^2 \leq -a^2$

$e^{-x^2} \leq e^{-a^2}$

$$\underbrace{|u(x,t) - 0|}_{L} = \frac{e^{\frac{-x^2}{4t}}}{\sqrt{4\pi t}} \leq \frac{e^{\frac{-a^2}{4t}}}{\sqrt{4\pi t}}$$

Now,

$$\boxed{\lim_{t \to 0^+} \frac{e^{\frac{-a^2}{4t}}}{\sqrt{4\pi t}} = 0} \quad \left(\tfrac{0}{0}\right)$$

by L'Hopital Rule (Exercise)

[illegible]

$$|u(x,t) - 0| \leq \frac{e^{\frac{-a^2}{4t}}}{\sqrt{4\pi t}} \to 0 \text{ as } t \to 0^+$$

$$\Rightarrow \lim_{t \to 0^+} u(x,t) = 0 \text{ uniformly}$$

(indep. of x).